THE NON-PERTURBATIVE QCD POTENTIAL

Antonio Vairo[1]

Institut für Theoretische Physik, Universität Heidelberg, Heidelberg, Germany

The potential between heavy quarks can be rigorously defined in a QCD effective field theories framework. I discuss the general situation when the coupling constant may not be considered small and provide an explicit expression for the potential valid up to order $1/m^2$. Spin dependent and spin independent terms are expressed in terms of Wilson loops, completing an ideal journey started twenty years ago with the classical work of Eichten and Feinberg.

INTRODUCTION

The mass of the b and the c quarks is usually considered large enough to treat as non-relativistic (NR) heavy-quark–antiquark systems made up by them (heavy quarkonia: ψ, Υ, B_c, ...). Therefore, these systems are characterized by, at least, three separated scales: hard (the mass m of the heavy quarks), soft (the momentum scale $|\mathbf{p}| \sim mv$, $v \ll 1$), and ultrasoft (the typical kinetic energy $E \sim mv^2$ of the heavy quark in the bound-state system). This NR picture is supported by the success of traditional potential models. In this case a description of heavy quarkonia systems in terms of a NR Schrödinger equation is assumed and the interaction potential is shaped in order to reproduce as best as possible the data (for some reviews see [1, 2, 3]). The fact that such a description turns out to work reasonably well even with simple *ansätze* on the potential (like, for instance, the Cornell potential) suggests that a NR quantum-mechanical description of heavy quarkonium systems, may, indeed, be appropriate and justified from QCD.

Fig. 1, taken from [4], shows the static potential versus the size of different heavy quarkonium systems. It suggests that heavy quarkonium ground states may belong to the region of validity of perturbative QCD (the region where the static potential has a $1/r$ behaviour). The perturbative QCD potential has been calculated up to order α_s^4 (with respect to the energy level) in [5] (the two-loop static potential) and in [6] (the one-loop $1/m$ potential and the tree-level $1/m^2$ potentials). Non-potential effects show up in perturbation theory at the NNNLO and have been investigated in [7, 8]. Fig. 1 shows, however, that most of the quarkonia states lie in a region where the inverse of the size of the system is close to the scale Λ_{QCD} of non-perturbative physics (i.e. in the region where the static potential starts to rise linearly in r). In this situation the potential can no longer be expressed as an expansion in α_s, but, starting from the seminal work of Wilson [9], it has been related to Wilson loop operators. These operators can be eventually calculated on the lattice [10] or in QCD vacuum models [11].

Despite the fact that the non-perturbative QCD potential has been investigated for more than twenty years [12, 13, 14, 15, 16, 17, 18, 19] only recently the complete (in pure gluodynamics) and correct expression valid up to order $1/m^2$ has been calculated [20, 21]. Here, I will report about these last progresses. The theoretical framework is NRQCD [22] and pNRQCD [23, 7], which are the suitable effective field theories for systems made up by two heavy quarks. NRQCD is obtained from QCD by integrating out the hard scale m. It is characterized by an ultraviolet cut-off much smaller than the mass m and much larger than any other scale, in particular much larger than Λ_{QCD}. This means that the matching from QCD to NRQCD can be done perturbatively, as well as within an expansion in $1/m$ [24, 25]. By integrating out the momentum scale mv one is left with the effective field theory called pNRQCD, where the soft and ultrasoft scales have been disentangled and

---

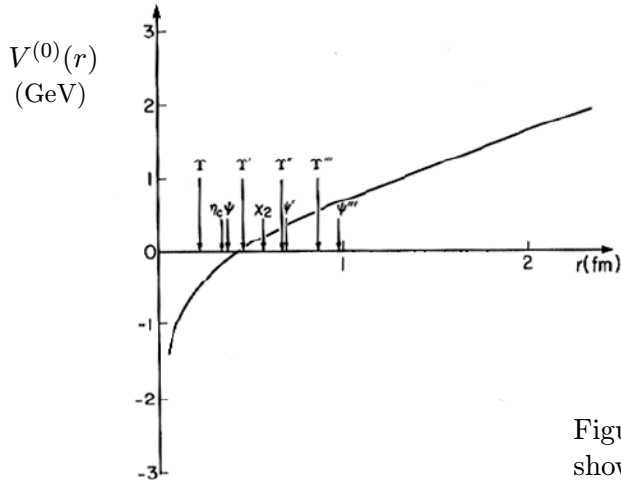[1]antonio.vairo@cern.ch

$V^{(0)}(r)$
(GeV)

Figure 1: The size of some heavy quarkonia is shown with respect to the Cornell potential [4].

where the connection between NRQCD and a NR quantum-mechanical description of the system may be formalized in a systematic way. We will assume that the matching between NRQCD and pNRQCD can be performed order by order in the $1/m$ expansion. Whereas this can be justified within a perturbative framework, in the non-perturbative case, we cannot, in general, guaranty the validity of the $1/m$ expansion. For instance, a situation where certain degrees of freedom cannot be integrated out in this way has been considered in [26]. The possibility of a failure of the $1/m$ expansion has never been studied in this context and also here we will not deal with this possibility, which, however, deserves further investigations.

