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On Nuclear Forces

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Abstract

The various types of exchange forces that are being used in current discussions of nuclear structure may all be simply expressed in terms of a formalism which attributes five coordinates to each "heavy" particle and applies the Pauli exclusion principle to all the particles in the system. The simplest assumption for the interaction law is that which implies equality of proton-proton and neutron-neutron forces and also equality with the proton-neutron forces of corresponding symmetry. This is in accord with the empirical knowledge of these interactions at present.

In this paper we show how the use of a coordinate having two proper values which tells whether a particle is a proton or a neutron, together with the assumption of the Pauli exclusion principle for all the particles, gives a unified description of the various types of exchange forces used in nuclear structure theories. Such a coordinate was first introduced by Heisenberg[1] and also plays a role in the Fermi-Konopinski-Uhlenbeck[2] theory of beta disintegration.

We suppose that each heavy particle (proton or neutron) is described by five coordinates. These are three for its position in space, a spin coordinate a giving the component of its angular momentum along some direction in space, and a fifth coordinate, τ, which can have the values ± 1. If τ has the value $+1$ the particle is a proton, while the value -1 indicates that it is a neutron.

[1]Heisenberg, Zeits. f. Physik **77,** 1 (1932).

[2]Fermi, Zeits. f. Physik **88,** 161 (1934); Konopinski and Uhlenbeck, Phys. Rev. **48,** 7 (1935).

The spin angular momentum is a vector equal to $\frac{1}{2}\hbar$ times the vector, $\boldsymbol{\sigma}$, which is represented by

$$\boldsymbol{\sigma} = \begin{pmatrix} \mathbf{k} & \mathbf{i} - i\mathbf{j} \\ \mathbf{i} + i\mathbf{j} & -\mathbf{k} \end{pmatrix},$$

the rows and columns referring to states which are labeled by precise values of the z component of $\boldsymbol{\sigma}$. This nonrelativistic description of the spin was introduced by Pauli and by Darwin.

In the same way can be considered purely formally like the z component of a vector. The analogy is purely formal in that the three "components" of $\boldsymbol{\tau}$ do not refer to directions in space. Formally we may write

$$\boldsymbol{\tau} = \begin{pmatrix} \mathbf{n} & \mathbf{1} - i\mathbf{m} \\ \mathbf{1} + i\mathbf{m} & -\mathbf{n} \end{pmatrix},$$

where $\mathbf{1}, \mathbf{m}$ and $\mathbf{n}$ behave algebraically like the three unit vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. The third component of $\boldsymbol{\tau}$ may be called the character coordinate and the whole expression $\boldsymbol{\tau}$ the character vector.

We postulate that in an assembly of heavy particles the wave function has to be antisymmetric in all particles with regard to exchange of all five of their coordinates. We want to show that this gives a convenient formalism for working with nuclear problems.

Let us first consider any attribute of a single heavy particle such as its mass, its charge or its magnetic moment. If A is the arithmetic mean of the two values for proton and neutron and B is half the difference, proton value minus neutron value, then that attribute will appear in the equations as a term involving,

$$(A + B\tau).$$

For example, the electrostatic charge will appear as $\frac{1}{2}(1 + \tau)$ where e is the electronic charge.

Next, let us consider the scalar product $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ of the character vectors associated with two particles. We have

$$(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2)^2 = \boldsymbol{\tau}_1^2 + \boldsymbol{\tau}_2^2 + 2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2,$$

since the operators for two different particles commute. Now as defined $\boldsymbol{\tau}$ is formally like twice an angular momentum vector of magnitude $\frac{1}{2}$. Therefore, the possible values of the vector sum are twice 1 and zero. Letting $2T$ stand for the magnitude of the resultant we have

$$4T(T + 1) = 3 + 3 + 2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2,$$

so the allowed values of $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ are $+1$ and -3. The value $+1$ corresponds to the case of parallel character vectors and so to a wave function that is symmetric in τ_1 and τ_2 while the value -3 corresponds to resultant zero of the two character vectors and hence to an antisymmetric dependence of the wave function on τ_1 and τ_2.

Therefore, the expression

$$\frac{1}{2}(1 + \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$$

has the allowed values $+1$ and -1, the positive value going with wave functions symmetric in τ_1 and τ_2, while the negative value has for its proper function a wave function antisymmetric in these two character coordinates.

These results are, of course, exactly analogous to the well-known results for the vector sum of two spin angular momenta and their connection with the symmetry properties of the wave function with regard to exchange of σ_1 and σ_2.

The applicability of the Pauli exclusion principle to a dynamical system requires that the Hamiltonian function for the system be a symmetric function of the coordinates of the particles. In looking for possible interaction laws we therefore have to confine ourselves to symmetric functions.

So far, four types of exchange forces have been proposed for description of the interaction between heavy particles. These are:

1. Ordinary (Wigner) potential[3]. This is the familiar kind and is simply a function of the distance between the two particles.

