MARKOV-YUKAWA TRANSVERSALITY PRINCIPLE AND 3D-4D INTERLINKAGE OF BETHE-SALPETER AMPLITUDES

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This article is designed to focus attention on the Markov-Yukawa Transversality Principle (MYTP) as a novel paradigm for an exact 3D-4D interlinkage between the corresponding BSE amplitudes, with a closely parallel treatment of $q\bar{q}$ and qqq systems, stemming from a common 4-fermion Lagrangian mediated by gluon (vector)-like exchange. This unique feature of MYTP owes its origin to a Lorentz- covariant 3D support to the BS kernel, which acts as a sort of 'gauge principle' and distinguishes it from most other 3D approaches to strong interaction dynamics. Some of the principal approaches in the latter category are briefly reviewed so as to set the (less familiar) MYTP in their context. Two specific types of MYTP which provide 3D support to the BSE kernel, are considered: a) Covariant Instantaneity Ansatz (CIA); b) Covariant LF/NP ansatz (Cov.LF). Both lead to formaly identical 3D BSE reductions but produce sharply different 4D structures: Under CIA, the 4D loop integrals suffer from Lorentz mismatch of the vertex functions, leading to ill-defined time-like momentum integrals, but Cov LF is free from this disease. This is illustrated by the pion form factor under Cov LF. The reconstruction of the 4D qqq wave function is achieved by Green's function techniques.
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1. A Review Of Strong Interaction Dynamics

Ever since the success of the Tomonaga-Schwinger-Feynman-Dyson formalism in QED [1], corresponding field-theoretic formulations have been in the forefront of strong interaction dynamics since the early fifties, the main strategy being to device various 'closed' form approaches which are represented as appropriate 'integral equations'. One of the earliest efforts in this direction was the Tamm-Dancoff formalism [2] which showed a great intuitive appeal. In this method, the state vector of the system under consideration is Fock-expanded in terms of a complete set of eigen- functions of the free field Hamiltonian, which was first systematically applied by Dyson (+ Cornell collaborators) in the early fifties, to the meson-nucleon scattering problem, for a dynamical understanding of the 'Delta' and other low-energy resonances [3]. It leads to 3D integral equations connecting amplitudes for successively higher numbers of meson (and/or nucleon-pair) quanta, much as the familiar 4D Schwinger-Dyson equations of QED connect (via Ward identities) vertex amplitudes of successively higher orders [4].

1.1. 3D Reduction of BSE: Quasipotentials, Etc

The 3D Tamm-Dancoff equation (TDE) [3] and the 4D Schwinger-Dyson equation (SDE) [4] have been the source of much wisdom underlying the formulation of many approaches to strong interaction dynamics. To these one should add the Bethe-Salpeter equation (BSE) [5], which is an approximation to SDE for the dynamics of a 4D two-particle amplitude, characterized by an effective (gluon-exchange-like) pairwise interaction, on the lines of a "Bethe Second Principle" of the Fifties for the effective N-N interaction, but now adapted to the quark level. Although not a fundamental (first principle) approach, it has attracted more attention in the contemporary literature (as the 4D counterpart of the Schroedinger equation) than many other approaches.

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A major bottleneck for the BSE approach has been its resistance to a probability interpretation, since the logical demands of its 4D content are incompatible with its approximate nature. This has led to many attempts at 3D reductions [6-9]: Instantaneous approximation [6]; Quasi-potential approaches [7,8]; variants of on-shellness of the associated propagators [9]. In [7,9], the starting BSE is 4D in all details, *including its kernel*, but the associated propagators are manipulated in various ways to reduce the 4D BSE to a 3D form as a fresh starting point of the dynamics; in [8], the old-fashioned 3D perturbation theory is reformulated covariantly to give a 3D quasipotential equation. These methods are briefly sketched in Sect 2.

At the 4D level, the BSE [5] and its SDE counterpart are still the most widely used form of 2-particle dynamics [10], despite the problems of probabilistic interpretation. Such equations have been widely employed [10] as prototypes of strong interaction dynamics, addressing issues of gauge and chiral symmetries, as well as dynamical breaking of chiral symmetry ($DB\chi S$) via an NJL-type mechanism [11]. In such full-fledged field-theoretic approaches, the NJL- mechanism of contact interaction logically gives way to space-time extended interactions, and the ($DB\chi S$) corresponds to the use of SDE for the self-energy operator [12]. As a general remark, while for conceptual issues of formulational self-consistency, there is little alternative to the full-fledged 4D equations, their applications to physical systems must recognize some ground realities. For example, the mass spectra of hadrons (which are revealed in Nature as $O(3)$-like [13]), suggest that the role of the time-like dimension (although an integral part of the dynamics) is not on the same footing as that of the space-like dimensions, so that a naive expection of $O(4)$-like spectra [14] may be quite misleading. Indeed this issue is quite central to the very theme of this article, viz., 3D-4D interlinkage of BS-amplitudes, and will claim attention throughout.

An alternative form of 2-particle dynamics (which also contributes to reducing the effective degrees of freedom from 4D to 3D) is that of Dirac constrained Hamiltonian formalism [15], developed by Komar and others [16]. The logic of this approach is that constraints H_i have a twin role, viz., they not only 'constrain the motion' in phase space, but also generate it in their 'Hamiltonian' capacity. These 'constraints' must be mutually compatible in the sense $[H_i, H_j] = 0$. Such compatibility relations restrict the dependence of the interaction on the relative time t, and require a 'reciprocity relation' between the constituent potentials, something akin to Newton's III Law. Such descriptions are valid for both two-boson and two-fermion systems [16], in the sense of coupled K-G and Dirac equations respectively [17]. (This approach will not be pursued further.)

1.2. Light Front(LF)/ Null Plane(NP) Dynamics

A powerful form of 3D dynamics came into prominence after Weinberg's discovery that the dynamics of the infinite momentum frame [18] serves as a cure for many ills in the theory of current algebras, by greatly simplifying the rules of calculation of Feynman diagrams of old fashioned perturbation theory. In the present context of strong interaction dynamics, the great virtue of Weinberg's infinite momentum method [18] lies in the simplicity and transparency of the integral equations for multiparticle potential scattering problems [19]. Indeed, the structure of the 3-momenta $(p_\perp, p_\parallel)$ appearing in this formalism is but a paraphrase for the standard null-plane variables first introduced by Dirac to project his theme [20] that a relativistically invariant Hamiltonian theory can be based on 3 different classes of initial surfaces (space-like, time-like, and null-like). The structure of such a Hamiltonian theory is strongly dependent on these respective surface forms whose "stability groups" (i.e., those generators of the Poincare group that leave the initial surface *invariant*), are $6, 6, 7$ respectively, thus giving the 'highest score' (7) to the null-plane dynamics ($x_0 = x_3$) whose 'kinematical' generators form a closed algebra, and include among others the quantity $P_+ = P_0 + P_3$ (which plays the role of the 'mass' term η in the Weinberg notation [18]). On the other hand, the dual generator $P_- = P_0 - P_3$ is the 'Hamiltonian' of the theory.

Leutwyler and Stern [21] gave a covariant 3D formulation of the Dirac theme [20] in terms of null-plane variables. A more explicit covariant formmulation in the null-plane language was given by Karmanov [22a] using diagrammatic techniques with on-shell propagators and spurions, on the lines of the Kadychevsky approach [8], which has been recently reviewed by Carbonell et al [22b],(referred to as KK). All these methods, including the Wilson group's [23], give rise to 3D integral equations for a (strongly interacting) two-body system, bearing strong resemblance to the other 3D BSE forms [6-9] above. Again, there is no getting back to the original 4D BSE form, the nearest connection being a one-way reduction from 4D BSE to 3D on the covariant null-plane [22b].

1.3. Markov–Yukawa Transversality: 3D-4D Interlinkage

Finally we come to a rather novel approach of more recent origin [24–25], based on the Markov-Yukawa Transversality Principle (MYTP) [26]. To motivate this approach, it is necessary to go back in time to Yukawa's non-local field theory [26b] according to which the field variable is a function of both x and p (unlike in local field theory where p is absent). Although unacceptable for an elementary particle/field, the non-local field theory is ideally suited to a *composite* particle, whose extended structure effectively provides for a momentum dependence in the direction of the total 4-momentum P_μ. Indeed the Yukawa theory [26b] was in a way the forerunner of a later theory of bi-local fields $\mathcal{M}(x, y)$ [27] for the formulation of the Effective Action for a 2-body dynamical system [27]. This approach was employed by the Prevushin group [25] in their formulation of the relativistic Coulomb problem in the Salpeter approximation [6b] in a covariant form, with the choice of the preferred direction governed by the 4-momentum P_μ of the composite as the canonical conjugate to its c.m. position $X = (x + y)/2$: $P = -i\partial_X$. More specifically the MYTP [26] is expressed by the condition [25]:

$$z_\mu \frac{\partial}{\partial X_\mu} \mathcal{M}(z, X) = 0; \quad z = x - y, \tag{1.1}$$

where the direction P_μ guarantees an irreducible representation of the Poincare' group for the bilocal field $\mathcal{M}$ [26c]. This condition may be viewed as a sort of *gauge principle* [26c] which demands "redundance" of the "relative time" variable for the bilocal field $\mathcal{M}(x, P)$ w.r.t. the translation of the relative coordinate:

$$x_\mu \to x_\mu + \xi P_\mu,$$

which induces the "gauge trnsformation"

$$\mathcal{M}(x_\mu, P_\mu) \to \mathcal{M}(x_\mu + \xi P_\mu, P_\mu).$$

The "gauge invariance" now demands that this transformation leave the bilocal field invariant, which is precisely the condition (1.1) above.

The condition (1.1) may also be viewed as equivalent to a covariant 3D support to the input 4-quark Lagrangian, whence follow the SDE and BSE as equations of motion with 3D support to the effective BSE kernel under covariant instaneity. More simply, the 3D support ansatz may be postulated at the outset for the pairwise BSE kernel K [24] by demanding that it be a function of only $\hat{q}_\mu = q - q.PP_\mu/P^2$, which implies that $\hat{q}.P \equiv 0$. In this approach, the propagators are left untouched in their full 4D forms. This is somewhat complementary to the approaches [6–9] (propagators manipulated but kernel left untouched), so that the resulting equations [24–25] look rather unfamiliar vis-a-vis 3D BSE's [6–9], but it has the advantage of allowing a *simultaneous* use of both 3D and 4D BSE forms via their interlinkage. Indeed what distinguishes the MYTP-motivated [26] Covariant Instantaneity Ansatz (CIA) [24] from the more familiar 3D reductions of the BSE [6-9] is its capacity for a 2-way linkage: an *exact* 3D BSE reduction, and an equally *exact reconstruction* of the original 4D BSE form without extra charge [24]: the former to access the observed O(3)-like spectra [13], and the latter to give transition amplitudes as 4D quark loop integrals [24]. (In the approach of the Pervushin Group [25], however, the built-in 3D-4D interconnection which follows from MYTP [26], apparently remained unnoticed in their final equation). In contrast the more familiar methods of 3D [6-9], including the covariant LF [22-23], give at most a one-way connection, viz., a $4D \rightarrow 3D$ reduction, but not vice versa.

At this point it is perhaps worth noting that even the Salpeter equation [6b] has (in principle) the ingredients for a reconstruction of the 4D BS amplitude Ψ in terms of 3D ingredients, insofar as the 'instant' form, (see eq.(2.1) in Sect 2), of the interaction kernel is employed on the RHS of the 4D BSE form for the 4D BS amplitude Ψ, and duly eliminated in favour of the 3D BS amplitude ψ, exactly as in CIA [24]. This is indeed tantamount to the use of the Transversality Principle [26] (albeit non-covariantly), but the possibility of reconstruction of the 4D BS amplitude had apparently remained unnoticed by subsequent workers who continued to employ the Salpeter equation [6b] in its 3D form only.

1.4. QCD-motivated Effective Lagrangian

The Transversality Principle (MYTP) [26] underlying the 3D-4D interconnection [24], termed 3D-4D-BSE in the following, of course needs supplementing by physical ingredients to govern the structure of the BSE kernel, much as a Hamiltonian needs a properly defined 'potential'. However its canvas is broad enough to accommodate a wide variety of kernels which must in turn be governed by independent physical principles. In this respect, short of a full-fledged QCD Lagrangian approach, the orthodox view (which we adopt) is to stick to an effective 4-fermion Lagrangian as a starting point of the dynamics, from which the successive equations of motion (SDE, BSE, etc) follow in the standard manner. In this regard, a basic proximity to QCD is ensured through a vector-type interaction [12], which while maintaining the correct one-gluon-exchange structure in the perturbative region, may be fine-tuned to give any desired structure to the mediating gluon propagator in the infrared domain as well. Although empirical, it captures a good deal of physics in the non-perturbative domain while retaining a broad QCD orientation, albeit short of a full-fledged QCD formulation. More importantly, the non-trivial solution of the SDE corresponding to this generalized gluon propagator [12] gives rise to a dynamical mass function $m(p)$ [12] as a result of $DB\chi S$, w.r.t. an input Lagrangian whose chiral invariance stems from a vector-type 4-fermion interaction between almost massless $u - d$ quarks. These considerations form the standard basis for a Lagrangian-based BSE-SDE framework [10] for Dynamical Breaking of Chiral Symmetry ($DB\chi S$) [11], for a space-time extended 4-quark Lagrangian mediated by vector exchange [12]. This generates a mass-function $m(p)$ via SDE, which accounts for the bulk of the constituent mass of ud quarks. The same BSE-SDE formalism [12, 10] can be simply adapted [28] to the MYTP [26]-based 3D-4D-BSE formalism [24] which reproduces 3D spectra of both hadron types [29] under a common parametrization [28] for the gluon propagator, with a self-consistent SDE determination [28] of the constituent mass; see Sect 3.

A BSE-SDE formulation [10] on QCD lines represents a 4D field- theoretic generalization of 'potential models'[30], wherein the generalized 4-fermion kernel [12] represents the non-perturbative gluon propagator, which can be easily adapted [28] to MYTP [26]). The 4D feature of BSE-SDE gives this framework a ready access to high energy amplitudes, while its 'potential' features give it a natural access to hadronic spectra [13]. It has thus an interpolating role between (low energy) quarkonia models [30], and (high energy) QCD-SR [31] techniques whose domains are largely complementary; details may be found in a recent review [32].