In [20, 21] the matching of NRQCD to pNRQCD has been performed at order $1/m^2$ in the general situation $\Lambda_{\mathrm{QCD}} \lesssim mv$. This has been proved to be equivalent to compute the heavy quarkonium potential at order $1/m^2$. More precisely, a pure potential picture emerges at the leading order in the ultrasoft expansion under the condition that all the gluonic excitations have a gap of $O(\Lambda_{\mathrm{QCD}})$. Higher order effects in the $1/m$ expansion as well as extra ultrasoft degrees of freedom such as hybrids and pions can be systematically included and may eventually affect the leading potential picture (like in the perturbative regime ultrasoft gluons [7]). The obtained expression for the potential is also correct at any power in α_{s} in the perturbative regime. In the following I will introduce the formalism and outline the derivation of the quarkonium potential in pNRQCD. I will discuss the results and briefly the non-perturbative power counting.

1. NRQCD

After integrating out the hard scale m, we obtain NRQCD [22]. Neglecting operators involving light quark fields [27], the NRQCD Hamiltonian for a quark of mass m_1 and an antiquark of mass m_2 up to $O(1/m^2)$ is given by:

$$H = H^{(0)} + \sum_{\ell+n=1,2} \frac{H^{(\ell,n)}}{m_1^\ell m_2^n}, \tag{1}$$

$$H^{(0)} = \int d^3\mathbf{x} \frac{1}{2} \left(\mathbf{\Pi}^a \mathbf{\Pi}^a + \mathbf{B}^a \mathbf{B}^a \right), \tag{2}$$

$$H^{(1,0)} = -\frac{1}{2} \int d^3\mathbf{x}\, \psi^\dagger \left(\mathbf{D}^2 + g c_F^{(1)} \boldsymbol{\sigma} \cdot \mathbf{B} \right) \psi, \tag{3}$$

$$H^{(0,1)} = -H^{(1,0)}(\psi \leftrightarrow \chi; 1 \leftrightarrow 2) \tag{4}$$

128

$$H^{(2,0)} = \int d^3\mathbf{x}\,\psi^\dagger \left\{ -c_D^{(1)}{}' g\frac{[\mathbf{D}\cdot,\mathbf{E}]}{8} - ic_S^{(1)} g\frac{\boldsymbol{\sigma}\cdot[\mathbf{D}\times,\mathbf{E}]}{8} \right\}\psi - \int d^3\mathbf{x}\,d_3^{(1)}{}' g f_{abc} G_{\mu\nu}^a G_{\mu\alpha}^b G_{\nu\alpha}^c, \quad (5)$$

$$H^{(0,2)} = H^{(2,0)}(\psi \leftrightarrow \chi; 1 \leftrightarrow 2), \tag{6}$$

$$H^{(1,1)} = -\int d^3\mathbf{x} \left(d_{ss}\psi^\dagger\psi\chi^\dagger\chi + d_{sv}\psi^\dagger\boldsymbol{\sigma}\psi\chi^\dagger\boldsymbol{\sigma}\chi + d_{vs}\psi^\dagger \mathrm{T}^a\psi\chi^\dagger \mathrm{T}^a\chi + d_{vv}\psi^\dagger \mathrm{T}^a\boldsymbol{\sigma}\psi\chi^\dagger \mathrm{T}^a\boldsymbol{\sigma}\chi \right), \quad (7)$$

where ψ is the Pauli spinor field that annihilates the fermion and χ is the Pauli spinor field that creates the antifermion, $i\mathbf{D} = i\boldsymbol{\nabla} + g\mathbf{A}$, $[\mathbf{D}\cdot,\mathbf{E}] = \mathbf{D}\cdot\mathbf{E} - \mathbf{E}\cdot\mathbf{D}$ and $[\mathbf{D}\times,\mathbf{E}] = \mathbf{D}\times\mathbf{E} - \mathbf{E}\times\mathbf{D}$. $\boldsymbol{\Pi}^a = \mathbf{E}^a + O(1/m^2)$ is the canonical momentum conjugated to $\mathbf{A}^a$ and the physical states are constrained to satisfy the Gauss law:

$$\mathbf{D}\cdot\boldsymbol{\Pi}^a|\text{phys}\rangle = g(\psi^\dagger T^a\psi + \chi^\dagger T^a\chi)|\text{phys}\rangle.$$

The coefficients c_F, c_D', c_S, d_2 and d_3' can be found in Ref. [24, 21] and d_{ij} $(i,j = s,v)$ in [25] for the $\overline{MS}$ scheme.

In the static limit the one-quark–one-antiquark sector of the Fock space may be spanned by

$$|\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} = \psi^\dagger(\mathbf{x}_1)\chi_c^\dagger(\mathbf{x}_2)|n; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)},$$

where $|\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$ is a gauge-invariant eigenstate (up to a phase) of $H^{(0)}$, as a consequence of the Gauss law, with energy $E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2)$, and $\chi_c(\mathbf{x}) = i\sigma^2\chi^*(\mathbf{x})$. $|n; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$ encodes the gluonic content of the state, i.e. it is annihilated by $\chi_c(\mathbf{x})$ and $\psi(\mathbf{x})$ ($\forall\mathbf{x}$). The normalization condition is

$$^{(0)}\langle\underline{\mathrm{m}}; \mathbf{x}_1, \mathbf{x}_2|\underline{\mathrm{n}}; \mathbf{y}_1, \mathbf{y}_2\rangle^{(0)} = \delta_{nm}\prod_{j=1}^2 \delta^{(3)}(\mathbf{x}_j - \mathbf{y}_j).$$

The positions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the quark and antiquark respectively are good quantum numbers for the static solution $|\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$; n generically denotes the remaining quantum numbers, which are classified by the irreducible representations of the symmetry group $D_{\infty h}$ (substituting the parity generator by CP). We also choose the basis such that $T|\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)} = |\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$, where T is the time-inversion operator. The ground-state energy $E_0^{(0)}(\mathbf{x}_1, \mathbf{x}_2)$ can be associated to the static potential of the heavy quarkonium under some circumstances (see Sec. 2.). The remaining energies $E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2)$, $n \neq 0$, are usually associated to the potentials describing heavy hybrids or heavy quarkonium (or other heavy hybrids) plus glueballs. They can be computed on the lattice (see, for instance, [28] and [29]). Translational invariance implies that $E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2) = E_n^{(0)}(r)$, where $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$.