2. Heisenberg potential[4]. This is of the form of a function of the distance multiplied by an operator H. This operator is defined as having the value $+1$ when applied to a wave function that is symmetric with regard to exchange of both position and spin coordinates of the two particles whose interaction is being considered, and the value -1 for the antisymmetric case.

3. Bartlett potential[5]. This is a function of the distance multiplied by an operator B. This operator is defined as having the value $+1$ when applied to a wave function that is symmetric in the spin coordinates alone and -1 for the antisymmetric case.

4. Majorana potential[6]. This is a function of the distance multiplied by an operator M. This operator is defined as having the value $+1$ when applied to a wave function that is symmetric with regard to exchange of the

[3] Wigner, Phys. Rev. **43,** 252 (1933).
[4] Heisenberg, Zeits. f. Physik **77,** 1 (1932).
[5] Bartlett, Phys. Rev. **49,** 102 (1936).
[6] Majorana, Zeits. f. Physik **82,** 137 (1933).

positional coordinates only of the two particles in question, and -1 when applied to an antisymmetric function in the positional coordinates.

Evidently the Majorana type can be expressed in terms of the preceding two:

$$M = HB = BH.$$

Since the operator H exchanges both position and spin, and the Bartlett operator B exchanges spin only, the product will be equivalent to an exchange of position only, for the double exchange of spin provided by the combined action of H and B cancels out and is the same as no exchange of spin.

We now point out that the four operators, $1, H, B$ and M are readily expressible in terms of the spin and character vectors, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ of the two particles. This follows from the requirement of over-all antisymmetry of the wave functions in the five coordinates of each of the particles. Let the letters $a, b, c, d \ldots$ stand for the different particles and consider a general wave function ψ that is a function of all five of the coordinates of each particle. More explicitly

$$\psi = \psi(\boldsymbol{\tau}_a, \tau_a, \sigma_a; \quad \boldsymbol{\tau}_b, \tau_b, \sigma_b; \quad \boldsymbol{\tau}_c, \tau_c, \sigma_c; \; \ldots).$$

Whatever the functional form of ψ this can be written

$$\begin{aligned}\psi = \tfrac{1}{2}[\psi(\boldsymbol{\tau}_a, \tau_1, \sigma_a; \; \ldots) + \psi(\boldsymbol{\tau}_b, \tau_a, \sigma_b; \quad \boldsymbol{\tau}_a, \tau_b, \sigma_a; \; \ldots)] \\ + \tfrac{1}{2}[\psi(\tau_a, \tau_a, \sigma_a; \quad \boldsymbol{\tau}_b, \tau_b, \sigma_b; \; \ldots) - \psi(\boldsymbol{\tau}_b, \tau_a, \sigma_b; \quad \tau_a, \tau_b, \sigma_a; \; \ldots)],\end{aligned}$$

that is, as the sum of a function symmetric in the position and spin coordinates of particles a and b and one antisymmetric in these same coordinates. As we require ψ to be antisymmetric in all five coordinates of a and b we know that the first term here must be antisymmetric in τ_a and τ_b and the second term must be symmetric in τ_a and τ_b. Therefore the operator H has the value -1 for symmetry in τ_a and τ_b, and $+1$ for antisymmetry in τ_a and τ_b. Using the earlier calculation of $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ we have

$$H_{ab} = -\frac{1}{2}(1 + \boldsymbol{\tau}_a \cdot \boldsymbol{\tau}_b)$$

which expresses the Heisenberg exchange operator in terms of the two character vectors.

Similarly it is easy to see that

$$B_{ab} = +\frac{1}{2}(1 + \boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b)$$

and therefore, in view of the relation, $M = HB$, we have

$$M_{ab} = -\frac{1}{4}(1 + \boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b)(1 + \boldsymbol{\tau}_a \cdot \boldsymbol{\tau}_b),$$

which completes the expression of each of the exchange operators in terms of symmetric functions of the coordinates of the two particles.

With the different types of exchange operators written in this simple way it suggests itself that the general law of interaction for the specifically nuclear forces can be written in the form:

$$U = V + V_h H + V_b B + V_m M.$$

Here the four V's may be quite different functions of the separation distance but the simplest assumption is that the entire dependence of the interaction on $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ is contained in the operators $\mathbf{1}, H, B$ and M.

Of course, this simple result is not required by the formalism. It is simply the simplest form for the exchange operators. The mere requirement of a symmetric function would be met if any one or all of the V's were replaced by

$$A + B(\tau_1 + \tau_2) + C\tau_1\tau_2,$$

where A, B and C are functions of the distance of separation. In fact, this more general form is necessary even for the description of the Coulomb interaction between the particles which for two particles is expressed as

$$\frac{1}{4}(e^2/r)(1 + \tau_1)(1 + \tau_2).$$

The expression above, involving A, B and C, has the value $(A+2B+C)$ for two protons, the value $(A - C)$ for a proton and a neutron, and the value $(A - 2B + C)$ for two neutrons. If proton-proton forces are the same as neutron-neutron forces we may conclude that $B = 0$, and if like-particle forces are the same as proton-neutron forces in states of corresponding symmetry then we can conclude that $C = 0$. With both B and C equal to zero the dependence on the components τ_1 and τ_2 is gone and we are reduced to the simpler original form.