1.5. MYTP: Cov Instantaneity vs Cov LF/NP

While MYTP [26] ensures 3D-4D interconnection [24] under covariant instantaneity ansatz (CIA) in the composite's rest frame [24], its applicability is limited by the ill-defined nature of 4D loop integrals which acquire time-like momentum components in the exponential/gaussian factors associated with the different vertex functions, due to a 'Lorentz-mismatch' among the rest-frames of the participating hadronic composites. This problem is especially serious for triangle loops and above, such as the pion form factor, while 2-quark loops [33] just escape this pathology. This problem was not explicitly encountered in the light-front (LF/NP) ansatz [34] in an earlier study of 4D triangle loop integrals, but this approach was criticized [35] on grounds of non-covariance. The CIA approach [24] which made use of MYTP [26], was an attempt to rectify the Lorentz covariance defect, but the presence of time-like components in the gaussian factors inside triangle loop integrals, e.g., in the pion form factor [36], impeded further progress.

In an attempt to remedy this situation, a generalization of MYTP [26] was proposed recently [37] to ensure formal covariance without having to encounter time-like components in the gaussian wave functions appearing inside the 4D loop integrals. The desired generalization was achieved by extending the Transversality Principle [26] from the covariant rest frame of the (hadron) composite [24], to a *covariantly defined* light-front [37] (Cov LF). It was found that while preserving the 3D-4D BSE interconnection, the resulting 3D equation under Cov LF [37] turns out to be formally identical to the old-fashioned null-plane formalism [34,38], so that the latter enjoys *ipso facto* covariance (despite its 'looks'). This 'covariant' LF/NP method [37] compares well with other covariant LF approaches [22-23].

1.6. Scope of the Article

This article has a 3-fold objective: A) a bird's eye view of some principal 3D vs 4D dynamical methods for the strong interaction problem that have been proposed over the last half century; B) Putting in perspective a novel property of the Markov-Yukawa Transversality Principle (MYTP), viz., a 2-way 3D-4D interconnection in the BS dynamics of 2- and 3-quark hadrons; C) Stressing a close parallelism between $q\bar{q}$ and qqq BSE's which stem from a common 4-fermion Lagrangian mediated by gluon (vector)-like exchange. Especially noteworthy is the capacity of MYTP [26] to achieve a 3D-3D interlinkage, a property which has remained obscured from view in the contemporary literature, vis-a-vis more familiar approaches to BSE and allied forms of dynamics [6–10, 18–23] which are either 3D or 4D in content, but have no provision for any interlinkage between these two dimensions. Our effort will be to stress the practical uses of MYTP in the strong interaction regime by virtue of its simplicity in evaluating 4D loop integrals with arbitrary vertex functions, while providing easy access to the spectroscopy sector. To that end, an outline of the dynamical structure of some principal 3D methods, [7–9], [18–23], is provided in Sect.2 as a background for comparison on 4D loop integral techniques, while on the issue of hadron spectra, which are basically $O(3)$-like [13], a comparison between non-MYTP [7–9] and MYTP [24–26] forms of dynamics does not bring out new physics beyond calibration of the respective parameters.

For a better understanding of the working of MYTP, it will be necessary to present two types in parallel for comparison, viz., Covariant Instantaneity or CIA [24–25], and Covariant Light-front (Cov LF) [37], which demand that the BSE kernel K for pairwise interaction be a function of relative momentum q *transverse* to the composite 4-momentum in the first case [24], and to the Covariant Null-plane in the second [37]. It will be shown that both types lead to identical 3D BSE reductions (so that their spectral predictions are formally the same), but their reconstructed 4D vertex functions reveal profound differences in structure: The Lorentz mismatch of individual wave functions that characterizes the CIA form [24], leading to complex amplitudes [34], disappears in the alternative Cov LF approach [37], but in general such integrals are dependent on the light-front orientation n_μ, as in other covariant approaches [22-23]. To eliminate such terms, a simple prescription of 'Lorentz completion' seems to suffice to produce an explicitly Lorentz invariant quantity such as was shown for the pion form factor [37]; (alternative prescriptions exist in other LF/NP formulations [22b]). For a historical perspective, it is useful to recall that in the old-fashioned NPA approach too [38], a very similar result had been found for various types of triangle loop amplitudes [36], despite a lack of manifest covariance [35] in that approach, but now MYTP [26] on the covariant null-plane [37] fills this formal gap.

1.7. Outline of Contents

Sect.2 briefly outlines some historical approaches to an effective 3D form of strong interaction dynamics: Levy-Salpeter [6]; Logunov- Tavkhelidze [7a]; Blankenbecler-Sugar [9]; Todorov [7d] ; Weinberg [18]; Feynman et al [39]. Sect.3 provides the theoretical framework with a short derivation of the BSE-SDE from an input chirally invariant Lagrangian, incorporating the original CIA form [25, 24] of MYTP [26], on the lines of ref.[25] in terms of bilocal fields [27]. It also includes a derivation [28] of the dynamical mass function $m(p)$ for an understanding of the constituent mass via Politzer additivity [40]. Sect.4 collects some basic results on the null-plane formalism due to Leutwyler-Stern [21], especially the role of the 'Angular Condition' in ensuring a formal $O(3)$-like invariance. From Sect.5 onwards, the focus is on MYTP [26]-orientation for bringing out its unique property which distinguishes it from most other approaches, viz., the *3D-4D interlinkage* of BS amplitudes.

Sect.5 gives a comparative view of the working of MYTP on the BSE forms in CIA [24] versus Cov LF [37], and outlines the derivation of the 3D BSE, as well as an explicit reconstruction of the 4D BS wave function in terms of 3D ingredients, with 3-momentum $\hat{q} = (q_\perp, q_3)$, where the third component emerges as a P-dependent one, suitably adapted to the CIA [24] or Cov LF [37] respectively. Sec.6 gives a corresponding derivation for the 3D-4D interconnection for a qqq BSE structure under CIA conditions. Sec.7 illustrates, through the calculation of triangle-loop integrals, the relative advantage of Cov LF [37] over the CIA [24] version of MYTP [26], in producing a well-defined structure for the pion e.m. form factor in a fully gauge invariant manner, and illustrating in the process the method of 'Lorentz-completion' for explicit Lorentz invariance, with the expected k^{-2} behaviour at high k^2. MYTP also gives a more general structure of triangle loop integrals for three-hadron form factors. Sect.8 summarises our conclusions.

2. Quasipotentials And Other 3D Equations

The reduction of the 4D BSE for an $N - N$ pair to the 3D level in the Instantaneous Approximation was first investigated in the non-adiabatic domain of pseudoscalar meson theory (effect of pair-creations included) by Levy [6a], who showed that this 3D BSE form is entirely equivalent to the corresponding Tamm-Dancoff equations [2] in the same (non-adiabatic) limit.

On the other hand, Salpeter [6b] employed the adiabatic approximation (no pair creation effects) to give a systematic 3D reduction of the fermionic BSE, using projection operators for large and small components. The adiabatic approximation amounts to replacing the propagator $\Delta_F(x - x')$ for the exchanged meson by

$$\Delta_F(x - x') \Rightarrow \delta(x_0 - x_0') \int_{-\inf}^{\inf} \Delta_F(\mathbf{x} - \mathbf{x}', x_0 - x_0') d(x_0 - x_0') \tag{2.1}$$

and simply gives the Yukawa potential between two particles. Similarly, in the Instantaneous (adiabatic) Approximation, IA for short, the 4D wave function $\Psi(x) = \Psi(\mathbf{x}, t)$ for relative motion of two particles becomes simply $\Psi(\mathbf{x}, 0)$. In the momentum representation, these statements read respectively as

$$\Delta_F(k) \Rightarrow \Delta_F(\mathbf{k}); \quad \psi(\mathbf{q}) = \int dq_0 \Psi(\mathbf{q}, q_0). \tag{2.2}$$

The Salpeter 3D BSE in the IA for a relativistic hydrogen-atom is [6b]:

$$(E - H_1(\mathbf{q}) - H_2(\mathbf{q}))\chi(\mathbf{q}) = \int d^3k \frac{e^2}{2\pi^2(\mathbf{k}-\mathbf{q})^2}[\Lambda_{1+}\Lambda_{2+} - \Lambda_{1-}\Lambda_{2-}]\chi(\mathbf{k}), \tag{2.3}$$

where the 3D wave function $\chi(\mathbf{q})$ is related to the corresponding 4D quantity by an equation of the form (2.2), and the symbols $\Lambda_\pm$ are energy projection operators for the large/small components, etc.

2.1. Logunov-Tavkhelidze Quasipotentials

A different form of 3D reduction of the 4D BSE was proposed by Logunov- Tavkhelidze [7a] in the language of Green's functions (G-fns) for 2-particle scattering whose momentum representation my be written as $G(p_1p_2; p_1'p_2')$ (with indicated 4-momenta before and after), which satisfies a 4D BSE [7a]:

$$(2\pi)^8\Delta(p_1)\Delta(p_2)G(p_1p_2; p_1'p_2') = \delta(p_1 - p_1')\delta(p_2 - p_2') + \int dp_1''dp_2''K(p_1p_2; p_1''p_2'')G(p_1''p_2''; p_1'p_2'), \tag{2.4}$$

where $\Delta(p_i) = p_i^2 + m_i^2$, etc. Expressing this equation in c.m. (P) and relative (q) 4-momenta, and taking out the δ-fns due to the c.m. motion, this equation simplifies to

$$(2\pi)^4\Delta(p_1)\Delta(p_2)G(q, q'; P) = \delta(q - q') + \int dq''K(q, q'')G(q'', q; P). \tag{2.5}$$

Next, they defined the 3D G-fn for the relative motion as a double integral w.r.t. the two time-like momenta:

$$\hat{G}(\mathbf{q}, \mathbf{q}'; P) = \int q_0 \int q_0' G(q, q'; P). \tag{2.6}$$

Now in operator notation, the 4D BSE (2.5) may be written as $G = G_0 + G_0KG$, from which the kernel K has the formal representation $K = G_0^{-1} - G^{-1}$. The L-T trick [7a] now consists in using the double integrals on the time-like momenta as in eq.(2.6) to formally define the 3D quasipotential $\hat{K}$ as

$$\hat{K} \equiv \hat{G_0}^{-1} - \hat{G}^{-1}, \tag{2.7}$$

which can be expanded perturbatively in the symbolic form [7a]

$$\hat{K} = \hat{G}_0^{-1}G_0\hat{K}G_0\hat{G}_0^{-1} - \hat{G}_0^{-1}G_0K\hat{G}_0KG_0\hat{G}_0^{-1} - .... \tag{2.8}$$

to any desired order of accuracy; [note that the inverse G-fns are just the self-energy operators]. If $V(\hat{q}, \hat{q}'; E)$ is the quasi- potential to a given order of accuracy, then, the BSE satisfied by the 3D BS wave function $\psi(\hat{q})$ is of the form [7a]:

$$(E^2 - \hat{q}^2 - m^2)\psi(\hat{q}) = \int d^3\hat{q}'V(\hat{q}, \hat{q}'; E)\psi(\hat{q}'), \tag{2.9}$$

where the 'denominator function on the LHS arises from integrating $G_0 = \Delta(p_1)^{-1}\Delta(p_2)^{-1}$ w.r.t. q_0 and rearranging.

2.1.1. Narrow resonances in charged particle systems

Within the last decade, the L-T theory [7a] has witnessed some interesting applications [41] to the understanding of 'new' narrow e^+e^- resosances observed in heavy ion collisions [42]. To that end, the authors [41] have employed an equation of the form (2.9) which reads for this system as [41]:

$$2\omega(M-2\omega)\phi(\mathbf{p}) = \frac{(2me)^2}{(2\pi)^3}\int \frac{d^3\mathbf{p}'\phi(\mathbf{p}')}{2\omega' q(M-\omega-\omega'-q+i0)}, \tag{2.10}$$

where $\omega = \sqrt{m^2+\mathbf{p}^2}$, and $q = \mid \mathbf{p}-\mathbf{p}' \mid$. The results indicate a possible interpretation of the observed peaks [42] as new quasi-stationary levels arising from the solution of the quasi- potential equation. More interestingly, they also suggest a close relationship of the observed states [42] with the von Neumann-Wigner [43] levels embedded in the continuum.

2.2. Blankenbecler-Sugar Equation

Another type of quasipotential was proposed by Blankenbecler-Sugar [9], as follows. The 2-particle scattering amplitude $T(q,q')$ due to a 4D potential $V(q,q')$ in the ladder approximation satisfies the BSE [9]:

$$T(q,q') = -i(2\pi)^{-4}\int d^4q''V(q,q'')[m^2+(P/2+q'')^2]^{-1}[m^2+(P/2-q'')^2]^{-1}T(q'',q'), \tag{2.11}$$

where the 2-particle 'free' G-fn is exhibited as the product of the two propagators inside the integral on the RHS. To express this equation in 3D form, the B-S [9] trick consists first in putting q' on the energy shell, which means that $q'_0 = 0.$ and $q'^2 = s/4 - m^2$, where $s = -P^2$ is the square of the c.m. energy. Next, the on-shell part E_2 of the free 2-particle G-fn is obtained by taking only the δ-fn parts of the two propagators which gives rise only to two-particle cuts in the physical region

$$E_2(q'') = 2\pi\int ds'(s'-s)^{-1}\delta[m^2+(P'/2+q'')^2]\delta[m^2+(P'/2-q'')^2], \tag{2.12}$$

where $s' = -P'^2$, and P' has only a fourth component. This works out as

$$E_2(q'') = \frac{1}{2}\pi\delta(q''_0)[\sqrt{(q''^2+m^2)}(q''^2-q^2)]{-1}. \tag{2.13}$$

The balance R_2 of the free G-fn is not singular along the positive cut of the s-variable. If it is neglected in the first approximation, and only E_2 from (2.13) is substituted in (2.11), then after a trivial integration over q''_0, the resultant 3D equation has the form

$$T(q,q') = V(q,q') + \frac{1}{4}\int \frac{d^3q''}{(2\pi)^3}\frac{V(q,q'')T(q'',q')}{\sqrt{m^2+q''^2}(q''^2-q'^2)}. \tag{2.14}$$

A comparison of (2.9) and (2.14) shows that although both equations are formally 3D in looks, there is a vast difference in their contents: The L-T [7a] form (2.9) involves only 3-momenta $\hat{q} \equiv \mathbf{q}$, since the Hilbert space has been 'truncated' by integrating out over their fourth components. The B-S [9] form (2.14) on the other hand has 4-momenta formally throughout (no truncation of Hilbert space), except that they are on their mass shells ! Thus formal covariance is violated in both equations, although in different ways.