Beyond the static limit, but still working order by order in $1/m$, the normalized eigenstates, $|\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle$, and eigenvalues, $E_n(\mathbf{x}_1, \mathbf{x}_2; \mathbf{p}_1, \mathbf{p}_2)$, of the Hamiltonian H satisfy the equations

$$H|\underline{\mathrm{n}}; \mathbf{x}_1, \mathbf{x}_2\rangle = \int d^3x_1' d^3x_2'|\underline{\mathrm{n}}; \mathbf{x}_1', \mathbf{x}_2'\rangle E_n(\mathbf{x}_1', \mathbf{x}_2'; \mathbf{p}_1', \mathbf{p}_2')\prod_{j=1}^2 \delta^{(3)}(\mathbf{x}_j' - \mathbf{x}_j), \tag{8}$$

$$\langle\underline{\mathrm{m}}; \mathbf{x}_1, \mathbf{x}_2|\underline{\mathrm{n}}; \mathbf{y}_1, \mathbf{y}_2\rangle = \delta_{nm}\prod_{j=1}^2 \delta^{(3)}(\mathbf{x}_j - \mathbf{y}_j). \tag{9}$$

Note that the positions $\mathbf{x}_1$ and $\mathbf{x}_2$ of the static solution still label the states even if the position operator does not commute with H beyond the static limit. We are interested in the eigenvalues E_n. E_0 corresponds to the quantum-mechanical Hamiltonian of the heavy quarkonium (in some specific situation). The other energies E_n for $n > 0$ are related to the quantum-mechanical Hamiltonian of higher gluonic excitations between heavy quarks. Expanding Eqs. (8) and (9) around the static solution we get up to $O(1/m^2)$

$$E_n(\mathbf{x}_1, \mathbf{x}_2; \mathbf{p}_1, \mathbf{p}_2) \prod_{j=1}^{2} \delta^{(3)}(\mathbf{x}'_j - \mathbf{x}_j) = E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2) \prod_{j=1}^{2} \delta^{(3)}(\mathbf{x}'_j - \mathbf{x}_j)$$

$$+ {}^{(0)}\langle \underline{n}; \mathbf{x}_1, \mathbf{x}_2 | \sum_{\ell+j=1,2} \frac{H^{(\ell,j)}}{m_1^\ell m_2^j} | \underline{n}; \mathbf{x}'_1, \mathbf{x}'_2 \rangle^{(0)}$$

$$- \frac{1}{2} \sum_{k \neq n} \int d^3 y_1 \, d^3 y_2 \, {}^{(0)}\langle \underline{n}; \mathbf{x}_1, \mathbf{x}_2 | \sum_{\ell+j=1} \frac{H^{(\ell,j)}}{m_1^\ell m_2^j} | \underline{k}; \mathbf{y}_1, \mathbf{y}_2 \rangle^{(0)} \, {}^{(0)}\langle \underline{k}; \mathbf{y}_1, \mathbf{y}_2 | \sum_{\ell+j=1} \frac{H^{(\ell,j)}}{m_1^\ell m_2^j} | \underline{n}; \mathbf{x}'_1, \mathbf{x}'_2 \rangle^{(0)}$$

$$\times \left(\frac{1}{E_k^{(0)}(\mathbf{y}_1, \mathbf{y}_2) - E_n^{(0)}(\mathbf{x}'_1, \mathbf{x}'_2)} + \frac{1}{E_k^{(0)}(\mathbf{y}_1, \mathbf{y}_2) - E_n^{(0)}(\mathbf{x}_1, \mathbf{x}_2)} \right). \tag{10}$$

Explicit expressions for the energies E_n, obtained from the above formula, can be found in [20, 21].

2. pNRQCD

In the static limit, the gap between different states at fixed $\mathbf{r}$ depends on the dimensionless parameter $\Lambda_{\mathrm{QCD}} r$. In general there will be a set of states $\{n_{\mathrm{us}}\}$ such that $E_{n_{\mathrm{us}}}^{(0)}(r) \sim mv^2$ for the typical r of the actual physical system. We call these states ultrasoft. The aim of pNRQCD is to describe the behaviour of the ultrasoft states. Therefore, in order to obtain pNRQCD all the physical degrees of freedom with energies larger than mv^2 are integrated out from NRQCD. In this context one may work order by order in $1/m$ (in particular for the kinetic energy), and the calculation of the previous section becomes the matching calculation between NRQCD and pNRQCD.