The assumption that B and C are zero seems to be in accord with the facts about nuclear interactions as far as these are known[7]. The assumption

[7] The consequences of assuming equality of the various specifically nuclear forces for like and unlike particles are considered in detail in a paper by Feenberg and Breit in this issue which we had the pleasure of seeing in manuscript after this paper was sent in.

makes the unification that there are only four different force laws, corresponding to the four possible types of symmetry in σ and τ. These four types are describable in terms of more usual notation by giving the symmetry in position and spin, since this determines the symmetry in character. A state that is symmetric in spin is called a triplet, one antisymmetric a singlet. Symmetry for exchange of position will be denoted by S and antisymmetry by P, since these are the standard notations for states of least orbital angular momentum in the two-body problem which have these positional symmetry properties. Here, however, we use S and P in a more general sense.

The distinct laws of interaction are given in Table I. Using the values of the operators $\mathbf{1}, H, B$ and M we can write for the four interaction laws:

$$\begin{aligned}
U(^3S) &= V + V_h + V_b + V_m, \\
U(^1S) &= V - V_h - V_b - V_m, \\
U(^3P) &= V - V_h + V_b - V_m, \\
U(^1P) &= V + V_h - V_b - V_m.
\end{aligned}$$

These are readily solved for explicit expressions for each V in terms of the four empirically occurring combinations.

We shall only make a few brief remarks about the empirical facts as they are known as these have been recently reviewed by Bethe and Bacher[8]. Ideally one would like to learn all eight force laws (there are eight if we do not make the simple formal assumption of the previous section) from studies based wholly on the two-body problem. So far this is not possible.

The situation with regard to the two-body problems is this:

PROTON-NEUTRON

$U(^3S)$: Normal state of deuteron. Observed binding energy gives a relation between depth and width of a potential well.

Scattering of neutrons by protons: This involves all four laws in principle, but in fact owing to short range of the forces only the two S laws enter in an important way for neutron energies less than some tens of millions of volts. Slow neutron scattering cross section indicates a ^1S level of deuteron near to zero binding energy, according to Wigner.

Photodissociation of the deuteron: Electric dipole effect involves transitions from bound ^3S normal state to continuum of ^3P, hence these two

[8] Bethe and Bacher, Rev. Mod. Phys. **8,** 82 (1936).

laws. In addition to Bethe and Bacher, the problem is discussed by Breit and Condon[9]. Magnetic dipole effect produces transitions from ^3S normal state to ^1S continuum. This is important near the photoelectric threshold.

Radiative capture of neutrons by protons: Here the important effect is for slow neutrons principally by action of magnetic dipole radiation from ^1S continuum to ^3S normal state, according to Fermi[10].

None of these involve the ^1P law in an essential way. Apparently this can only be studied by scattering very high energy neutrons with protons.

PROTON-PROTON

Here a little evidence comes from the probable nonexistence of He^2. But mainly the knowledge comes from the recent work of Tuve, Heyden-burg and Hafstad as analyzed by Breit, Condon and Present[11]. The analysis indicates that up to 1 Mev the departures from coulomb scattering may be described entirely in terms of effects of the ^1S law and gives strong indication that this is the same as the ^1S law in the deuteron.

NEUTRON-NEUTRON

No positive evidence from two-body interactions. Absence of a double neutron is in accord with assumption of the same ^1S law as in proton-neutron since the ^1S level is now supposed to be virtual (see reference 10 for details).

All other knowledge of the force laws comes from approximate calculations of binding energies of many-body nuclei as fully reviewed by Bethe and Bacher. These are in accord with assumption of equality of the interaction laws for various kinds of particles so far as specifically nuclear forces are concerned.

This paper grew out of association at the 1936 summer symposium on theoretical physics of the University of Michigan. We wish to express here to Professor H. M. Randall our deep appreciation of the opportunity of working in the stimulating atmosphere of the Michigan laboratory.

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[9]Breit and Condon, Phys. Rev. **49,** 904 (1936). See also Morse, Fisk and Schiff, Phys. Rev. **50**, 748 (1936).

[10]Fermi, Phys. Rev. **48,** 570 (1935).

[11]Breit, Condon and Present, Phys. Rev. this issue.

TABLE I.

Symmetry in Notation	Spin	Character	Notation	Occurrence
s	s	a	^3S	proton-neutron
s	a	s	^1S	proton-neutron
				proton-neutron
				neutron-neutron
a	s	s	^3P	proton-neutron
				proton-proton
				neutron-neutron
a	a	a	^1P	proton-neutron