2.3. Kadychevsky-Todorov Equation

Still another form of 3D (Lippmann-Schwinger) equation was given by Todorov [7d], following the Covariant method of Kadychevsky [8]. In the Todorov approach [7d], the potential V_w is defined as an infinite power series in the coupling constant which fits the perturbative expansion of the scattering amplitude T_w for two particles of masses m_1, m_2 and 4-momenta p_1, p_2 and q_1, q_2 before and after respectively. The quantity T_w in the off-shell regime satisfies the L-S equation [7d]

$$T_w(\mathbf{p},\mathbf{q}) + V_w(\mathbf{p},\mathbf{q}) + \frac{1}{\pi^2 w}\int d^3\mathbf{k}\frac{V_w(\mathbf{p},\mathbf{q})T_w(\mathbf{p},\mathbf{q})}{\mathbf{k}^2 - b^2 - i\epsilon}, \tag{2.15}$$

where the 3D quantities in the c.m. frame are defined as

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}; \quad \mathbf{q}_1 = -\mathbf{q}_2 = \mathbf{q} \tag{2.16}$$

and on the energy shell, the corresponding time-like quantities are

$$p_{10} + p_{20} = w = q_{10} + q_{20}; \quad -p^2 = -q^2 = w^2; \quad 4w^2 b^2(w) = \lambda(w^2, m_1^2, m_2^2) \tag{2.17}$$

This equation too has strong resemblance to the L-T equation [7a]. The corresponding equation for the bound state wave function $\phi(\mathbf{p})$ is

$$\pi^2 w(\mathbf{k}^2 - b^2(w))\phi(\mathbf{p}) = -\int d^3\mathbf{k} V(\mathbf{p},\mathbf{k})\phi(\mathbf{k}). \tag{2.18}$$

Both B-S [9] and Todorov[7d] equations have been extensively employed in the literature.

2.4. Infinite-Momentum Frame: Weinberg Equation

Weinberg [18] observed some remarkable simplifications that occur when the results of old-fashioned perturbation theory are expressed in a reference frame in which the total 3-momentum $\mathbf{P}$ is very large. In this limit, the 3-momentum $\mathbf{p}_n$ of the n-th particle may be projected parallel and perpendicular to $\mathbf{P}$, and the results collected as follows:

$$\mathbf{p}_n = \eta_n \mathbf{P} + \mathbf{q}_n; \quad \mathbf{q}_n.\mathbf{P} = 0; \quad \sum_n \eta_n = 1; \quad \sum_n \mathbf{q}_n = 0. \tag{2.19}$$

The quantity $\eta_n > 0$ in this theory, plays the role of 'mass' of the n-th particle (in a 3D Schroedinger-type equation), and in the $P \to \infty$ limit, the rules of calculation become very simple: all old-fashioned perturbative diagrams passing through negative energy intermediate states vanish, while for the contributing diagrams, the propagator for an intermediate state c in a transition from a to b, has the form $2[s_a - s_c + i\epsilon]^{-1}$, where s for any state is the usual total c.m. energy squared:

$$s = \sum_n [\mathbf{q}_n^2 + m_n^2]/\eta_n; \quad s_a = s_b = s_c, etc. \tag{2.20}$$

Momentum conservation at each vertex is 3D in content:

$$(2\pi)^3 \delta(\Delta \sum \eta)\delta^2(\Delta \sum \mathbf{q}) \tag{2.21}$$

in accordance with the conservation of η and $\mathbf{q}$, eq.(2.15). The Weinberg counterpart of the L-T [7a], B-S [9] and Todorov [7d] equations (2.9), (2.14) and (2.18) respectively, is the integral equation [18]

$$< \mathbf{q}'\eta' \mid T \mid \mathbf{q}\eta > = < \mathbf{q}'\eta' \mid V \mid \mathbf{q}\eta > + \int d^2q'' \int d\eta'' \frac{< \mathbf{q}'\eta' \mid V \mid \mathbf{q}\eta >< \mathbf{q}'\eta' \mid V \mid \mathbf{q}\eta >}{2(2\pi)^3[s\eta''(1-\eta'') - \mathbf{q}''^2 - m^2 + i\epsilon]}. \tag{2.22}$$

Although this equation is effectively 3D, and has considerable similarity to the corresponding equations of [7a,7d,9] above, there is a big difference, viz., the angular momentum is no longer a well-defined concept in this 3D description. This gap was bridged later by Leutwyler-Stern [21] by invoking the 'angular condition' [21], after it became clear that the Weinberg approach is equivalent to Dirac's [20] null-plane dynamics; see Sect.3 below.

2.5. The FKR Model For 2- And 3-Quark Dynamics

Before ending this Section, we draw attention for historical reasons, to a unique paper by Feynman and collaborators [39], FKR for short, which gave an integrated view of 2- and 3-quark hadron dynamics, and played a big role in shaping the direction of strong interaction physics to come. The importance of the FKR approach stems, among other things, from the fact that these authors were the first to show the way to a unified treatment of both 2- and 3-quark hadrons within a common dynamical framework, which was to serve as a model for the future. This paper effectively incorporated all the relevant aspects of quark dynamics that had been generated piecemeal in the Sixties, and had by and large come to be accepted, viz., the group structure $SU(6) \times O(3)$, the symmetrical quark model, and harmonic oscillator classification of hadron states (based on their linear M^2-N plots) on the one hand [44], and the mechanism of single-quark transitions, quark recoil effects, etc, on the other [45].

The FKR model, which made essential use of harmonic confinement, sought to give a relativistic meaning to the internal motion of light quarks through the following definitions of 2- and 3-quark Hamiltonians [39, 38]:

$$- K_M = 2(p_1^2 + p_2^2) + \frac{1}{16}\Omega^2 (x_1 - x_2)^2 + Const \equiv P^2 + M_M^2; \tag{2.23}$$

$$- K_B = 3({p_1}^2 + {p_2}^2 + {p_3}^2) + \frac{1}{36}\Omega^2 \sum_{123} (x_1 - x_2)^2 + Const \quad \equiv (P^2 + M_B^2), \tag{2.24}$$

where $x_{1\mu}$=$i\partial_{1\mu}$; $p_1^2 = p_{1\mu}p_{1\mu}$, etc. The quantity Ω which is postulated to be the *same* for *both* systems, has the significance of the universal Regge slope ($\approx 1GeV^2$) as observed [44] in their respective spectra; [Note the geometrical factors as cofficients in front of the respective kinetic and potential terms above]. The operators $K_{M,B}^{-1}$ are the 'free' propagators (albeit with h.o. confinement) for the mesons and baryons, whose 'poles' correspond to the eigenvalues (spectra) of their squared masses. The presence of a perturbation δK can be simulated in a standard gauge-invariant manner, by the substitutions $p \rightarrow p_\mu - eV_\mu$ or $p_\mu \rightarrow p_\mu - g\gamma_5 A_\mu$ for vector and axial vector couplings respectively, after rewriting p_μ^2 as $(\gamma.p)^2$, while the $i \rightarrow j$ transition amplitudes are just $< h_j \mid \delta K \mid h_i >$, by standard rules of quantum mechanics.

A major achievement of the FKR model was its success in giving two distinct types of unification, viz., a common framework for Spectroscopy and transition amplitudes; and ii) a unified dynamical treatment of $q\bar{q}$ and qqq hadrons. Both these features represented landmarks in a dynamical understanding of the quark model, yet the FKR model failed on another count: the 'wrong' sign of the time-like momenta in the gaussian wave functions for the hadrons was a disease which pointed to an asymmetric role of time-like (1D) momenta vis-a-vis the space-like (3D) ones. Attempts to cure this disease by a Euclidean treatment (via Wick rotation) [46a] failed on the spectroscopy front [13] which reveal only $O(3)$-like spectra, while other non-covariant treatments [46b] were not very successful either. Nevertheless the lessons from the FKR model were significant pointers to the need to treat the 1D time-like and 3D space-like d.o.f.'s on different footings in a future quest for a covariant theory [24-26].

3. Self-Energy And Vertex Fns Under MYTP

As a first step towards introducing the MYTP [26] theme, we collect in this Section some essential machinery for the interconnection between self-energy and vertex functions via Schwinger-Dyson (SDE) and Bethe-Salpeter (BSE) equations, starting from a chirally invariant Lagrangian characterized by a vector-type interaction [12] as a prototype for a gluon-exchange propagator in the non-perturbative QCD regime [28]. To that end, we shall first outline the method of bilocal fields [27] to derive the equations of motion (SDE and BSE) from such a Lagrangian, following the Pervushin

Group's [25] bilocal field method, under MYTP [26] conditions of covariant instantaneity [24]. This will be followed by a general result connecting self-energy and pion- quark vertex functions in the chiral limit, i.e., when the pion mass vanishes. This result in turn paves the way to a derivation [28], under MYTP [26] conditions of Covariant Instantaneity [24], of the mass function $m(p)$ whose low momentum limit is the main contributor to the constituent mass, via Politzer additivity [40].

3.1. Method of Bilocal Fields for BSE-SDE

The effective action for a system of two interacting massless fermions constrained by MYTP [26] is given by [25]

$$W_{eff}[\psi,\bar{\psi}] = \int d^4x[\bar{\psi}(x)(i\gamma\dot{\partial} - m_0)\psi(x) + \frac{1}{2}\int d^4y(\psi(y)\bar{\psi}(x))K(z^\perp, X)(\psi(x)\bar{\psi}(y))], \tag{3.1}$$

where $z = x - y$; $X = (x + y)/2$. $z^\perp$ is the component of z transverse to the P-direction. Now redefine the action (3.1) in terms of bilocal fields $\mathcal{M}$ via the Legendre transformation [27] on the second term to give

$$-\frac{1}{2}\int d^4xd^4y\mathcal{M}(x,y)K^{-1}(z^\perp, X)\mathcal{M}(x,y) + \int d^4xd^4y(\psi(x)\bar{\psi}(y)\mathcal{M}(x,y). \tag{3.2}$$

Then in an obvious short-hand notation [25b], (3.1) may be written as

$$W_{eff}[\mathcal{M}] = (\psi\bar{\psi}, (-G_0^{-1} + \mathcal{M}) - \frac{1}{2}(\mathcal{M}, K^{-1}\mathcal{M}), \tag{3.3}$$

where G_0 is the inverse Dirac operator for the free fermion field. After quantization over N_c fermion fields and normal ordering, the action takes the form [25b]

$$W_{eff}[\mathcal{M}] = -\frac{1}{2}N_c(\mathcal{M}, K^{-1}\mathcal{M}) + iN_c\sum_{n=1}^{\inf}\frac{1}{n}\Phi^n, \tag{3.4}$$

where $\Phi = G_0\mathcal{M}$ is a matrix in (x, y) space, and its successive powers are defined in the standard matrix fashion. Now for the quantization of the action (3.4), its minimum is given by

$$N_c^{-1}\frac{\delta W_Q(\mathcal{M})}{\delta\mathcal{M}} \equiv -K^{-1}\mathcal{M} + \frac{1}{G_)^{-1} - \mathcal{M}} = 0. \tag{3.5}$$

The corresponding 'classical' (lowest order) solution for the bilocal field is $\Sigma(x - y)$ which depends only on the difference $x - y$ due to the translation invariance of the vacuum solutions. Next expand the action (3.4) around the point of minimum $\mathcal{M} = \Sigma + \mathcal{M}'$, and denote the small fluctuations $\mathcal{M}'$ as a sum over the complete set of 'classical' solutions Γ. Then in the next order of extremum, we have:

$$\frac{\delta^2 W_Q(\Sigma + \mathcal{M}')}{\delta\mathcal{M}^2}\mid_{\mathcal{M}'=0}\dot{\Gamma} = 0. \tag{3.6}$$

Eqs.(3.5-6) give respectively the SDE for Σ and BSE for Γ:

$$\begin{aligned}\Sigma(x-y) &= m_0\delta^4(x-y) + iK(z^\perp, X)G_\Sigma(x-y); \qquad (3.7)\\ \Gamma &= iK(z^\perp, X)\int d^4z_1d^4z_2G_\Sigma(x-z_1)\Gamma(z_1,z_2)G_\Sigma(z_2-y),\end{aligned}$$

which describe the spectrum of the fermions and composites respectively. In momentum space these equations for the mass operator and vertex function are

$$\Sigma(\hat{p}) = m_0 + i\int\frac{d^4q}{(2\pi)^4}V(\hat{p} - \hat{q})\gamma\dot{\eta}G_\Sigma(q)\gamma\dot{\eta}; \quad \eta_\mu = P_\mu/|P|, \tag{3.8}$$

$$\Gamma(\hat{k}) = i \int \frac{d^4q}{(2\pi)^4} V(\hat{k} - \hat{q}) \gamma\dot{\eta} [G_\Sigma(q + P/2)\Gamma(q^\perp)G_\Sigma(q - P/2)]\gamma\dot{\eta}, \tag{3.9}$$

where $G_\Sigma(q)$=$(\gamma\dot{q} - \Sigma(q^\perp))^{-1}$; V is the scalar part of the kernel K with 3D support; $\hat{k}$ is the transverse part of k w.r.t. the direction η_μ of the total 4-momentum P_μ.

3.2. Self-Energy vs Vertex Fn in Chiral Limit

The formal equivalence of the mass-gap equation (3.8) and the BSE (3.9) for a pseudoscalar meson in the chiral limit [12] is now demonstrated by adapting them to a non-perturbative gluon exchange propagator [28] with an arbitrary confining form $D(k)$ (not just the perturbative form k^{-2}). The SDE, eq.(3.8), after replacing the color factor $\lambda_1.\lambda_2/4$ by its Casimir value 4/3, and a relabelling of symbols [28], now reads

$$\Sigma(p) = \frac{4}{3} i (2\pi)^{-4} \int d^4k D(\hat{k}) \gamma\dot{\eta} S'_F(p - k)\gamma\dot{\eta}; \tag{3.10}$$

S'_F is the full propagator related to the mass operator $\Sigma(p)$ by

$$\Sigma(p) + i\gamma.p = S_F^{-1}(p) = A(p^2)[i\gamma.p + m(p^2)] \tag{3.11}$$

thus defining the mass function $m(p^2)$ in the chiral limit $m_c = 0$. In the same way, eq.(3.9) for the vertex function Γ_H of a $q\bar{q}$ hadron (H) of 4-momentum P_μ made up of quark 4-momenta $p_{1,2} = P/2 \pm q$ reads as

$$\Gamma_H(q, P) = -\frac{4}{3} i (2\pi)^{-4} \int d^4q' D(\hat{q} - \hat{q}')\gamma\dot{\eta} S_F(q' + P/2)\Gamma_H(q', P)S_F(q' - P/2)\gamma\dot{\eta}. \tag{3.12}$$

The complete equivalence of (3.10) and (3.12) for the pion case in the chiral limit $P_\mu \to 0$ is easily established. Indeed, with the self-consistent ansatz Γ_H =$\gamma_5\Gamma(q)$, eq.(3.12) simplifies to

$$\Gamma(q) = \frac{4}{3} i (2\pi)^{-4} \int d^4k \gamma\dot{\eta} S'_F(k - q)\Gamma(q - k)S'_F(q - k)\gamma\dot{\eta} D(\hat{k}), \tag{3.13}$$

where the replacement $q' = q - k$ has been made. Substitution for S'_F from (3.11) in (3.13) gives

$$\Gamma(p) = -\frac{4}{3} i (2\pi)^{-4} \int d^4k \frac{D(\hat{k})\Gamma(p - k)}{A^2(p - k)(m^2((p - k)^2) + (p - k)^2)}, \tag{3.14}$$

where we have relabelled $q \to p$. On the other hand substituting for S'_F (3.11) in (3.10) gives for the mass term of $\Sigma(p)$ the result

$$A(p^2)m(p^2) = -\frac{4}{3} i (2\pi)^{-4} \int d^4k \frac{D(\hat{k})A(q')m(q'^2)}{A^2(q')(m^2(q'^2) + q'^2)}, \tag{3.15}$$

where $q' = p - k$. A comparison of (3.14) and (3.15) shows their equivalence with the identification $\Gamma(q) = A(q)m(q^2)$, i.e. the identity of the vertex and mass functions in the chiral limit, provided $A = 1$, (which corresponds to the Landau gauge; see [32]). Although obtained here in the context of MYTP [26] this result is independent of this ansatz. A more explicit gauge theoretic derivation of the equations for the self-energy and vertex functions is given in [32].