In the perturbative situation, $\Lambda_{\mathrm{QCD}} r \ll 1$, $\{n_{\mathrm{us}}\}$ corresponds to a heavy-quark–antiquark state, in either a singlet or an octet configuration, plus gluons and light fermions, all of them with energies of $O(mv^2)$ [7]. In a non-perturbative situation, $\Lambda_{\mathrm{QCD}} r \sim 1$, it is not obvious what $\{n_{\mathrm{us}}\}$ is. One may think of different possibilities. In particular, one could consider the situation where, because of a mass gap in QCD, the energy splitting between the ground state and the first gluonic excitation is larger than mv^2, and, because of chiral symmetry breaking of QCD, Goldstone bosons (pions/kaons) appear. Hence, in this situation, $\{n_{\mathrm{us}}\}$ would be the ultrasoft excitations about the static ground state, which we will call the singlet, plus the Goldstone bosons. If one switches off the light fermions (pure gluodynamics), only the singlet survives and pNRQCD reduces to a pure two-particle NR quantum-mechanical system. Therefore, the situation *assumed* by all potential models, may be now rigorously *derived* under a specific set of circumstances.

Here, I shall discuss the pure singlet sector with no further reference to ultrasoft degrees of freedom. In this situation, pNRQCD only describes the ultrasoft excitations about the static ground state of NRQCD. In terms of static NRQCD eigenstates, this means that only $|\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$ is kept as an explicit degree of freedom, whereas $|\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle^{(0)}$ with $n \neq 0$ are integrated out. This provides the only dynamical degree of freedom of the theory. It is described by means of a bilinear colour singlet field, $S(\mathbf{x}_1, \mathbf{x}_2, t)$, which has the same quantum numbers and transformation properties under symmetries as the static ground state of NRQCD in the one-quark–one-antiquark sector. In the above situation, the Lagrangian of pNRQCD reads

$$\mathcal{L}_{\mathrm{pNRQCD}} = S^\dagger \left(i\partial_0 - h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) \right) S, \tag{11}$$

where h_s is the Hamiltonian of the singlet, $\mathbf{p}_1 = -i\boldsymbol{\nabla}_{\mathbf{x}_1}$ and $\mathbf{p}_2 = -i\boldsymbol{\nabla}_{\mathbf{x}_2}$. It has the following expansion up to order $1/m^2$

$$h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V^{(0)} + \frac{V^{(1,0)}}{m_1} + \frac{V^{(0,1)}}{m_2} + \frac{V^{(2,0)}}{m_1^2} + \frac{V^{(0,2)}}{m_2^2} + \frac{V^{(1,1)}}{m_1 m_2}. \tag{12}$$

The integration of the higher excitations is trivial using the basis $|\underline{n}; \mathbf{x}_1, \mathbf{x}_2\rangle$ since, in this case, they are decoupled from $|\underline{0}; \mathbf{x}_1, \mathbf{x}_2\rangle$. The matching of NRQCD to pNRQCD simply consists in a renaming of things in a way such that pNRQCD reproduces the matrix elements of NRQCD for the ground state, in particular the energy. This fixes the matching condition

$$E_0(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2) = h_s(\mathbf{x}_1, \mathbf{x}_2, \mathbf{p}_1, \mathbf{p}_2). \tag{13}$$

Let us note that, since, in practice, we are integrating over all the states, in the situation where some states, different from the singlet, are ultrasoft, we would need to subtract them later on. This is analogous to what happens in the perturbative situation where the subtraction is done order by order in the multipole expansion. Therefore, in the perturbative situation our calculation should be understood as the leading term in the multipole expansion.

3. HEAVY QUARKONIUM POTENTIAL

To express the heavy quarkonium potential in terms of Wilson loops is quite convenient for lattice simulations [10]. We shall use the following notations: $\langle \ldots \rangle$ will stand for the average over the Yang–Mills action, $W_\square$ for the rectangular static Wilson loop of dimensions $r \times T_W$ and $\langle\!\langle \ldots \rangle\!\rangle \equiv \langle \ldots W_\square \rangle / \langle W_\square \rangle$. We define $\langle\!\langle O_1(t_1) O_2(t_2) ... O_n(t_n) \rangle\!\rangle_c$ as the *connected* Wilson loop with $O_1(t_1)$, $O_2(t_2)$, ... $O_n(t_n)$ operators insertions for $T_W/2 \geq t_1 \geq t_2 \geq \ldots \geq t_n \geq -T_W/2$. We also define in a short-hand notation $\lim_{T \to \infty} \equiv \lim_{T \to \infty} \lim_{T_W \to \infty}$, where $T \leq T_W$ is the time-length appearing in the time integrals. By performing first the limit $T_W \to \infty$, the averages $\langle\!\langle \ldots \rangle\!\rangle$ become independent of T_W and thus invariant under global time translations.

Using the matching condition (13) and Eq. (10) we get in terms of Wilson loops [20]

$$V^{(0)}(r) = \lim_{T \to \infty} \frac{i}{T} \ln\langle W_\square \rangle, \tag{14}$$

$$V^{(1,0)}(r) = -\frac{1}{2} \lim_{T \to \infty} \int_0^T dt\, t\, \langle\!\langle g\mathbf{E}_1(t) \cdot g\mathbf{E}_1(0) \rangle\!\rangle_c = V^{(0,1)}(r), \tag{15}$$

where $\mathbf{E}_j \equiv \mathbf{E}(\mathbf{x}_j)$.

