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Planck's Law and Light Quantum Hypothesis.

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Planck's formula for the distribution of energy in the radiation from a black body was the starting point of the quantum theory, which has been developed during the last 20 years and has borne a wealth of fruit in energy domain of physics. Since its publication in 1901 many methods for deriving this law have been proposed. It is recognized that basic assumptions of the quantum theory are irreconcilable with the laws of classical electrodynamics. All derivations up to now use the relation

$$\rho_\nu d\nu = \frac{8\pi\nu^2 d\nu}{c^3}\, E,$$

that is, the relation between the radiation density and the mean energy of an oscillator, and they make assumptions about the number of degrees of freedom of the ether, which appear in the above formula (the first factor on the right– hand side). This factor, however, can be derived only from classical theory. This is the unsatisfactory feature in all derivations and it is therefore no wonder that attempts are being made to obtain a derivation that is free of this logical flaw.

Einstein has given a remarkably elegant derivation. He recognized the logical defect of all previous derivations and tried to deduce the formula independently of classical theory. From very simple assumptions about the energy exchange between molecules and a radiation field he found the relation

$$\rho_\nu = \frac{\alpha_{mn}}{e^{\frac{\epsilon_m - \epsilon_n}{kT}} - 1}.$$

To make this formula agree with Planck's he had to use Wien's displacement law and Bohr's correspondence principle. Wien's law is based on classical theory and the correspondence principle assumes that the quantum theory and the classical theory coincide in centrum limits.

In all cases it appears to me that the derivations have not been sufficiently justified from a logical point of view. As opposed to these the light quantum hypothesis combined with statistical mechanics (as it was formulated to meet the needs of the quantum theory) appears sufficient for the derivation of the law independent of classical theory. In the following I shall sketch the method briefly.

Let the radiation be enclosed in the volume V and let its total energy be E. Let various types of quanta be present of abundances N_8 and energy $h\nu_8(s = 0 \quad \text{to} \quad s = \infty)$. The total energy is then

$$E = \sum_s N_s h\nu_s = V \int \rho_\nu d\nu \tag{1}$$

The solution of the problem therefore requires the determination of the N_S, which, in turn, determine ρ_ν. If we can give the probability for each distribution characterized by arbitrary values of N_s then the solution is given by the condition that this probability is to be a maximum, keeping in mind the condition (1) which is a constraint on the problem. We now seek this probability.

The quantum has the momentum $\frac{h\nu_s}{c}$ in the direction of its motion. The momentary state of the quantum is characterized by its coordinates x, y, z and the corresponding components of the momentum p_x, p_y, p_z. These six quantities can be considered as point coordinates in a six–dimensional space, where we have the relation

$$p_{x^2} + p_{y^2} + p_{z^2} = \frac{h^2\nu^2}{c^2},$$

in virtue of which point representing the quantum in our six–dimensional space is forced to lie on a cylindrical surface determined by the frequency. To the frequency range d_{ν_s} there belongs in this sense the phase space

$$\int dx\ dy\ dz\ dp_x\ dp_y\ dp_z = V \cdot 4\pi(h_\nu/c)^2 h d_\nu/c = 4\pi \cdot h^3\nu^3/c^3 \cdot V \cdot d\nu$$

If we divide the total phase volume into cells of size h^3, there are then $4\pi \cdot \nu^2/c^3 \cdot d_\nu$ cells in the frequency range d_ν. Nothing definite can be said about the method of dividing the phase space in this manner. However, the total number of cells must be considered as equal to the number of possible ways of placing a quantum in this volume. To take into account polarization it appears necessary to multiply this number by 2 so that we obtain $8\pi V\nu^2 d\nu/c^3$ as the number of cells belonging to d_ν.

It is now easy to calculate the thermodynamic probability of a (macroscopically defined) state. Let N^s be number of quanta belonging to the frequency range $d\nu^s$. In how many ways can these be distributed among the cells that belong to $d\nu^s$? Let p_0^s be number of empty cells., p_1^s the number containing 1 quantum, p_2^s the number containing 2 quanta, and so on. The number of possible distributions is then

$$\frac{A^s!}{p_0^s! p_1^s! \dots} \qquad \text{where} \qquad A^s = \frac{8\pi\nu^2}{c^3} \cdot V d\nu^s$$

and where

$$N^s = 0 \cdot p_0^s + 1 \cdot p_1^s + 2p_2^s + \dots$$

is the number of quanta belonging to $d\nu^s$.

The probability W of the state defined by all p_r^s is clearly

$$\Pi \frac{A^s!}{{}_s p_0^s! p_1^s \dots}$$

Taking into account that the p_r^s are large numbers we have

$$\log W = \sum_s A^s \log A^s - \sum_s \sum_r p_r^s \log p_r^s$$

where

$$A^s = \sum_r p_r^s.$$

This expression must be a maximum under the constraints

$$E = \sum_s N^s h\nu^s; \quad N^s = \sum_r = rp_r^s.$$

Carrying through the variations we obtain the conditions

$$\sum_s \sum_r \delta p_r^s (1 + \log p_r^s) = 0, \ \sum \delta N^s h\nu^s = 0$$

$$\sum_r \delta p_r^s = 0 \qquad\qquad \delta N^s = \sum_r r\delta p_r^s.$$

From this we obtain

$$\sum_r \sum_s \delta p_r^s (1 + \log p_r^s + \lambda^s) + \frac{1}{\beta} \sum_s r \delta p_r^s = 0$$

From this we first see that

$$p_r^s = B^s e^{-\dfrac{rh\nu^s}{\beta}}.$$

Since, however,

$$A^s = \sum_r B^s e^{-\dfrac{rh\nu^s}{\beta}} = B^s (1 - e^{-\dfrac{h\nu^s}{\beta}})^{-1}$$

then

$$B^s = A^s (1 - e^{-\dfrac{h\nu^s}{\beta}}).$$

We further have

$$N^s = \sum_r r p_r^s = \sum_r r A^s (1 - e^{-\dfrac{h\nu^s}{\beta}}) e^{-\dfrac{rh\nu^s}{\beta}} = \frac{A^s e^{-\dfrac{h\nu^s}{\beta}}}{1 - e^{\dfrac{h\nu^s}{\beta}}}$$

Taking into account the value of A^s found above, we have

$$E = \sum_S \frac{8\pi h \nu^{s3} d\nu^s}{c^3} V \frac{e^{-\dfrac{h\nu^s}{\beta}}}{1 - e^{h\nu^s/\beta}}$$

Using the result obtained previously

$$S = k \left[\frac{E}{\beta} - \sum_s A^s \log(1 - e^{h\nu^s/\beta}) \right]$$

and nothing that

$$\frac{\partial S}{\partial E} = \frac{1}{T}$$

we obtain

$$\beta = kT$$

Hence

$$E = \sum_S \frac{8\pi h \nu^{s3}}{c^3} V \frac{1}{e^{h\nu^s/kT} - 1} d\nu^s$$

which is Planck's formula.

Comment of translator. Bose's derivation of Planck's formula appears to me to be an important step forward. The method used here gives also the quantum theory of an ideal gas, as I shall show elsewhere. [A. Einstein]