3.3. Dynamical Mass As $DB\chi S$ Solution of SDE

We end this Section with the definition of the 'dynamical' mass function of the quark in the chiral limit ($M_\pi = 0$) of the pion-quark vertex function $\Gamma(\hat{q})$, in the 3D-4D BSE framework [24,28]. The logic of this follows from the BSE-SDE formalism outlined above, eqs.(3.10-15), for the connection between eq.(3.15) for $m(p)$ and eq.(3.14) for $\Gamma(q)$ in the limit of zero mass of the pseudoscalar. So, setting $M = 0$ in the (unnormalized) $Hq\bar{q}$ vertex function Γ_H this quantity may be identified with the mass function $m(\hat{p})$, in the limit $P_\mu = 0$, where p_μ is the 4-momentum of either quark; (note the appearance of the 'hatted' momentum). The result is [28,32]

$$m(\hat{p}) = [\omega(\hat{p}); \sqrt{2}p.n]\frac{m_q^2 + \hat{p}^2}{m_q^2}\phi(\hat{p}) \tag{3.16}$$

under CIA and Cov LF respectively. The normalization is such that in the low momentum limit, the constituent ud mass m_q is recovered under CIA [28], while the corresponding 'mass' under Cov LF is p_+ [32].

4. Null–Plane Preliminaries

The Weinberg infinite momentum method received a major boost through an understanding of Bjorken scaling [47a] in deep inelastic scattering, as well of the Feynman parton picture [47b] in the same process. The similarity of the $P \to \inf$ and the null-plane descriptions became clear with the demonstration by Susskind [48] of the $U(2)$ structure of the LF/NP language wherein the role of 'mass' is played by the combination $p_+ = p_0 + p_3$, and subsequently a more complete formulation of null-plane dynamics by Kogut and Soper [49] within the Hamiltonian formalism in the context of field theory.

In a different direction, efforts were made to extend the Lorentz contraction ideas to finite momentum frames, designed to bring out the effect of Lorentz contraction on cluster form factors as a result of motion [50]. In the respect, the role of the Breit frame is particularly interesting since it gives the best possible overlap [50b] between the initial and final clusters. The Lorentz contraction factors [50] in turn are the key to an understanding of 'dimensional scaling' [51], especially in a 'symmetrized' version [50d] of the Breit frame [50b] which exactly reproduces the correct 'power counting' [51]. And the Weinberg result [18] is duly recovered in the $P \to \inf$ limit, giving rise to the null-plane dynamics; (for more details of these results, see [38]). A more complete formulation of LF/NP dynamics, albeit with a *finite* number of degrees of freedom was given by Leutwyler-Stern [21] which comes closest to the original Dirac [20] spirit, and is summarized below.

4.1. Leutwyler-Stern Formalism

Leutwyler-Stern [21] employed a Hamiltonian approach for investigating the properties of a relativistic 2-body system with a *finite* number of d.o.f.'s, and postulating a general criterion of 'covariance' in the form of an operator relation among the mass and spin operators of the system. Their formalism is based on the maximum 'stability group' (in the Dirac [20] sense) for the null-plane surface $x_0 + x_3 = 0$, which gives rise to the following seven 'kinematical' generators [38]:

$$\begin{aligned} \mathcal{K} &= (P_1, P_2, P_+, E_1, E_2, K_3, J_3); \quad 2P_\pm = P_0 \pm P_3; \\ 2E_1 &= K_1 + J_2, \quad 2E_2 = K_2 - J_1; \quad J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}; \quad K_i = M_{0i}. \end{aligned} \tag{4.1}$$

The matrix elements of $\mathcal{K}$ form a closed algebra ($r, s = 1, 2$):

$$\begin{aligned} [K_3, E_r] &= -iE_r; \quad [K_3, P_+] = -iP_+; \quad [J_3, E_r] = +i\epsilon_{rs}E_s; \\ [J_3, P_r] &= +i\epsilon_{rs}P_s; \quad [E_r, P_s] = -i\delta_{rs}P_+ \,. \end{aligned} \tag{4.2}$$

On the other hand, there are 3 'Hamiltonians' which can be chosen in one of several different ways. To that end, it is necessary to introduce certain rotation operators I_i defined by $I_i \mid n >= J_i \mid n >$, on a rest system $\mid n >$, but extended to states $\mid \mathbf{p}, n >$ defined by

$$\mid \mathbf{p}, n >= Exp[-i\beta_1 E_1 - i\beta_2 E_2 - i\beta_3 K_3] \mid n >; \quad \beta_r = p_r/p_+; \quad \beta_3 = \ln(2p_+/M) \tag{4.3}$$

such that $\mathbf{I}$ commutes with the algebra of $\mathcal{K}$. More explicitly,

$$\begin{aligned} I_i \mid \mathbf{p}, n > &= Exp[-i\beta_1 E_1 - i\beta_2 E_2 - i\beta_3 K_3] J_i \mid n >; \\ [I_i, I_j] &= i\epsilon_{ijk} I_k; \quad [J_i, I_i] = 0. \end{aligned} \tag{4.4}$$

In particular,

$$I_3 = J_3 + (E_1 P_2 - E_2 P_1)/P_+ = (W_0 + W_3)/P_+; \quad W_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} P^\nu M^{\alpha\beta}. \tag{4.5}$$

W_μ is the Pauli-Lubanski operator, and $MI_r = W_r - P_r W_+/P_+$, where $r = 1, 2$. One thus arrives at the 'dynamical group' $D = (M, I_i)$, or (M^2, MI_i), which has the structure of $U(2)$ [48], because of (3.4). D is really a 3-member group, since I_3 already belongs to $\mathcal{K}$ by virtue of (3.5). For a particular significance of $\mathbf{I}$, note its connection with the non-relativisic Galilei-invariant algebra generated by the momentum $\mathbf{P}$ and Galilei boosts $\mathbf{K}$, viz.,

$$\mathbf{I} = \mathbf{J} + m^{-1}\mathbf{K} \times \mathbf{P}; \quad (m = mass). \tag{4.6}$$

Now Galilei invariance of a system is equivalent to the condition that the corresponding dynamical algebras constitute a unitary representation of $U(2)$. In the relativistic case, there is a superficial similarity to the $U(2)$ structure of D, but unlike the NR case, only the component I_3 is now 'kinematical', by virtue of (3.5), while $I_{1,2}$ are 'dynamical', and do not have explicit representations by themselves. L-S [21] sought to bridge this gap by imposing the 'covariance' requirement in the form of an 'angular condition' for the operators I_i as follows:

$$x_1 MI_1 + x_2 MI_2 + x_L I_3; \quad x_L = P_1 x_1 + p_2 x_2 + P_+ x_- \,, \tag{4.7}$$

which can be shown to be valid on the null-plane $x_+ = x_0 + x_3 = 0$ [38].

The L-S formalism [21] provides a compact support to the longitudinal momentum z of 2-particle system with constituent 4-momenta $p_{1,2}$:

$$2zP_+ = p_{1+} - p_{2+}; \quad P_+ = p_{1+} + p_{2+}; \quad -\frac{1}{2} \leq x \leq +\frac{1}{2}. \tag{4.8}$$

The internal wave function ϕ is defined by

$$< \mathbf{p}_1, \mathbf{p}_2 \mid \mathbf{P}, \phi >= 2P_+ \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P})\phi(\mathbf{q}_\perp, x), \tag{4.9}$$

where ϕ is a matrix in helicity space (h', h''), with the norm:

$$< \phi \mid \phi >= \frac{1}{2}\int \frac{d^2 q_\perp dz}{\frac{1}{4} - z^2} \times \sum_{h'h''} \mid \phi_{h'h''}(\mathbf{q}_\perp, z) \mid^2. \tag{4.10}$$

The L-S [21] structure formally allows the introduction of a 3-vector $\mathbf{q}$ and angular momentum $\mathbf{L}$ for the internal motion of a composite of mass M with (equal) constituent masses m_q as

$$\mathbf{q} = (\mathbf{q}_\perp, Mx); \quad \mathbf{L} = -i\mathbf{q} \times \nabla_q \tag{4.11}$$

with the phase space

$$\frac{d^2q_\perp dz}{\frac{1}{4}-x^2} = \frac{4d^3\mathbf{q}}{M}; \quad M^2 = 4(m_q^2 + \mathbf{q}^2). \tag{4.12}$$

With these definitions of $\mathbf{q}$ and $\mathbf{L}$, the theory formally preserves the concept of L-invariance of a $q\bar{q}$ system despite the apparently asymmetric treatment meted out to the transverse and longitudinal components of 3-momenta in the NP or P = inf formalism [18, 25, 48-9]. This invariance can be traced to angular condition (4.7). Incidentally, the historic FKR model [39], despite its other defects, was found by L-S [21] to satisfy the angular condition (4.7).

An alternative but more pedigogical recipe to achieve the same end was given in [38] via the simpler condition $P.q = 0$, to be consistently imposed between the total (P_μ) and relative (q_μ) 4-momenta of a $q\bar{q}$ system, as outlined in subsection 4.2 below [38].

4.2. An Alternative "Angular Condition" $P.q = 0$

For unequal masses $m_{1,2}$ of the (quark) constituents with 4-momenta $p_{1,2}$, the total (P) and relative (q) 4-momenta are given by the Wight-Gaerding definitions [52]

$$p_{1,2} = \hat{m}_{1,2}P \pm q; \quad 2\hat{m}_{1,2} = 1 \pm \frac{m_1^2 - m_2^2}{M^2}, \tag{4.13}$$

where $M = \sqrt{-P^2}$ is the composite mass. The condition $P\dot{q} = 0$ is satisfied on the mass shells $m_{1,2}^2 + p_{1,2}^2 = 0$ of the respective constituents, by virtue of the Wightman-Gaerding definition (4.13).

To link the condition $P.q = 0$ with the construction of an effective 3-vector in the null-plane language, so as to preserve the invariance of the angular momentum concept, note that this condition translates to the relation $q_- = -q_+ M^2/P_+^2$ which expresses the component q_- in terms of q_+, in a frame $P_\perp = 0$, since in this frame, $P_+P_- = M^2$ on the mass shell of the composite. [The collinearity condition is not a restriction for a two-body system]. This relation then allows a definition of the 3-momentum $\mathbf{q}$ with the components $(\mathbf{q}_\perp, q_3)$, with $q_3 = Mq_+/P_+$, which preserves the meaning of $\mathbf{L}$ in the sense of L-S [21], together with NP covariance. For any other internal 4-vector A_μ for the composite, a similar 3-vector $\mathbf{A}$ may be defined as $(\mathbf{A}_\perp, A_3)$, with $A_3 = MA_+/P_+$, via the condition $A\dot{P} = 0$. Examples of A_μ are polarization vectors, Dirac matrices, etc. Using these techniques, null-plane wave functions of the L-S type [21] have been constructed and applied to hadronic processes via quark loops [34]. A more formal mathematical basis for this prescription comes from MYTP [26] on the covariant null-plane [37]; see Sect.5 below.

5. 3D-4D BSE Under MYTP: Scalar/Fermion Quarks

We now come to our objectives (B) and (C), viz., 3D-4D interlinkage of BS amplitudes brought about by MYTP [26], and a unified treatment of $q\bar{q}$ [24, 37] and qqq [53] systems under MYTP conditions. The full calculational details with 4-fermion couplings via gluonic propagators have been collected in a recent review [32]. However, to bring out the basic mathematical structures, we shall derive the 3D-4D interconnection with spinless quarks for 2- and 3-quark systems in this and the next sections respectively. Further, we shall consider two MYTP methods in parallel for comparison: i) Covariant Instantaneity Ansatz (CIA) [24-25]; ii) Covariant LF/NP Ansatz (Cov LF) [37], to bring out the structural identity of the resulting BSE's for a $q\bar{q}$ system. This will be followed by a reconstruction of the 4D BS vertex functions for both types [24, 37] as basic building blocks for 4D quark loop calculations.

5.1. 3D-4D BSE Under CIA: Spinless Quarks

For a self-contained presentation, with unequal mass kinematics [24], let the quark constituents of masses $m_{1,2}$ and 4-momenta $p_{1,2}$ interact to produce a composite hadron of mass M and 4-momentum P_μ. The relative 4-momentum q_μ is related to these by

$$p_{1,2} = \hat{m}_{1,2}P \pm q; \quad P^2 = -M^2; \quad 2\hat{m}_{1,2} = 1 \pm (m_1^2 - m_2^2)/M^2. \tag{5. 1}$$

These Wightman-Garding definitions [52] of the fractional momenta $\hat{m}_{1,2}$ ensure that $q.P = 0$ on the mass shells $m_i^2 + p_i^2 = 0$ of the constituents, though not off-shell. Now define $\hat{q}_\mu = q_\mu - q.PP_\mu/P^2$ as the relative momentum *transverse* to the hadron 4-momentum P_μ which automatically gives $\hat{q}.P \equiv 0$, for all values of $\hat{q}_\mu$. If the BSE kernel K for the 2 quarks is a function of only these transverse relative momenta, viz. $K = K(\hat{q}, \hat{q}')$, this is called the "Cov. Inst.Ansatz (CIA)" [24] which accords with MYTP [26]. For two scalar quarks with inverse propagators $\Delta_{1,2}$, this ansatz gives rise to the following BSE for the wave fn $\Phi(q, P)$ [24]:

$$i(2\pi)^4\Delta_1\Delta_2\Phi(q,P) = \int d^4q' K(\hat{q},\hat{q}')\Phi(q',P); \quad \Delta_{1,2} = m_{1,2}^2 + p_{1,2}^2. \tag{5. 2}$$

The quantities $m_{1,2}$ are the 'constituent' masses which are strictly momentum dependent since they contain the mass function $m(p)$ [12,28], but may be regarded as constant for low energy phenomena: $m(p) \cong m(0)$. Further, under CIA, $m(p) = m(\hat{p})$, a momentum-dependence which is governed by the $DB\chi S$ condition [28] (see below).