Let us now consider the terms of order $1/m^2$. We define

$$V^{(2,0)} = \frac{1}{2} \left\{ \mathbf{p}_1^2, V_{\mathbf{p}^2}^{(2,0)}(r) \right\} + \frac{V_{\mathbf{L}^2}^{(2,0)}(r)}{r^2} \mathbf{L}_1^2 + V_r^{(2,0)}(r) + V_{LS}^{(2,0)}(r) \mathbf{L}_1 \cdot \mathbf{S}_1,$$

where $\mathbf{L}_j \equiv \mathbf{r} \times \mathbf{p}_j$. Analogous definitions held for $V^{(0,2)}$, for which, by using invariance under charge conjugation plus $m_1 \leftrightarrow m_2$ transformation, we have $V_{\mathbf{p}^2}^{(2,0)}(r) = V_{\mathbf{p}^2}^{(0,2)}(r)$, $V_{\mathbf{L}^2}^{(2,0)}(r) = V_{\mathbf{L}^2}^{(0,2)}(r)$ and $V_{r,LS}^{(2,0)}(r) = V_{r,LS}^{(0,2)}(r; m_2 \leftrightarrow m_1)$. Using Eqs. (13) and (10) we get, in terms of Wilson loops,

$$V_{\mathbf{p}^2}^{(2,0)}(r) = \frac{i}{2} \hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \lim_{T \to \infty} \int_0^T dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0) \rangle\!\rangle_c, \tag{16}$$

$$V_{\mathbf{L}^2}^{(2,0)}(r) = \frac{i}{4} \left(\delta^{ij} - 3\hat{\mathbf{r}}^i \hat{\mathbf{r}}^j \right) \lim_{T \to \infty} \int_0^T dt\, t^2 \langle\!\langle g\mathbf{E}_1^i(t) g\mathbf{E}_1^j(0) \rangle\!\rangle_c, \tag{17}$$

$$V_r^{(2,0)}(r) = -\frac{c_D^{(1)\prime}}{8} \lim_{T \to \infty} \int_0^T dt\, \langle\!\langle [\mathbf{D}_1, g\mathbf{E}_1(t)] \rangle\!\rangle_c$$

$$-\frac{ic_F^{(1)\,2}}{4}\lim_{T\to\infty}\int_0^T dt\,\langle\!\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_1(0)\rangle\!\rangle_c$$

$$-\frac{i}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\,(t_2-t_3)^2\langle\!\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2)g\mathbf{E}_1(t_3)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c$$

$$+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\,(t_1-t_2)^2\langle\!\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c\right)$$

$$-\frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right)\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\,(t_1-t_2)^3\langle\!\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c$$

$$-\frac{1}{2}\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\,(t_1-t_2)^2\langle\!\langle [\mathbf{D}_1.,g\mathbf{E}_1](t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c$$

$$+\frac{i}{8}\lim_{T\to\infty}\int_0^T dt\,t^2\langle\!\langle [\mathbf{D}_1.,g\mathbf{E}_1](t)[\mathbf{D}_1.,g\mathbf{E}_1](0)\rangle\!\rangle_c$$

$$-\frac{i}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^2\langle\!\langle g\mathbf{E}_1^i(t)[\mathbf{D}_1.,g\mathbf{E}_1](0)\rangle\!\rangle_c\right)$$

$$-\frac{1}{4}\lim_{T\to\infty}\int_0^T dt\,t^3\langle\!\langle [\mathbf{D}_1.,g\mathbf{E}_1](t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c(\boldsymbol{\nabla}_r^j V^{(0)})$$

$$+\frac{1}{4}\left(\boldsymbol{\nabla}_r^i\lim_{T\to\infty}\int_0^T dt\,t^3\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c(\boldsymbol{\nabla}_r^j V^{(0)})\right)$$

$$+\frac{1}{2}(\boldsymbol{\nabla}_r^2 V_{\mathbf{p}^2}^{(2,0)})-\frac{i}{12}\lim_{T\to\infty}\int_0^T dt\,t^4\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c(\boldsymbol{\nabla}_r^i V^{(0)})(\boldsymbol{\nabla}_r^j V^{(0)})$$

$$-d_3^{(1)\,\prime}f_{abc}\int d^3\mathbf{x}\,\lim_{T_W\to\infty}g\langle\!\langle G_{\mu\nu}^a(x)G_{\mu\alpha}^b(x)G_{\nu\alpha}^c(x)\rangle\!\rangle,\tag{18}$$

$$V_{LS}^{(2,0)}(r)=\frac{c_S^{(1)}}{2r^2}\mathbf{r}\cdot(\boldsymbol{\nabla}_r V^{(0)})-\frac{c_F^{(1)}}{r^2}i\mathbf{r}\cdot\lim_{T\to\infty}\int_0^T dt\,t\,\langle\!\langle g\mathbf{B}_1(t)\times g\mathbf{E}_1(0)\rangle\!\rangle.\tag{19}$$