To make a 3D reduction of eq.(5.2), define the 3D wave function $\phi(\hat{q})$ in terms of the longitudinal momentum $M\sigma$ as

$$\phi(\hat{q}) = \int M d\sigma \Phi(q,P); \quad M\sigma = Mq.P/P^2 \tag{5. 3}$$

using which, eq.(5.2) may be recast as

$$i(2\pi)^4\Delta_1\Delta_2\Phi(q,P) = \int d^3\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}'); \quad d^4q' \equiv d^3\hat{q}' M d\sigma'. \tag{5. 4}$$

Next, divide out by $\Delta_1\Delta_2$ in (5.4) and use once again (5.3) to reduce the 4D BSE form (5.4) to the 3D form

$$(2\pi)^3 D(\hat{q})\phi(\hat{q}) = \int d^3\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}'); \quad \frac{2i\pi}{D(\hat{q})} \equiv \int \frac{M d\sigma}{\Delta_1\Delta_2}. \tag{5. 5}$$

Here $D(\hat{q})$ is the 3D denominator function associated with the like wave function $\phi(\hat{q})$. The integration over $d\sigma$ is carried out by noting pole positions of $\Delta_{1,2}$ in the σ-plane, where

$$\Delta_{1,2} = {\omega_{1,2}}^2 - M^2(\hat{m}_{1,2} \pm \sigma)^2; \quad {\omega_{1,2}}^2 = m_{1,2}^2 + \hat{q}^2. \tag{5. 6}$$

The pole positions are given for $\Delta_{1,2} = 0$ respectively by

$$M(\sigma + \hat{m}_1) = \pm\omega_1 \mp i\epsilon; \quad M(\sigma - \hat{m}_2) = \pm\omega_2 \mp i\epsilon, \tag{5. 7}$$

where the $(\pm)$ indices refer to the lower/upper halves of the σ- plane. The final result for $D(\hat{q})$ is expressible symmetrically [24]:

$$D(\hat{q}) = M_\omega D_0(\hat{q}); \quad \frac{2}{M_\omega} = \frac{\hat{m}_1}{\omega_1} + \frac{\hat{m}_2}{\omega_2}, \tag{5. 8}$$

$$\frac{1}{2}D_0(\hat{q}) = \hat{q}^2 - \frac{\lambda(m_1^2, m_2^2, M^2)}{4M^2}; \quad \lambda = M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2. \tag{5. 9}$$

The crucial thing for MYTP[26] is now to observe the *equality* of the RHS of eqs (5.4) and (5.5), thus leading to an *exact interconnection* between the 3D and 4D BS wave functions [24]:

$$\Gamma(\hat{q}) \equiv \Delta_1\Delta_2\Phi(q,P) = \frac{D(\hat{q}\phi(\hat{q})}{2i\pi}. \tag{5. 10}$$

Eq.(5.10) determines the hadron-quark vertex function $\Gamma(\hat{q})$ as a product $D\phi$ of the 3D denominator and wave functions, satisfying a relativistic 3D Schroedinger-like equation (5.5). The simultaneous appearance of the 3D form (5.5) and the 4D form (5.4), leading to their interconnection (5.10), reveals a two-tier character: The 3D form (5.5) gives the basis for making contact with the 3D spectra [13], while the reconstructed 4D wave (vertex) function (5.10) in terms of 3D ingredients D and ϕ can be used for 4D quark-loop integrals in the standard Feynman fashion. Note that the vertex function $\Gamma = D\phi/(2i\pi)$ has a general structure, independent of the details of the input kernel K. Further, the D-function, eq.(5.8), is universal and well-defined off the mass shell of either quark. The 3D wave function ϕ is admittedly model- dependent, but together with $D(\hat{q})$, it controls the 3D spectra via (5.5), thus offering a direct experimental check on its structure. Both functions depend on the single 3D Lorentz-covariant quantity $\hat{q}^2$ whose most important property is its positive definiteness for time-lke hadron momenta ($M^2 > 0$).

5.2. Cov LF/NP for 3D-4D BSE: Fermion Quarks

As a preliminary to defining a 3D support to the BS kernel on the light-front (LF/NP), on the lines of CIA [24], a covariant LF/NP orientation [37] may be represented by the 4-vector n_μ, as well as its dual $\tilde{n}_\mu$, obeying the normalizations $n^2 = \tilde{n}^2 = 0$ and $n.\tilde{n} = 1$. In the standard NP scheme (in euclidean notation), these quantities are $n = (001; -i)/\sqrt{2}$ and $\tilde{n} = (001; i)/\sqrt{2}$, while the two other perpendicular directions are collectively denoted by the subscript $\perp$ on the concerned momenta. We shall try to maintain the n-dependence of various momenta to ensure explicit covariance; and to keep track of the usual NP notation $p_\pm = p_0 \pm p_3$, our covariant notation is normalized to the latter as $p_+ = n.p\sqrt{2}$; $p_- = -\tilde{n}.p\sqrt{2}$, while the perpendicular components continue to be denoted by $p_\perp$ in both notations.

In the same notation as for CIA [24], the 4th component of the relative momentum $q = \hat{m}_2 p_1 - \hat{m}_1 p_2$, that should be eliminated for obtaining a 3D equation, is now proportional to $q_n \equiv \tilde{n}.q$, as the NP analogue [37] of $P.qP/P^2$ in CIA [24], where $P = p_1 + p_2$ is the total 4-momentum of the hadron. However the quantity $q - q_n n$ is still only $q_\perp$, since its square is $q^2 - 2n.q\tilde{n}.q$, as befits $q_\perp^2$ (readily checked against the 'special' NP frame). We still need a third component p_3, for which the correct definition turns out to be [37] $q_{3\mu} = zP_n n_\mu$, where P_n=$P.\tilde{n}$ and $z = q.n/P.n$, which checks with $\hat{q}^2 = q_\perp^2 + z^2M^2$. We now collect the following definitions/results:

$$\begin{aligned} q_\perp &= q - q_n n; \quad \hat{q} = q_\perp + xP_n n; \quad x = q.n/P.n; \quad P^2 = -M^2; \\ q_n, P_n &= \tilde{n}.(q,P); \hat{q}.n = q.n; \quad \hat{q}.\tilde{n} = 0; \quad P_\perp.q_\perp = 0; \\ P.q &= P_n q.n + P.n q_n; \quad P.\hat{q} = P_n q.n; \quad \hat{q}^2 = q_\perp^2 + M^2x^2. \end{aligned} \tag{5. 11}$$

Now in analogy to CIA, the reduced 3D BSE (wave-fn ϕ) may be derived from the 4D BSE (5.2) for spinless quarks (wave-fn Φ) when its kernel K is *decreed* to be independent of the component q_n, i.e., $K = K(\hat{q}, \hat{q}')$, with $\hat{q} = (q_\perp, P_n n)$, in accordance with MYTP [26] condition imposed on the null-plane (NP), so that $d^4q = d^2q_\perp dq_3 dq_n$. Now define a 3D wave-fn $\phi(\hat{q}) = \int dq_n \Phi(q)$, as the CNPA counterpart of the CIA definition (5.3), and use this result on the RHS of (5.2) to give

$$i(2\pi)^4\Phi(q) = {\Delta_1}^{-1}{\Delta_2}^{-1}\int d^3\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}'), \tag{5. 12}$$

which is formally the same as eq.(5.4) for CIA above. Now integrate both sides of eq.(5.12) w.r.t. dq_n to give a 3D BSE in the variable $\hat{q}$:

$$(2\pi)^3 D_n(\hat{q})\phi(\hat{q}) = \int d^2q_\perp{}' dq_3{}' K(\hat{q}, \hat{q}')\phi(\hat{q}'), \tag{5. 13}$$

which again corresponds to the CIA eq.(5.5), except that the function $D_n(\hat{q})$ is now defined by

$$\int dq_n \Delta_1{}^{-1}\Delta_2{}^{-1} = 2\pi i D_n^{-1}(\hat{q}) \tag{5. 14}$$

and may be obtained by standard NP techniques [38] (Chaps 5-7) as follows. In the q_n plane, the poles of $\Delta_{1,2}$ lie on opposite sides of the real axis, so that only *one* pole will contribute at a time. Taking the Δ_2-pole, which gives

$$2q_n = -\sqrt{2}q_- = \frac{m_2^2 + (q_\perp - \hat{m}_2 P)^2}{\hat{m}_2 P.n - q.n} \tag{5. 15}$$

the residue of Δ_1 works out, after a routine simplification, to just $2P.q = 2P.nq_n + 2P_n q.n$, after using the collinearity condition $P_\perp.q_\perp = 0$ from (5.11). And when the value (5.15) of q_n is substituted in (5.14), one obtains (with $P_n P.n = -M^2/2$):

$$D_n(\hat{q}) = 2P.n(\hat{q}^2 - \frac{\lambda(M^2, m_1^2, m_2^2)}{4M^2}); \quad \hat{q}^2 = q_\perp^2 + M^2 x^2; \quad x = q.n/P.n\,. \tag{5. 16}$$

Now a comparison of (5.12) with (5.13) relates the 4D and 3D wave-fns:

$$2\pi i\Phi(q) = D_n(\hat{q})\Delta_1{}^{-1}\Delta_2{}^{-1}\phi(\hat{q}) \tag{5. 17}$$

as the Cov LF counterpart of (5.10) which is valid near the bound state pole. The BS vertex function now becomes $\Gamma = D_n \times \phi/(2\pi i)$. This result, though dependent on the LF/NP orientation, is nevertheless formally *covariant*, and closely corresponds to the pedagogical result of the old LF/NP formulation [38], with $D_n \Leftrightarrow D_+$.

A 3D equation similar to the covariant eq.(5.13) above, also obtains in alternative LF formulations such as in Kadychevsky-Karmanov [22b] (see their eq.(3.48)). However the *independent* 4-vector $\tilde{n}_\mu$ (which has no counterpart in [22b]), makes this a manifestly covariant 4D formulation without need for explicit Lorentz transformations [22b]. The 'angular condition' [21] is also trivially satisfied by the effective 3-vector $\hat{q}_\mu$ appearing in the 3D BSE (5.13). A more important contrast from other null-plane approaches is that the inverse process of *reconstruction* of the 4D hadron-quark vertex, eq.(5.17)), has no counterpart in them [22–23], as these are basically 3D oriented. *not vice versa.*

For fermion quarks with gluonic propagators, the MYTP formulation needs no new principles, except for certain technical details involving slight modifications [54] of the BSE structure for easier handling; see [32] for detailed steps. The full 4D wave function $\Psi(P, q)$ may be expressed as a 4x4 matrix [38, 32]:

$$\Psi(P, q) = S_F(p_1)\Gamma(\hat{q})\gamma_D S_F(-p_2); \quad \Gamma(\hat{q}) = N_H[1; P_n/M]D(\hat{q})\phi(\hat{q})/2i\pi, \tag{5. 18}$$

where γ_D is a Dirac matrix which equals γ_5 for a P-meson, $i\gamma_\mu$ for a V-meson, $i\gamma_\mu\gamma_5$ for an A-meson, etc. The factors in square brackets stand for CIA and Cov LF values respectively. N_H represents the hadron normalization.

6. The *qqq* BSE: 3D-4D Interlinkage

We now come to the aspect of MYTP [26] that governs the inter-relation of 3D and 4D Bethe-Salpeter amplitudes for qqq-systems [53], in keeping with a perceived 'duality' between meson ($q\bar{q}$) and baryon (qqq) systems which necessitates a parallel treatment between them. In this respect a fairly comprehensive review of baryon dynamics as a 3-body relativistic system with full permutation symmetries in all relevant degrees of freedom [55] has been given recently [32]. These include: A detailed correspondence [56] between qqq and quark-diquark wave functions; Complex HO techniques for the qqq problem [57]; fermionic qqq BSE with the same gluon propagator for pair qq interactions [29] as employed for $q\bar{q}$ systems [28], except for reduction by half due to the color factor; and Green's function methods for 3D reduction of the 4D BSE form, plus reconstruction of the 4D qqq wave function [53], on the lines of the $q\bar{q}$ problem [24]. Within the formalistic scope of this Article however, we shall merely dwell on the last item, viz., Green's fn techniques [53] for a 3D reduction of the 4D BSE, plus *reconstruction* of the 4D wave function, for a qqq system for three identical spinless quarks, keeping in forefront the issue of *connectedness* [58] in a 3-particle amplitude whose signal is the *absence of any δ-function* in its structure; (for a detailed perspective, see [32]).

6.1. Two-Quark Green's Function Under CIA

As a warm up to the method of Green's functions (G-fns), we first derive the 3D-4D interconnection for the corresponding G-fns for 2-particle scattering of two identical spinless particles, before moving on to the 3-body problem in the next 2 subsections. For simplicity we shall consider the G-fns near the bound state poles, so that the inhomogeneous terms may be dropped. In the notation and phase convention of Section 5, the 4D qq Green's fn $G(p_1p_2;{p_1}'{p_2}')$ near a *bound* state satisfies a 4D BSE (no inhomogeneous term):

$$i(2\pi)^4G(p_1p_2;{p_1}'{p_2}') = {\Delta_1}^{-1}{\Delta_2}^{-1}\int dp_1''dp_2''K(p_1p_2;p_1''p_2'')G(p_1''p_2'';{p_1}'{p_2}'), \tag{6.1.1}$$

where

$$\Delta_1 = {p_1}^2 + {m_q}^2, \tag{6.1.2}$$

and m_q is the mass of each quark. Now using the relative 4- momentum $q = (p_1 - p_2)/2$ and total 4-momentum $P = p_1 + p_2$ (similarly for the other sets), and removing a δ-function for overall 4-momentum conservation, from each of the G- and K- functions, eq.(6.1.1) reduces to the simpler form

$$i(2\pi)^4G(q.q') = {\Delta_1}^{-1}{\Delta_2}^{-1}\int d\hat{q}''Md\sigma''K(\hat{q},\hat{q}'')G(q'',q'), \tag{6.1.3}$$

where $\hat{q}_\mu = q_\mu - \sigma P_\mu$, with $\sigma = (q.P)/P^2$, is effectively 3D in content (being orthogonal to P_μ). Here we have incorporated the ansatz of a 3D support for the kernel K (independent of σ and σ'), and broken up the 4D measure dq'' arising from (6.1.1) into the product $d\hat{q}''Md\sigma''$ of a 3D and a 1D measure respectively. We have also suppressed the 4-momentum P_μ label, with ($P^2 = -M^2$), in the notation for $G(q.q')$.

Now define the fully 3D Green's function $\hat{G}(\hat{q},\hat{q}')$ as [53]

$$\hat{G}(\hat{q},\hat{q}') = \int\int M^2 d\sigma d\sigma' G(q,q') \tag{6.1.4}$$

and two (hybrid) 3D-4D Green's functions $\tilde{G}(\hat{q},q')$, $\tilde{G}(q,\hat{q}')$ as

$$\tilde{G}(\hat{q},q') = \int Md\sigma G(q,q'); \quad \tilde{G}(q,\hat{q}') = \int Md\sigma' G(q,q'). \tag{6.1.5}$$

Next, use (6.1.5) in (6.1.3) to give

$$i(2\pi)^4\tilde{G}(q,\hat{q}') = {\Delta_1}^{-1}{\Delta_2}^{-1}\int dq'' K(\hat{q},\hat{q}'')\tilde{G}(q'',\hat{q}'). \tag{6.1.6}$$

Now integrate both sides of (6.1.3) w.r.t. $Md\sigma$ and use the result

$$\int Md\sigma {\Delta_1}^{-1}{\Delta_2}^{-1} = 2\pi i D^{-1}(\hat{q}); \quad D(\hat{q}) = 4\hat{\omega}(\hat{\omega}^2 - M^2/4); \quad \hat{\omega}^2 = {m_q}^2 + \hat{q}^2 \tag{6.1.7}$$

to give a 3D BSE w.r.t. the variable $\hat{q}$, while keeping the other variable q' in a 4D form:

$$(2\pi)^3\tilde{G}(\hat{q},q') = D^{-1}\int d\hat{q}'' K(\hat{q},\hat{q}'')\tilde{G}(\hat{q}'',q'). \tag{6.1.8}$$

A comparison of (6.1.3) with (6.1.8) gives the desired connection between the full 4D G-function and the hybrid $\tilde{G}(\hat{q},q')$-function:

$$2\pi i G(q,q') = D(\hat{q}){\Delta_1}^{-1}{\Delta_2}^{-1}\tilde{G}(\hat{q},q'). \tag{6.1.9}$$

Again, the symmetry of the left hand side of (6.1.9) w.r.t. q and q' allows rewriting the right hand side with the roles of q and q' interchanged. This gives the dual form

$$2\pi i G(q,q') = D(\hat{q}'){\Delta_1'}^{-1}{\Delta_2'}^{-1}\tilde{G}(q,\hat{q}'), \tag{6.1.10}$$

which on integrating both sides w.r.t. $Md\sigma$ gives

$$2\pi i\tilde{G}(\hat{q},q') = D(\hat{q}'){\Delta_1'}^{-1}{\Delta_2'}^{-1}\hat{G}(\hat{q},\hat{q}'). \tag{6.1.11}$$

Substitution of (6.1.11) in (6.1.9) then gives the symmetrical form

$$(2\pi i)^2 G(q,q') = D(\hat{q}){\Delta_1}^{-1}{\Delta_2}^{-1}\hat{G}(\hat{q},\hat{q}')D(\hat{q}'){\Delta_1'}^{-1}{\Delta_2'}^{-1}. \tag{6.1.12}$$

Finally, integrating both sides of (6.1.8) w.r.t. $Md\sigma'$, we obtain a fully reduced 3D BSE for the 3D Green's function:

$$(2\pi)^3\hat{G}(\hat{q},\hat{q}') = D^{-1}(\hat{q}\int d\hat{q}'' K(\hat{q},\hat{q}'')\hat{G}(\hat{q}'',\hat{q}'). \tag{6.1.13}$$

Eq.(6.1.12) which is valid near the bound state pole, expresses the desired connection between the 3D and 4D forms of the Green's functions; and eq(6.1.13) is the determining equation for the 3D form. A spectral analysis can now be made for either of the 3D or 4D Green's functions in the standard manner, viz.,

$$G(q,q') = \sum_n \Phi_n(q;P)\Phi_n^*(q';P)/(P^2+M^2), \tag{6.1.14}$$

where Φ is the 4D BS wave function. A similar expansion holds for the 3D G-function $\hat{G}$ in terms of $\phi_n(\hat{q})$. Substituting these expansions in (6.1.12), one immediately sees the connection between the 3D and 4D wave functions in the form:

$$2\pi i\Phi(q,P) = {\Delta_1}^{-1}{\Delta_2}^{-1}D(\hat{q})\phi(\hat{q}), \tag{6.1.15}$$

whence the BS vertex function becomes $\Gamma = D \times \phi/(2\pi i)$ as found in [24]. We shall make free use of these results, taken as qq subsystems, for our study of the qqq G-functions in subsects. 6.2-3.

6.2. 3D BSE Reduction for *qqq* G-fn

As in the two-body case, and in an obvious notation for various 4-momenta (without the Greek suffixes), we consider the most general Green's function $G(p_1p_2p_3;p_1'p_2'p_3')$ for 3-quark scattering *near the bound state pole* (for simplicity) which allows us to drop the various inhomogeneous terms from the beginning. Again we take out an overall delta function $\delta(p_1+p_2+p_3-P)$ from the G-function and work with *two* internal 4-momenta for each of the initial and final states defined as follows [54b]:

$$\sqrt{3}\xi_3 = p_1 - p_2\ ; \quad 3\eta_3 = -2p_3 + p_1 + p_2, \tag{6.2.1}$$

$$P = p_1 + p_2 + p_3 = p_1' + p_2' + p_3' \tag{6.2.2}$$

and two other sets ξ_1, η_1 and ξ_2, η_2 defined by cyclic permutations from (6.2.1). Further, as we shall consider pairwise kernels with 3D support, we define the effectively 3D momenta $\hat{p}_i$, as well as the three (cyclic) sets of internal momenta $\hat{\xi}_i, \hat{\eta}_i$, (i = 1,2,3) by [54b]:

$$\hat{p}_i = p_i - \nu_i P\ ; \quad \hat{\xi}_i = \xi_i - s_i P\ ; \quad \hat{\eta}_i - t_i P, \tag{6.2.3}$$

$$\nu_i = (P.p_i)/P^2\ ; \quad s_i = (P.\xi_i)/P^2\ ; \quad t_i = (P.\eta_i)/P^2, \tag{6.2.4}$$

$$\sqrt{3}s_3 = \nu_1 - \nu_2\ ; \quad 3t_3 = -2\nu_3 + \nu_1 + \nu_2 \quad (+\text{cyclicpermutations}). \tag{6.2.5}$$

The space-like momenta $\hat{p}_i$ and the time-like ones ν_i satisfy [54b]

$$\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 0\ ; \quad \nu_1 + \nu_2 + \nu_3 = 1. \tag{6.2.6}$$

Strictly speaking, in the spirit of covariant instantaneity, we should have taken the relative 3D momenta $\hat{\xi}, \hat{\eta}$ to be in the instantaneous frames of the concerned pairs, i.e., w.r.t. the rest frames of $P_{ij} = p_i + p_j$; however the difference between the rest frames of P and P_{ij} is small and calculable [54b], while the use of a common 3-body rest frame ($P = 0$) lends considerable simplicity and elegance to the formalism.

We may now use the foregoing considerations to write down the BSE for the 6-point Green's function in terms of relative momenta, on closely parallel lines to the 2-body case. To that end note that the 2-body relative momenta are $q_{ij} = (p_i - p_j)/2 = \sqrt{3}\xi_k/2$, where (ijk) are cyclic permutations of (123). Then for the reduced *qqq* Green's function, when the *last* interaction was in the (ij) pair, we may use the notation $G(\xi_k\eta_k;\xi_k'\eta_k')$, together with 'hat' notations on these 4-momenta when the corresponding time-like components are integrated out. Further, since the pair ξ_k, η_k is *permutation invariant* as a whole, we may choose to drop the index notation from the complete G-function to emphasize this symmetry as and when needed. The G-function for the *qqq* system satisfies, in the neighbourhood of the bound state pole, the following (homogeneous) 4D BSE for pairwise *qq* kernels with 3D support:

$$i(2\pi)^4 G(\xi\eta;\xi'\eta') = \sum_{123} {\Delta_1}^{-1}{\Delta_2}^{-1} \int d\hat{q}_{12}'' M d\sigma_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') G({\xi_3}''{\eta_3}''; {\xi_3}'{\eta_3}'), \tag{6.2.7}$$

where we have employed a mixed notation (q_{12} versus ξ_3) to stress the two-body nature of the interaction with one spectator at a time, in a normalization directly comparable with eq.(6.1.3) for the corresponding two-body problem. Note also the connections

$$\sigma_{12} = \sqrt{3}s_3/2; \quad \hat{q}_{12} = \sqrt{3}\hat{\xi}_3/2; \quad \hat{\eta}_3 = -\hat{p}_3, \quad etc. \tag{6.2.8}$$

The next task is to reduce the 4D BSE (6.2.7) to a fully 3D form through a sequence of integrations w.r.t. the time-like momenta s_i, t_i applied to the different terms on the right hand side, *provided both* variables are simultaneously permuted. We now define the following fully 3D as well as mixed (hybrid) 3D-4D G-functions according as one or more of the time-like ξ, η variables are integrated out:

$$\hat{G}(\hat{\xi}\hat{\eta}; \hat{\xi}'\hat{\eta}') = \int\int\int\int ds dt ds' dt' G(\xi\eta; \xi'\eta'), \tag{6.2.9}$$

which is S_3-symmetric.

$$\tilde{G}_{3\eta}(\xi\hat{\eta}; \xi'\hat{\eta}') = \int\int dt_3 dt_3{}' G(\xi\eta; \xi'\eta'); \tag{6.2.10}$$

$$\tilde{G}_{3\xi}(\hat{\xi}\eta; \hat{\xi}'\eta') = \int\int ds_3 ds_3{}' G(\xi\eta; \xi'\eta'). \tag{6.2.11}$$

The last two equations are however *not* symmetric w.r.t. the permutation group S_3, since both the variables ξ, η are not simultaneously transformed; this fact has been indicated in eqs.(8.2.10-11) by the suffix "3" on the corresponding (hybrid) $\tilde{G}$-functions, to emphasize that the 'asymmetry' is w.r.t. the index "3". We shall term such quantities "S_3-indexed", to distinguish them from S_3-symmetric quantities as in eq.(6.2.9). The full 3D BSE for the $\hat{G}$-function is obtained by integrating out both sides of (6.2.7) w.r.t. the *st*-pair variables $ds_i ds_j{}' dt_i dt_j{}'$ (giving rise to an S_3-symmetric quantity), and using (6.2.9) together with (6.2.8) as follows:

$$(2\pi)^3 \hat{G}(\hat{\xi}\hat{\eta}; \hat{\xi}'\hat{\eta}') = \sum_{123} D^{-1}(\hat{q}_{12}) \int d\hat{q}_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') \hat{G}(\hat{\xi}''\hat{\eta}''; \hat{\xi}'\hat{\eta}'). \tag{6.2.12}$$

This integral equation for $\hat{G}$ which is the 3-body counterpart of (6.1.13) for a qq system in the neighbourhood of the bound state pole, is the desired 3D BSE for the qqq system in a *fully connected* form, i.e., free from delta functions. Now using a spectral decomposition for $\hat{G}$

$$\hat{G}(\hat{\xi}\hat{\eta}; \hat{\xi}'\hat{\eta}') = \sum_n \phi_n(\hat{\xi}\hat{\eta}; P)\phi_n^*(\hat{\xi}'\hat{\eta}'; P)/(P^2 + M^2) \tag{6.2.13}$$

on both sides of (6.2.12) and equating the residues near a given pole $P^2 = -M^2$, gives the desired equation for the 3D wave function ϕ for the bound state in the connected form:

$$(2\pi)^3 \phi(\hat{\xi}\hat{\eta}; P) = \sum_{123} D^{-1}(\hat{q}_{12}) \int d\hat{q}_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') \phi(\hat{\xi}''\hat{\eta}''; P). \tag{6.2.14}$$

Now the S_3-symmetry of ϕ in the $(\hat{\xi}_i, \hat{\eta}_i)$ pair is a very useful result for both the solution of (6.2.14) *and* for the reconstruction of the 4D BS wave function in terms of the 3D wave function (6.2.14), as is done in the subsect.6.3 below.

6.3. Reconstruction of 4D qqq Wave Function

We now attempt to *re-express* the 4D G-function given by (6.2.7) in terms of the 3D $\hat{G}$-function given by (6.2.12), as the qqq counterpart of the qq results (6.1.12-13). To that end we adapt the result (6.1.12) to the hybrid Green's function of the (12) subsystem given by $\tilde{G}_{3\eta}$, eq.(6.2.10), in which the 3-momenta $\hat{\eta}_3, \hat{\eta}_3'$ play a parametric role reflecting the spectator status of quark #3, while the *active* roles are played by $q_{12}, q_{12}{}' = \sqrt{3}(\xi_3, \xi_3{}')/2$, for which the analysis of subsect.6.1 applies directly. This gives

$$(2\pi i)^2 \tilde{G}_{3\eta}(\xi_3\hat{\eta}_3; \xi_3{}'\hat{\eta}_3') = D(\hat{q}_{12})\Delta_1{}^{-1}\Delta_2{}^{-1}\hat{G}(\hat{\xi}_3\hat{\eta}_3; \hat{\xi}_3'\hat{\eta}_3')D(\hat{q}_{12}')\Delta_1{}'^{-1}\Delta_2{}'^{-1}, \tag{6.3.1}$$

where on the right hand side, the 'hatted' G-function has full S_3-symmetry, although (for purposes of book-keeping) we have not shown this fact explicitly by deleting the suffix '3' from its arguments. A second relation of this kind may be obtained from (6.2.7) by noting that the 3 terms on its right hand side may be expressed in terms of the hybrid $\tilde{G}_{3\xi}$ functions vide their definitions (6.2.11), together with the 2-body interconnection between $(\xi_3, \xi_3{}')$ and $(\hat{\xi}_3, \hat{\xi}_3')$ expressed once again via (6.3.1), but without the 'hats' on η_3 and $\eta_3{}'$. This gives

$$
\begin{aligned}
(\sqrt{3}\pi i)^2 G(\xi_3\eta_3;\xi_3{}'\eta_3{}') &= (\sqrt{3}\pi i)^2 G(\xi\eta;\xi'\eta') \\
&= \sum_{123} \Delta_1{}^{-1}\Delta_2{}^{-1}(\pi i\sqrt{3}) \int d\hat{q}_{12}'' M d\sigma_{12}{}'' K(\hat{q}_{12}, \hat{q}_{12}'') G(\xi_3{}''\eta_3{}''; \xi_3{}'\eta_3{}') \\
&= \sum_{123} D(\hat{q}_{12})\Delta_1{}^{-1}\Delta_2{}^{-1}\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3;\hat{\xi}_3'\eta_3{}')\Delta_1{}'^{-1}\Delta_2{}'^{-1}, \qquad (6.3.2)
\end{aligned}
$$

where the second form exploits the symmetry between ξ, η and ξ', η'.