For the $V^{(1,1)}$ potential we define

$$V^{(1,1)}=-\frac{1}{2}\left\{\mathbf{p}_1\cdot\mathbf{p}_2,V_{\mathbf{p}^2}^{(1,1)}(r)\right\}-\frac{V_{\mathbf{L}^2}^{(1,1)}(r)}{2r^2}(\mathbf{L}_1\cdot\mathbf{L}_2+\mathbf{L}_2\cdot\mathbf{L}_1)+V_r^{(1,1)}(r)$$

$$+V_{L_1S_2}^{(1,1)}(r)\mathbf{L}_1\cdot\mathbf{S}_2-V_{L_2S_1}^{(1,1)}(r)\mathbf{L}_2\cdot\mathbf{S}_1+V_{S^2}^{(1,1)}(r)\mathbf{S}_1\cdot\mathbf{S}_2+V_{\mathbf{S}_{12}}^{(1,1)}(r)\mathbf{S}_{12}(\hat{\mathbf{r}}),$$

where $\mathbf{S}_{12}(\hat{\mathbf{r}})\equiv 3\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$. Due to invariance under charge conjugation plus $m_1\leftrightarrow m_2$ transformation, we have $V_{L_1S_2}^{(1,1)}(r)=V_{L_2S_1}^{(1,1)}(r;m_1\leftrightarrow m_2)$. Using Eqs. (13) and (10) we get, in terms of Wilson loops,

$$V_{\mathbf{p}^2}^{(1,1)}(r)=i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j\lim_{T\to\infty}\int_0^T dt\,t^2\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c,\tag{20}$$

$$V_{\mathbf{L}^2}^{(1,1)}(r)=i\frac{\delta^{ij}-3\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j}{2}\lim_{T\to\infty}\int_0^T dt\,t^2\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c,\tag{21}$$

$$V_r^{(1,1)}(r)=-\frac{1}{2}(\boldsymbol{\nabla}_r^2 V_{\mathbf{p}^2}^{(1,1)})$$

$$-i\lim_{T\to\infty}\int_0^T dt_1\int_0^{t_1} dt_2\int_0^{t_2} dt_3\,(t_2-t_3)^2\langle\!\langle g\mathbf{E}_1(t_1)\cdot g\mathbf{E}_1(t_2)g\mathbf{E}_2(t_3)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c$$

132

$$+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c\right)$$

$$+\frac{1}{2}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle g\mathbf{E}_2^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c\right)$$

$$-\frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right)\lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^3 \langle\!\langle g\mathbf{E}_1^i(t_1)g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c$$

$$-\frac{i}{2}\left(\boldsymbol{\nabla}_r^i V^{(0)}\right)\lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^3 \langle\!\langle g\mathbf{E}_2^i(t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c$$

$$-\frac{1}{2}\lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle [\mathbf{D}_1\cdot,g\mathbf{E}_1](t_1)g\mathbf{E}_2(t_2)\cdot g\mathbf{E}_2(0)\rangle\!\rangle_c$$

$$+\frac{1}{2}\lim_{T\to\infty}\int_0^T dt_1 \int_0^{t_1} dt_2 (t_1-t_2)^2 \langle\!\langle [\mathbf{D}_2\cdot,g\mathbf{E}_2](t_1)g\mathbf{E}_1(t_2)\cdot g\mathbf{E}_1(0)\rangle\!\rangle_c$$

$$-\frac{i}{4}\lim_{T\to\infty}\int_0^T dt\, t^2 \langle\!\langle [\mathbf{D}_1\cdot,g\mathbf{E}_1](t)[\mathbf{D}_2\cdot,g\mathbf{E}_2](0)\rangle\!\rangle_c$$

$$+\frac{i}{4}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt\, t^2 \left\{\langle\!\langle g\mathbf{E}_1^i(t)[\mathbf{D}_2\cdot,g\mathbf{E}_2](0)\rangle\!\rangle_c - \langle\!\langle g\mathbf{E}_2^i(t)[\mathbf{D}_1\cdot,g\mathbf{E}_1](0)\rangle\!\rangle_c\right\}\right)$$

$$-\frac{1}{4}\lim_{T\to\infty}\int_0^T dt\, t^3 \left\{\langle\!\langle [\mathbf{D}_1\cdot,g\mathbf{E}_1](t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c - \langle\!\langle [\mathbf{D}_2\cdot,g\mathbf{E}_2](t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c\right\}(\boldsymbol{\nabla}_r^j V^{(0)})$$

$$+\frac{1}{4}\left(\boldsymbol{\nabla}_r^i \lim_{T\to\infty}\int_0^T dt\, t^3 \left\{\langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c + \langle\!\langle g\mathbf{E}_2^i(t)g\mathbf{E}_1^j(0)\rangle\!\rangle_c\right\}(\boldsymbol{\nabla}_r^j V^{(0)})\right)$$

$$-\frac{i}{6}\lim_{T\to\infty}\int_0^T dt\, t^4 \langle\!\langle g\mathbf{E}_1^i(t)g\mathbf{E}_2^j(0)\rangle\!\rangle_c(\boldsymbol{\nabla}_r^i V^{(0)})(\boldsymbol{\nabla}_r^j V^{(0)})$$

$$+(d_{ss}+d_{vs}\lim_{T_W\to\infty}\langle\!\langle T_1^a T_2^a\rangle\!\rangle)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2), \tag{22}$$

$$V_{L_2 S_1}^{(1,1)}(r) = -\frac{c_F^{(1)}}{r^2} i\mathbf{r}\cdot \lim_{T\to\infty}\int_0^T dt\, t\, \langle\!\langle g\mathbf{B}_1(t)\times g\mathbf{E}_2(0)\rangle\!\rangle, \tag{23}$$

$$V_{S^2}^{(1,1)}(r) = \frac{2c_F^{(1)}c_F^{(2)}}{3} i \lim_{T\to\infty}\int_0^T dt\, \langle\!\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\!\rangle$$
$$-4(d_{sv}+d_{vv}\lim_{T_W\to\infty}\langle\!\langle T_1^a T_2^a\rangle\!\rangle)\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}_2), \tag{24}$$

$$V_{\mathbf{S}_{12}}^{(1,1)}(r) = \frac{c_F^{(1)}c_F^{(2)}}{4} i\hat{\mathbf{r}}^i\hat{\mathbf{r}}^j \lim_{T\to\infty}\int_0^T dt\, \left[\langle\!\langle g\mathbf{B}_1^i(t)\,g\mathbf{B}_2^j(0)\rangle\!\rangle - \frac{\delta^{ij}}{3}\langle\!\langle g\mathbf{B}_1(t)\cdot g\mathbf{B}_2(0)\rangle\!\rangle\right]. \tag{25}$$