At this stage, unlike the 2-body case, the reconstruction of the 4D Green's function is *not yet* complete for the 3-body case, as eq.(6.3.2) clearly shows. This is due to the *truncation* of Hilbert space implied in the ansatz of 3D support to the pairwise BSE kernel K which, while facilitating a 4D to 3D BSE reduction without extra charge, does *not* have the *complete* information to permit the *reverse* transition (3D to 4D) without additional assumptions. To fill up this gap in this "inverse" mathematical problem, we look for a suitable ansatz for $\tilde{G}_{3\xi}$ on the RHS of (6.3.2) in terms of *known* quantities, so that the reconstructed 4D G-function satisfies the 3D equation (6.2.12) exactly, as a check-point. We therefore seek a structure of the form

$$
\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3;\hat{\xi}_3'\eta_3{}') = \hat{G}(\hat{\xi}_3\hat{\eta}_3;\hat{\xi}_3'\hat{\eta}_3') \times F(p_3, p_3{}'), \qquad (6.3.3)
$$

where the unknown function F must involve only the momentum of the spectator quark #3. A part of the $\eta_3, \eta_3{}'$ dependence has been absorbed in the $\hat{G}$ function on the right, so as to satisfy the requirements of S_3-symmetry for this 3D quantity [53].

As to the remaining factor F, it is necessary to choose its form in a careful manner so as to conform to the conservation of 4-momentum for the *free* propagation of the spectator between two neighbouring vertices, consistently with the symmetry between p_3 and $p_3{}'$. A possible choice consistent with these conditions is:

$$
F(p_3, p_3{}') = C_3\Delta_3{}^{-1}\delta(\nu_3 - \nu_3{}'). \qquad (6.3.4)
$$

Here $\Delta_3{}^{-1}$ represents the "free" propagation of quark #3 between successive vertices, while C_3 represents some residual effects which may at most depend on the 3-momentum $\hat{p}_3$, but must satisfy the main constraint that the 3D BSE, (6.2.12), be *explicitly* satisfied.

To check the self-consistency of the ansatz (6.3.4), integrate both sides of (6.3.2) w.r.t. $ds_3 ds_3{}' dt_3 dt_3{}'$ to recover the 3D S_3-invariant $\hat{G}$-function on the left hand side. Next, in the first form on the right hand side, integrate w.r.t. $ds_3 ds_3{}'$ on the G-function which alone involves these variables. This yields the quantity $\tilde{G}_{3\xi}$. At this stage, employ the ansatz (6.3.4) to integrate over $dt_3 dt_3{}'$. Consistency with the 3D BSE, eq.(6.2.12), now demands

$$
C_3 \int\int d\nu_3 d\nu_3{}' \Delta_3{}^{-1}\delta(\nu_3 - \nu_3{}') = 1; (since dt = d\nu). \qquad (6.3.5)
$$

The 1D integration w.r.t. $d\nu_3$ may be evaluated as a contour integral over the propagator Δ^{-1} , which gives the pole at $\nu_3 = \hat{\omega}_3/M$, (see below for its definition). Evaluating the residue then gives

$$
C_3 = i\pi/(M\hat{\omega}_3); \quad \hat{\omega}_3^2 = m_q{}^2 + \hat{p}_3^2, \qquad (6.3.6)
$$

which will reproduce the 3D BSE, eq.(6.2.12), *exactly*! Substitution of (6.3.4) in the second form of (6.3.2) finally gives the desired 3-body generalization of (6.1.12) in the form

$$
3G(\xi\eta;\xi'\eta') = \sum_{123} D(\hat{q}_{12})\Delta_{1F}\Delta_{2F}D(\hat{q}_{12}')\Delta_{1F}{}'\Delta_{2F}{}'\hat{G}(\hat{\xi}_3\hat{\eta}_3;\hat{\xi}_3'\hat{\eta}_3')[\Delta_{3F}/(M\pi\hat{\omega}_3)], \qquad (6.3.7)
$$

where for each index, $\Delta_F = -i\Delta^{-1}$ is the Feynman propagator. To find the effect of the ansatz (6.3.4) on the 4D BS wave function $\Phi(\xi\eta; P)$, we do a spectral reduction like (6.2.13) for the 4D Green's function G on the LHS of (6.3.2). Equating the residues on both sides gives the desired 4D-3D connection between Φ and ϕ:

$$\Phi(\xi\eta; P) = \sum_{123} D(\hat{q}_{12}){\Delta_1}^{-1}{\Delta_2}^{-1}\phi(\hat{\xi}\hat{\eta}; P) \times \sqrt{\frac{\delta(\nu_3 - \hat{\omega}_3/M)}{M\hat{\omega}_3\Delta_3}} \tag{6.3.8}$$

defines the 4D wave fn in terms of piecewise vertex fns V_i, as

$$\Phi(p_1p_2p_3) \equiv \frac{V_1 + V_2 + V_3}{\Delta_1\Delta_2\Delta_3}. \tag{6.3.9}$$

From (6.3.8-9), we infer the baryon-qqq vertex function V_3 corresponding to the 'last' interaction in the 12-pair as

$$V_3 = D(\hat{q}_{12}\phi(\hat{\xi}, \hat{\eta}) \times \sqrt{2\Delta_3\delta({\nu_3}^2M^2 - \hat{\omega}_3^2)} \tag{6.3.10}$$

and so on cyclically. (The argument of the δ-function inside the radical for V_3 simplifies to $p_3^2 + m_q^2$). This expression had been obtained earlier from intuitive considerations [54b].

To account for the appearance of the 1D δ-fn under radical in (6.3.10), it is explained elsewhere [53] that it has nothing to do with connectedness [58] as such, but merely reflects a 'dimensional mismatch' due to the 3D nature of the pairwise kernel K [24] imbedded in a 4D Hilbert space. (For a physical explanation, see [53]). A further self-consistency check on (6.3.10), is found by taking the limit of a point interaction, which amounts to setting $K = Constant$, when the radical (expectedly) disappears, and gives a Lorentz-invariant result [53], in agreement with the so-called NJL-Faddeev (contact) model [59] for 3-particle scattering. For the fermion qqq case with pairwise gluonic interactions, the details may be found in [60], wherein the strength of the 'color' qq interaction [29] is half of that of $q\bar{q}$ [28]. For brevity, we skip the MYTP [26] derivation of the 4D qqq vertex function under Cov LF [37] conditions, which parallels that for the 2-body case [Sect.5], except for the remark that the old-fashioned LF/NP treatment [38] gives the same results as the more formal Cov LF treatment in Sect.5, so that a similar Cov LF form for qqq dynamics should be expected [38], with $D(\hat{q}) \to D_n(\hat{q})$, etc in (6.3.10).

7. Triangle Loops Under MYTP On Cov LF/NP

In this Section, we shall illustrate the MYTP techniques on the covariant light-front to bring out the main feature, viz., structure of the triangle loop integrals free from the anomalies of time-like momenta in the product of gaussian vertex functions, such as complexities in the pion form factor [36] (see Sects. 1 and 5). To that end, we shall mainly consider the mathematical structure of the P-meson form factor, followed by a brief stetch of the structure of 3-hadron form factors, in the next few subsections, leaving routine calculational details to [37, 32].

7.1. Pion Form Factor by Cov LF/NP Method

Using Fig. 1 above, and an identical one with $1 \to 2$, (c.f., figs. 1a,1b of [34b]), the Feynman amplitude for the $h \to h' + \gamma$ transition is given by [34b]

$$2\bar{P}_\mu F(k^2) = 4(2\pi)^4 N_n(P)N_n(P')e\hat{m}_1 \int d^4T_\mu^{(1)} \frac{D_n(\hat{q})\phi(\hat{q})D_n(\hat{q}')\phi(\hat{q}')}{\Delta_1\Delta_1'\Delta_2} + [1 \Rightarrow 2]; \tag{7.1}$$

$$4T_\mu^{(1)} = Tr[\gamma_5(m_1 - i\gamma.p_1)i\gamma_\mu(m_1 - i\gamma.p_1')\gamma_5(m_2 + i\gamma.p_2)]; \quad \Delta_i = m_i^2 + p_i^2; \tag{7.2}$$

$$p_{1,2} = \hat{m}_{1,2}P \pm q; \quad p_{1,2}' = \hat{m}_{1,2}P' \pm q' \quad p_2 = p_2'; \quad P - P' = p_1 - p_1' = k; \quad 2\bar{P} = P + P'. \tag{7.3}$$

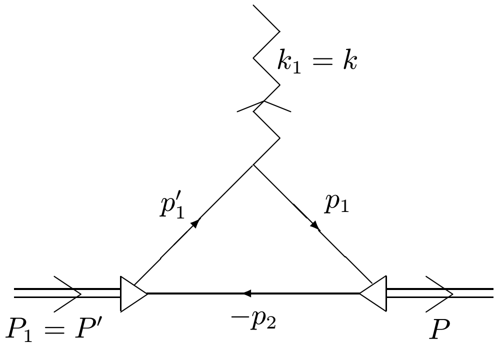

Figure 1: *Triangle loop for e.m. vertex.*

After evaluating the traces and simplifying, T_μ becomes

$$T^{(1)}_\mu = (p_{2\mu} - \bar{P}_\mu)[\delta m^2 - M^2 - \Delta_2] - k^2 p_{2\mu}/2 + (\Delta_1 - \Delta'_1)k_\mu/4. \tag{7.4}$$

The last term in (7.4) is non-gauge invariant, but it does not survive the integration in (7.1), since the coefficient of k_μ, viz., $\Delta_1 - \Delta'_1$ is antisymmetric in p_1 and p'_1, while the rest of the integrand in (7.1) is symmetric in these two variables. Next, to bring out the proportionality of the integral (7.1) to $\bar{P}_\mu$, it is necessary to resolve p_2 into the mutually perpendicular components $p_{2\perp}$, $(p_2.k/k^2)k$ and $(p_2.\bar{P}/\bar{P}^2)\bar{P}$, of which the first two will again not survive the integration, the first due to the angular integration, and the second due to the antisymmetry of $k = p_1 - p'_1$ in p_1 and p'_1, just as in the last term of (7.4). The third term is explicitly proportional to $\bar{P}_\mu$, and is of course gauge invariant since $\bar{P}.k = 0$. (This fact had been anticipated while writing the LHS of (7.4)). Now with the help of the results

$$p_2.\bar{P} = -\hat{m}_2 M^2 - \Delta_1/4 - \Delta'_1/4; \quad 2\hat{m}_2 = 1 - (m_1^2 - m_2^2)/M^2; \quad \bar{P}^2 = -M^2 - k^2/4, \tag{7.5}$$

it is a simple matter to integrate (7.1), on the lines of Sec.5, noting that terms proportional to $\Delta_1\Delta_2$ and $\Delta'_1\Delta_2$ will give zero, while the non-vanishing terms will get contributions only from the residues of the Δ_2-pole, eq.(5.15). Before collecting the various pieces, note that the 3D gaussian wave functions ϕ, ϕ', as well as the 3D denominator functions D_n, D'_n, do *not* depend on the time-like components p_{2n}, so that no further pole contributions accrue from these sources. (It is this problem of time-like components of the internal 4-momenta inside the gaussian ϕ-functions under the CIA approach [24], that had plagued a earlier CIA study of triangle diagrams [36]). To proceed further, it is now convenient to define the quantity $\bar{q}.n = p_2.n - \hat{m}_2\bar{P}.n$ to simplify the ϕ- and D_n- functions. To that end define the symbols:

$$(q, q') = \bar{q} \pm \hat{m}_2 k/2; \quad z_2 = \bar{q}.n/\bar{P}.n; \quad \hat{k} = k.n/\bar{P}.n; \quad (\theta_k, \eta_k) = 1 \pm \hat{k}^2/4 \tag{7.6}$$

and note the following results of pole integration w.r.t. p_{2n} [38]:

$$\int dp_{2n} \frac{1}{\Delta_2}[1/\Delta_1; 1/\Delta'_1; 1/(\Delta_1\Delta'_1)] = [1/D_n; 1/D'_n; 2p_2.n/(D_n D'_n)]. \tag{7.7}$$

Details of further calculation of the form factor are given in [37]. An essential result is the normalizer $N_n(P)$ of the hadron, obtained by setting $k_\mu = 0$, and demanding that $F(0) = 1$. The reduced (Lorentz-invariant) normalizer $N_H = N_n(P)P.n/M$ is given by [32, 37]:

$$N_H^{-2} = 2M(2\pi)^3 \int d^3\hat{q} e^{-\hat{q}^2/\beta^2}[(1 + \delta m^2/M^2)(\hat{q}^2 - \lambda/4M^2) + 2\hat{m}_1\hat{m}_2(M^2 - \delta m^2)], \tag{7.8}$$

where the internal momentum $\hat{q} = (q_\perp, Mz_2)$ is formally a 3-vector, in conformity with the 'angular condition' [21]. The corresponding expression for the form factor is [32, 37]:

$$F(k^2) = 2MN_H^2(2\pi)^3 exp[-(M\hat{m}_2\hat{k}/\beta)^2/4\theta_k](\pi\beta^2)^{3/2}\frac{\eta_k}{\sqrt{\theta_k}}\hat{m}_1 G(\hat{k}) + [1 \Rightarrow 2], \tag{7.9}$$

where $G(\hat{k})$ is a function of $\hat{k}$; see eqs.(A.12-13) of [32].

7.2. 'Lorentz Completion' for $F(k^2)$

The expression (7.9) for $F(k^2)$ still depends on the null-plane orientation n_μ via the dimensionless quantity $\hat{k} = k.n/P.n$ which while having simple Lorentz transformation properties, is nevertheless *not* Lorentz invariant by itself. To make it explicitly Lorentz invariant, we shall employ a simple method of 'Lorentz completion' which is merely an extension of the 'collinearity trick' empolyed at the quark level, viz., $P_\perp.q_\perp = 0$; see eq.(5.11). Note that this collinearity ansatz has already become reduntant at the level of the Normalizer N_H, eq.(7.8), which owes its Lorentz invariance to the integrating out of the null-plane dependent quantity z_2 in (7.8). This is of course because N_H depends only on one 4-momentum (that of a *single hadron*), so that the collinearity assumption is exactly valid. However the form factor $F(k^2)$ depends on *two independent* 4-momenta P, P', for which the collinearity assumption is non-trivial, since the existence of the perpendicular components cannot be wished away! Actually the quark-level assumption $P_\perp.q_\perp = 0$ has, so to say, got transferred, via the $\hat{q}$-integration in eq.(7.9), to the *hadron level*, as evidenced from the $\hat{k}$-dependence of $F(k^2)$; therefore an obvious logical inference is to suppose this $\hat{k}$-dependence to be the result of the collinearity ansatz $P_\perp.P'_\perp = 0$ at the hadron level. Now, under the collinearity condition, one has

$$P.P' = P_\perp.P'_\perp + P.nP'.\tilde{n} + P'.nP.\tilde{n} = P.nP'_n + P'.nP_n; \quad P.\tilde{n} \equiv P_n. \tag{7.10}$$

Therefore 'Lorentz completion'(the opposite of the collinearity ansatz) merely amounts to reversing the direction of the above equation by supplying the (zero term) $P_\perp.P'_\perp$ to a 3-scalar product to render it a 4-scalar! Indeed the process is quite unique for 3-point functions such as the form factor under study, although for more involved cases (e.g., 4-point functions), further assumptions may be needed.

In the present case, the prescription of Lorentz completion is relatively simple, being already contained in eq.(7.10). Thus since $P, P' = \bar{P} \pm k/2$, a simple application of (7.10) gives

$$\begin{aligned} k.nk_n &= +k^2; \quad \bar{P}.n\bar{P}_n = -M^2 - k^2/4; \\ \hat{k}^2 &= \frac{4k^2}{4M^2 + k^2} = 4\theta_k - 4 = 4 - 4\eta_k. \end{aligned} \tag{7.11}$$

This simple prescription for $\hat{k}$ automatically ensures the 4D (Lorentz) invariance of $F(k^2)$ at the hadron level. (For comparison with alternative methods [22b], see [37]).

7.3. QED Gauge Corrections to $F(k^2)$

While the 'kinematic' gauge invariance of $F(k^2)$ has already been ensured in Sec.7.1 above, there are additional contributions to the triangle loops - figs.1a and 1b of [34b] - obtained by inserting the photon lines at each of the two vertex blobs instead of on the quark lines themselves. These terms arise from the demands of QED gauge invariance, as pointed out by Kisslinger and Li [61] in the context of two-point functions, and are simulated by inserting exponential phase integrals with the e.m. currents. However, this method (which works ideally for *point* interactions) is not amenable to *extended* (momentum-dependent) vertex functions, and an alternative strategy is needed, as described below.

The way to an effective QED gauge invariance lies in the simple-minded substitution $p_i - e_i A(x_i)$ for each 4-momentum p_i (in a mixed p, x representation) occurring in the structure of the vertex function. This amounts to replacing each $\hat{q}_\mu$ occurring in $\Gamma(\hat{q}) = D(\hat{q})\phi(\hat{q})$, by $\hat{q}_\mu - e_q \hat{A}_\mu$, where $e_q = \hat{m}_2 e_1 - \hat{m}_1 e_2$, and keeping only first order terms in A_μ after due expansion. Now the first order correction to $\hat{q}^2$ is $-e_q \hat{q}.\hat{A} - e_q \hat{A}.\hat{q}$, which simplifies on substitution from eq.(7.11) to

$$- 2e_q \tilde{q}.A \equiv -2e_q A_\mu [\hat{q}_\mu - \hat{q}.n\tilde{n}_\mu + P.\tilde{n}\hat{q}.nn_\mu / P.n]. \tag{7.12}$$

The net result is a first order correction to $\Gamma(\hat{q})$ of amount $e_q j(\hat{q}).A$ where

$$j(\hat{q})_\mu = -4M_> \tilde{q}_\mu \phi(\hat{q})(1 - (\hat{q}^2 - \lambda/4M^2)/2\beta^2). \tag{7.13}$$

The contribution to the P-meson form factor from this hadron-quark-photon vertex (4-point) now gives the QED gauge correction to the triangle loops, in the form of a similar function $F_1(k^2)$ which works out as [37]:

$$F_1(k^2) = 4(2\pi)^4 N_H^2 e_q \hat{m}_1 M_>^2 \int d^4q (M^2 - \delta m^2)\phi\phi' [\frac{D'_n \hat{q}.P}{\Delta'_1 \Delta_2 P'.n} + \frac{D_n \hat{q}'.P}{\Delta_1 \Delta_2 P.n}] + \{1 \Rightarrow 2\}, \tag{7.14}$$

where $M_> = \sup\{M; m_1 + m_2\}$ [37], and the common factor $2\bar{P}_\mu$ has been extracted as for $F(k^2)$ in (7.1). Note that e_q is antisymmetric in '1' and '2', signifying a change of sign when $\{1 \Rightarrow 2\}$ is added on the RHS. The pole integration of $F_1(k^2)$ now yields a result like (7.9) for $F(k^2)$; see [37] for details.

The large and small k^2 limits of $F(k^2)$ and $F_1(k^2)$ are on expected lines, and we summarise only the final results for completeness [37]. For large k^2, the functions $F(k^2)$ and $F_1(k^2)$ both yield the correct asymptotic form C/k^2, where $C = 0.35 GeV^2$, to be compared with the experimental value [62a] 0.50 ± 0.10, and the (perturbative) QCD value [63] $8\pi\alpha_s f_\pi^2 = 0.296$.

For low k^2, on the other hand, an expansion of F, F_1 in powers of k^2 yields a value of the charge radius R according to $< R^2 > = -\nabla_k{}^2 (F(k^2) + F_1(k^2))$ in the $k^2 = 0$ limit. Of the two functions, only $F(k^2)$ contributes in this limit [37]. The numerical values for the kaon and pion radii, vis-a-vis experiment [62b], are

$$R_K = 0.63 fm \quad vs(0.53 fm); \quad R_\pi = 0.661 fm \quad vs(0.656 fm). \tag{7.15}$$

7.4. Three-Hadron Couplings Via Triangle-Loops

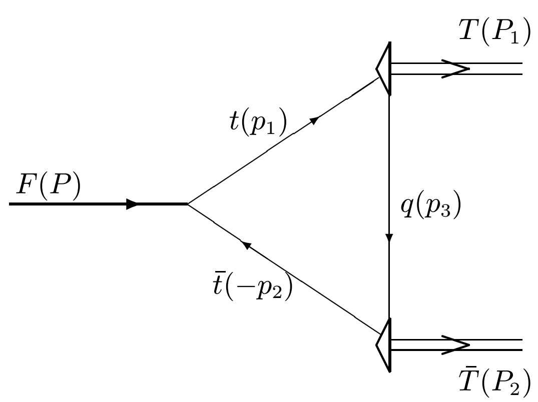

Figure 2: *3-hadron coupling.*

For a large class of hadronic processes like $H \to H' + H''$ and $H \to H' + \gamma$, the quark triangle loop [64] represents the lowest order "tree" diagram for their evaluation. Criss-cross gluonic exchanges inside the triangle-loop are not important for this description in which the hadron-quark vertices, as well as the quark propagators are *both non-perturbative*, and thus take up a lion's share of non- perturbative effects. This is somewhat similar to the 'dynamical perturbation theory' of Pagels-Stokar [65], in which criss-cross diagrams are neglected.

We now indicate in the barest outline, the structure of the 3-hadron loop integral for the most general case of unequal mass kinematics $m_1 \neq m_2 \neq m_3$, while referring for notational details to ref.[32]. The full structure of the 3-hadron amplitude may be written down from fig.2 above (c.f., fig.1 of [64]), just as for the e.m. form factor (7.1):

$$A(3H) = \frac{2i}{\sqrt{3}}(2\pi)^8 \int d^4p_i \Pi_{123} \frac{\Gamma_i(\hat{q}_i)}{\Delta_i(p_i)}, \tag{7.16}$$

exhibiting cyclic symmetry, where the normalized vertex function Γ_i in CNPA [41] is given in an obvious notation by eq.(5.18) as

$$\Gamma_i(\hat{q}_i) = N_i(2\pi)^{-5/2} D_i(\hat{q}_i)\phi_i(\hat{q}_i); \quad D_i = 2M_i(\hat{q}_i^2 - \frac{\lambda(M_i^2, m_j^2, m_k^2)}{4M_i^2}), \tag{7.17}$$

where the 'reduced' denominator function $D_i = D_{i+}M_i/P_{i+}$ and the (invariant) normalizer N_{iH} is N_i. The color factor and the effect of reversing the loop direction are given by $2/\sqrt{3}$, etc [38,64]. the overall BS normalizer [38].

To evaluate (7.16), we first write the cyclically invariant measure:

$$d^4p_i = d_\perp^p \frac{1}{2} d(x_i^2) M_i^2 dy_i; \quad x_i = p_{i+}/P_{i+}; \quad y_i = p_{i-}/P_{i-}. \tag{7.18}$$

The cyclic invariance of (7.18) ensures that it is enough to take any index, say 2, and first do the pole integration w.r.t. the y_2 variable which has a pole at $y_2 = \xi_2 \equiv \omega_{2\perp}^2/(M_2^2 x_2)$. The process can be repeated, by turn, over all the indices and the results added. Note that the ϕ-functions do *not* include the time-like y_i variables under CNPA [37], so that the residues from the poles arise from only the propagators. The crucial thing to note is that the denominator functions D_1 and D_3 sitting at the opposite ends of the p_2-line (c.f. Fig.1 of [64]) will *cancel out* the residues from the complementary (inverse) propagators Δ_3 and Δ_1 respectively. Indeed by substituting the pole value $y_2 = \xi_2$, in $\Delta_{1,3}$, the corresponding residues in an obvious notation work out as [32]:

$$\Delta_{1;2} = \xi_2 n_{32} M_2^2 + x_2 n_{23} M_3^2 - 2\hat{\mu}_{21} M_3^2; \Delta_{3;2} = -\xi_2 n_{12} M_2^2 - x_2 n_{21} M_1^2 - 2\hat{\mu}_{23} M_1^2. \tag{7.19}$$

It is then found, with a short calculation [32], that

$$\frac{D_3(\hat{q}_3)}{\Delta_{1;2}} = 2M_3 x_2 n_{23}; \quad \frac{D_1(\hat{q}_1)}{\Delta_{3;2}} = 2M_1 x_2 n_{21}, \tag{7.20}$$

which shows the precise cancellation mechanism between the D_i-functions and the residues of the propagators Δ_i at the Δ_2 pole. This mechanism thus eliminates [24, 64] the (overlapping) Landau-Cutkowsky poles that would otherwise have caused free propagation of quarks in the loops. The same procedure is then repeated cyclically for the other two terms arising from the $\Delta_{3,1}$ poles. Collecting the factors, the result of all the 3 contributions is compactly expressible as [64, 32]:

$$A(3H) = 8\sqrt{\frac{2\pi}{3}} \Sigma_{123} \int \int M_2 n_{23} n_{21} \pi^2 dx_2 d\xi_2 x_2^2 [TR]_2 D_2(\hat{q}_2) \Pi_{123} M_i N_i \phi_i, \tag{7.21}$$

where the limits of integration for both variables are $-\inf < (\xi_2, x_2) < +\inf$, since these are governed, not by the on-shell dynamics of standard LF methods [22–23], but by off-shell 3D-4D BSE. The difference from [64] (under CIA [24]) arises from using CNPA [37] which has ensured that the (gaussian) functions ϕ_i on the RHS of (7.21) are now free from time-like momenta (unlike in CIA [24, 64]).

Eq. (7.21) is the central result of this exercise. Its general nature stems from the use of unequal mass kinematics at both the quark and hadron levels, which greatly enhances its applicability to a wide class of problems involving 3-hadron couplings, either as complete processes by themselves (such as in decay processes) or as parts of bigger diagrams in which 3-hadron couplings serve as basic building blocks. What makes the formula particularly useful for general applications is its explicit Lorentz invariance which has been achieved through the simple method of 'Lorentz Completion' on the lines of sect.7.2 for the e.m. form factor of P-mesons; for more details, see [32].

As regards two- quark loops, such as for $SU(2)$ mass splittings of P-mesons [33b], and the mixing of ρ and ω off-shell propagators [33a], the distinction between CIA [24] and CNPA [37] is less sharp, (no time-like momentum problems in the overlap integrals). The same holds for one-quark loops, e.g., in the problem of vacuum condensates. For more details of these processes, as well as for other references, see [32].

8. Retrospect And Conclusions

To set the relatively unfamiliar MYTP [26] in the context of other BSE-SDE type approaches to strong interaction dynamics, Sections 1–2 have attempted a panoramic view of several standard approaches to 3D BSE reductions [6–9], as a background for putting in perspective its unique feature of *exact* 3D-4D interconnection of BS amplitudes of both $q\bar{q}$ and qqq types in closely parallel fashions. Further background methodology is provided in Sections 3–4, the former for a general derivation of the equations of motion in interlinked BSE-SDE form from an input 4-fermion Lagrangian for 'current' quarks, under MYTP conditions [25], and the latter for some essential Light-front techniques, such as the 'angular condition' [21]. With this background, Sect. 5 outlines a comparative derivation of the MYTP-controlled 3D-4D interlinkage of $q\bar{q}$ Bethe-Salpeter amplitudes under both CIA [24] and Cov LF [37] conditions. And in keeping with the parallelism between the 2- and 3-quark treatments, Sect. 6 gives a similar derivation for the qqq sustem. Now the 3D-4D interlinkage which charactizes MYTP, unlike other approaches [6–9, 22–23], gives rise to a natural two-tier description [38], the 3D BSE form being appropriate for making contact with the hadron spectra [13], while the reconstructed 4D BSE yields a vertex function which allows the use of standard Feynman diagrams for 4D loop integrals. The examples of the pion form factor and more general 3-hadron couplings in Section 7 illustrate the advantages of Cov-LF [37] over CIA [24] in producing well-defined triangle loop integrals, except for a (less serious) problem of dependence on the 'null-plane orientation' which can be handled through a simple device of 'Lorentz completion' to yield an explicitly Lorentz-invariant structure.

In keeping with its mathematical (formalistic) emphasis of this Article, we have refrained from discussing the phenomenological applications, but it has been shown that the canvas of MYTP [26] is broad enough to accommodate additional physical principles. In particular, the physical basis chosen for detailed presentation, has been a QCD motivated 4-fermion Lagrangian (with an effective gluonic propagator) which generates the BSE-SDE structure by breaking its chiral symmetry dynamically ($DB\chi S$) [11–12], formulated within an MYTP [26] framework.

Clearly, the **MYTP** is a powerful Gauge Principle which helps organize a whole spectrum of phenomena under a single umbrella. For its applications, only a few examples have been indicated, but its potential warrants many more. More importantly, the interlinked 3D-4D structure of BS dynamics under **MYTP** [26] premises, gives it access to a whole range of physical phenomena, from spectroscopy to diverse types of loop integrals. The emphasis on the spectroscopy sector as an integral part of quark physics was first given by Feynman et al [39].

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