A critical comparison of the above results with the previous literature will be done in the conclusions. Here, I mention that the potentials (15), (18) and (22) have been calculated for the first time in Ref. [20] and [21]. Since the potential we get here is a well defined quantity, derived from QCD via a systematic and unambiguous procedure, and *complete* up to order $1/m^2$, it is not affected by the usual ambiguities (ordering, retardation corrections, etc.), which affect all potential models and all phenomenological reductions of Bethe–Salpeter kernels [1]. For the same reason the above result may be relevant for the study of the properties of the QCD vacuum in the presence of heavy sources. While the lattice data for the spin structure of the potential seem to suggest the so-called "scalar confinement", the data for the momentum dependent and spin independent part of

the potential seem to support a flux-tube picture [1, 30]. It will be interesting to see how consistent these pictures are with the new momentum and spin independent potentials, once lattice data will be available for them. I note that some of them are not simply expressed by two field insertions on a static Wilson loop. In particular an extended object coming from the Yang–Mills sector is required (similar extended objects would also show up by taking into account operators with light quarks).

Perturbatively the leading contribution of Eq. (15) to the potential is at the one loop level ($\sim \alpha_s^2/r^2$). In Ref. [20] it has been checked that this result agrees with known results. The tree level contribution of Eq. (18) to the potential comes from the Darwin term (the term proportional to c_D), while the tree level contribution of Eq. (22) to the potential comes only from the first term, which gives a delta type contribution.

4.   POWER COUNTING

The standard power counting of NRQCD, as discussed, for instance, in [31], has been tested in the perturbative regime. However, even in this regime, due to the different dynamical scales still involved, the matrix elements of NRQCD do not have an unique power counting in v. In the non-perturbative regime the problem of the power-counting of NRQCD is still open. The above non-perturbative formulation of pNRQCD has translated this problem to obtaining the power counting of the different potentials. This is expected to be of some advantage. Since the scale mv has been integrated out, the power counting of pNRQCD is simpler. Since all the potentials are expressed in terms of Wilson loops, there are or there will be direct lattice measurements of them.

Here, let us only say few words about the expected behaviour of the potential using arguments of naturalness on the scale mv, i.e. assuming that the potentials scale with mv. We first consider $V^{(0)}$. In principle, $V^{(0)}$ counts as mv, but, by definition, the kinetic energy counts as mv^2. Therefore, the virial theorem constrains $V^{(0)}$ also to count as mv^2. In the perturbative case this extra v suppression comes from the factor $\alpha_s \sim v$ in the potential. Using naturalness, $V^{(1,0)}/m$ scales like mv^2. Therefore, it could be in principle as large as $V^{(0)}$. This makes a lattice calculation of this potential urgent. Perturbatively, due to the factor α_s^2, it is $O(mv^4)$. For what concerns the $1/m^2$ potentials, the naturalness argument suggests that they are of order mv^3. However, also here several constraints apply. Terms involving $\boldsymbol{\nabla} V^{(0)}$ are suppressed by an extra factor v, due to the virial theorem. The Gromes relations [14], $\frac{1}{2r}\frac{dV^{(0)}}{dr} + V^{(2,0)}_{LS} - V^{(1,1)}_{L_2 S_1} = 0$, suppresses by an extra factor v the combination $V^{(2,0)}_{LS} - V^{(1,1)}_{L_2 S_1}$. Similar constraints also exist for the spin-independent potentials [15]. Finally, it may be important to consider that some of the potentials are $O(\alpha_s)$ suppressed in the matching coefficients inherited from NRQCD.

CONCLUSIONS AND OUTLOOK

I have reported about a new derivation of the QCD potential in an effective field theory framework. Explicit expressions for the heavy quarkonium potential up to order $1/m^2$, valid beyond perturbation theory, have been written.

I now compare with previous results in the literature. For the spin-dependent potentials we find agreement with the Eichten–Feinberg results [13, 14] (once the NRQCD matching coefficients have been taken into account) *except* for the $1/m_1 m_2$ spin-orbit potential $V^{(1,1)}_{L_2 S_1}$. Our result gives back the well-known tree-level calculation, whereas the Eichten–Feinberg expression gives half of the expected result. Moreover, our perturbative result fulfills the Gromes relation. An analysis of this error, present in the original papers and in several others appeared afterwards, is done in [21]. The spin-independent potentials have been computed before only by Barchielli, Brambilla, Montaldi and Prosperi in [15] (the analysis done in [17], which appears to be inconclusive, has never been

published). We agree (once the NRQCD matching coefficients have been taken into account) with their result for the momentum-dependent terms but not for the momentum-independent terms, where new contributions are found. An approach similar to that one presented here has been used in [32] in order to derive, from the QCD Hamiltonian in Coulomb gauge, the spin-dependent part of the potential up to $O(1/m^2)$. Our expression (10) differs from that one used in [32], which, in general, would give incorrect spin-independent potentials. However, if we take our matching coefficients at tree level and neglect the tree-level annihilation contributions in the equal mass case, we find essential agreement for the spin-dependent potentials.

I conclude, commenting on two possible developments of the present work. First, it is worthwhile to explore the possibility of expressing the potentials associated with higher gluonic excitations in terms of Wilson loop operators as done here for the heavy quarkonium ground state. The corresponding quantum-mechanical expressions are known and have been calculated in [21]. Second, our results are complete at $O(1/m^2)$ in the case of pure gluodynamics. If we want to incorporate light fermions, the procedure to follow is analogous and our results still remain valid (considering now matrix elements and Wilson loops with dynamical light fermions incorporated) except for new terms appearing in the energies at $O(1/m^2)$ due to operators involving light fermions. They may be incorporated along the same lines as the terms discussed here.

Acknowledgments

References

[1] N. Brambilla and A. Vairo, hep-ph/9904330.

[2] F. J. Ynduráin, Nucl. Phys. **B** (Proc. Suppl.) **64**, 433 (1998).

[3] G. S. Bali, hep-ph/0001312.

[4] S. Godfrey and N. Isgur, Phys. Rev. **D 32**, 189 (1985).

[5] Y. Schröder, Phys. Lett. **B 447**, 321 (1999); M. Peter, Phys. Rev. Lett. **78**, 602 (1997).

[6] S. N. Gupta and S. F. Radford, Phys. Rev. **D 25**, 3430 (1982).

[7] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. **D 60**, 091502 (1999); N. Brambilla, A. Pineda, J. Soto and A. Vairo, Nucl. Phys. **B 566**, 275 (2000); A. Vairo, Nucl. Phys. **B** (Proc. Suppl.) **86**, 521 (2000).

[8] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Lett. **B 470**, 215 (1999); A. Penin and B. Kniehl, Nucl. Phys. **B 563**, 200 (1999); A. Pineda, Nucl. Phys. **B** (Proc. Suppl.) **86**, 517 (2000).

[9] K. G. Wilson, Phys. Rev. **D 10**, 2445 (1974).

[10] G. S. Bali, A. Wachter and K. Schilling, Phys. Rev. **D 56**, 2566 (1997).

[11] N. Brambilla and A. Vairo, Phys. Rev. **D 55**, 3974 (1997); M. Baker, J. S. Ball, N. Brambilla and A. Vairo, Phys. Lett. **B 389**, 577 (1996).

[12] L. Susskind, Les Houches lectures (1976); W. Fischler, Nucl. Phys. **B 129**, 157 (1977); L. S. Brown and W.I. Weisberger, Phys. Rev. **D 20**, 3239 (1979).

[13] E. Eichten and F. L. Feinberg, Phys. Rev. **D 23**, 2724 (1981); M. E. Peskin, in Proceeding of the 11th SLAC Institute, SLAC Report No. 207, 151, edited by P. Mc Donough (1983);

[14] D. Gromes, Z. Phys. **C 26**, 401 (1984).

[15] A. Barchielli, E. Montaldi and G. M. Prosperi, Nucl. Phys. **B 296**, 625 (1988); (E) *ibid.* **303**, 752 (1988); A. Barchielli, N. Brambilla, G. Prosperi, Nuovo Cimento **103 A**, 59 (1990).

[16] N. Brambilla, P. Consoli and G. M. Prosperi, Phys. Rev. **D 50**, 5878 (1994).

[17] R. Tafelmayer, Diploma Thesis, Heidelberg (1986).

[18] Y. Chen, Y. Kuang and R. J. Oakes, Phys. Rev. **D 52**, 264 (1995).

[19] N. Brambilla and A. Vairo, Nucl. Phys. **B** (Proc. Suppl.) **74**, 201 (1999).

[20] N. Brambilla, A. Pineda, J. Soto and A. Vairo, hep-ph/0002250.

[21] A. Pineda and A. Vairo, hep-ph/0009145.

[22] W. E. Caswell and G. P. Lepage, Phys. Lett. **B 167**, 437 (1986); G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. **D 51**, 1125 (1995); Erratum *ibid.* **D 55**, 5853 (1997).

[23] A. Pineda and J. Soto, Nucl. Phys. **B** (Proc. Suppl.) **64**, 428 (1998).

[24] A. V. Manohar, Phys. Rev. **D 56**, 230 (1997).

[25] A. Pineda and J. Soto, Phys. Rev. **D 58**, 114011 (1998).

[26] D. Eiras and J. Soto, Phys. Rev. **D 61**, 114027 (2000).

[27] C. Bauer and A. V. Manohar, Phys. Rev. **D 57** 337 (1998).

[28] C. Michael, hep-ph/9809211; K. J. Juge, J. Kuti and C. Morningstar, hep-lat/9809015.

[29] N. Brambilla, Nucl. Phys. **B** (Proc. Suppl.) **86**, 389 (2000).

[30] N. Brambilla, hep-ph/9809263.

[31] G. P. Lepage, L. Magnea, C. Nakhleh, U. Magnea and K. Hornbostel, Phys. Rev. **D 46**, 4052 (1992).

[32] A.P. Szczepaniak and E.S. Swanson, Phys. Rev. **D 55**, 3987 (1997).