QUANTIZATION OF GRAVITATIONAL FIELD

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The problem of field quantization in the vicinity of an arbitrary classical solution of the equation of motion is being discussed wide nowadays, and in principle the correct method of quasi-classical field expansion on the classical background has been solved by N.N.Bogoliubov in 1950.

The Bogoliubov method permits to perform quantization taking into account conservation laws precisely;

explicit separation of the classical component gives a possibility to avoid the zero-mode problem;

descriptive picture of interaction consents to calculate various quantum corrections to the classical parameters.

Nowadays there is large enough amount of works that specialize and develop the original idea. All of them underline the structure of the essential role of the Hamiltonian, however they are not applicable for the accurate account of conservation laws related to time transformation invariance, for example, because the Hamiltonian as the generator of time translations becomes clear only after solution of equations of motion, so the task was rather indefinite.

We have proposed such technique for the systems with time transformations: to define group variables together with developing of perturbation theory, and to specify formulae of variables substitutions step by step, in according with forthcoming to accurate solutions of field equations.

This scheme has been developed using the simplest model of self-acting Poincare-invariant scalar field. The application of this method to the nonstationary polaron is published in the Proceeding of the previous Workshop, and our approach is applicable to any system with symplectic structure.

One of such systems is the general relativity, and the main difficulty in gravity field quantization consists in the consecutive account of classical field.

There are 24 types of exact solution of Einstein equation, so in the present report we would like to apply the method of Bogoliubov group variables to the gravitational field quantization in the neighbourhood of such solutions.

We consider gravitational field in (3+1)-dimensioned formalism that has been proposed by *Arnowitt-Deser-Misner (ADM).*

Metrical tensor in this formalism looks like

$$g_{\alpha\beta} = \begin{pmatrix} -a^2 + b^t b_t & b_t \\ b_t & \gamma_{st} \end{pmatrix},$$

here γ_{st} is metrix of 3D-space in 4D-manyfold.

Canonical momentum π^{st} is determined as usual:

$$\pi^{st} = -\sqrt{\gamma}\left(K^{st} - \gamma^{st}K\right),$$

here

$$\sqrt{\gamma} = \sqrt{\det\|\gamma_{st}\|}; \qquad K_{tp} = -a\Gamma^0_{tp};$$

$$\Gamma^s_{tp} = \frac{1}{2}\gamma^{sp}\Gamma_{tlp}; \qquad \Gamma_{tlp} = \gamma_{pl,t} + \gamma_{pt,l} - \gamma_{tl,p}$$

denoting as usually

$$R_{\kappa\lambda} = \Gamma^\sigma_{\kappa\lambda,\sigma} - \Gamma^\sigma_{\kappa\sigma,\lambda} + \Gamma^\sigma_{\kappa\lambda}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\kappa\rho}\Gamma^\rho_{\lambda\sigma},$$

we can represent the action of gravitational field

$$S = \int d^3x \sqrt{g} g^{\kappa\lambda} R_{\kappa\lambda}$$

in the following form:

$$S = \int d^3x \left(\pi^{st} \gamma_{st,0} - aH - b_s H^s \right),$$

here

$$H = \frac{1}{\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} R,$$

$$H^s = -2\pi^{sl}_{;l}.$$

General principles of canonical formalism for the systems with constraints leads us to the following statement: *Lichnerowicz, Choquet-Bruhat, Dirac, Arnowitt-Deser-Misner:*

If the following evolutions equations

$$\gamma_{st,0} = \frac{2a}{\sqrt{\gamma}} \left(\pi_{st} - \frac{1}{2} \gamma_{st} \pi \right) + b_{s;t} + b_{t;s};$$

$$\pi^{st}_{,o} = -a\sqrt{\gamma} \left(R^{st} - \frac{1}{2} \gamma^{st} R \right) + \frac{a}{2\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) \gamma^{st} +$$

$$-\frac{a}{2\sqrt{\gamma}} \left(\pi^s_l \pi^{lt} - \frac{1}{2} \pi^{st} \pi \right) + \sqrt{\gamma} \left(\gamma^{sl} c^t_{;l} - \gamma^{st} c^l_{;l} \right) +$$

$$+ \left(\pi^{st} b^l \right)_{;l} - \pi^{sl} b^t_{;l} - \pi^{lt} b^s_{;l}; \qquad c^l = \gamma^{ls} a_{;s}$$

and constraint equations:

$$\frac{1}{\sqrt{\gamma}} \left(\pi_{st} \pi^{st} - \frac{1}{2} \pi^2 \right) - \sqrt{\gamma} R = 0;$$

$$\pi^{sl}_{;l} = 0$$

holds true on the 3D-space, then in 4D manifold the Einstein equations holds true:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \qquad \left(R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0 \right).$$

Suppose that 4D manifold with given metric permits to choose a space-like hypersurface and to set normal fields on this hypersurface. Those normals are tangent to geodesic and determine time coordinate. Hence geometry of 4D manifold could be described via Gaussian coordinates:

$$g_{\alpha\beta} = \begin{pmatrix} -a^2 & 0 \\ 0 & \gamma_{st} \end{pmatrix},$$

Let's variables $x^{'}$ are connected with x by space-time translations:

$$x^\alpha = x^{'\alpha} + \tau^\alpha.$$

Note that conservation laws performance in curved space-time is connected with Killing vectors existence, they are not straightforward sequence of system space-time transformation invariance.

In present case Bogoliubov transformation reconstructs translation invariance that has been violated due to presence of a classical field.

It means the following: if we have made quantization in some surface Σ in definite moment, application of group variables permits to state that we can move this surface Σ, including moving in time axis.

We define Bogoliubov transformation as following:

$$f_{st}(x) = v_{st}(x^{'}) + \epsilon u_{st}(x^{'}),$$

dimensionless parameter ϵ is assumed to be small, and τ are new independent variables.

The problem is how to formulate invariant conditions, which we have to impose on functions $u_{st}(x^{'})$. The substitution $g_{st}(x) \rightarrow \{u_{st}, (\tau)\}$ enlarges the number of independent variables to 4, so those conditions are necessary.

We consider systems in which there are invariant symplectic forms that look like the following

$$\omega\left(u_{st}, \left(N^{st}\right)^{k}\right) = \int_{\Sigma} (u_{n}^{st}(x^{'})\left(N_{st}(x^{'})\right)^{k} - u_{st}(x^{'})\left(N_{n}^{st}(x^{'})\right)^{k})dx$$

here Σ is some space-like surface and $u_{n}^{st}(x^{'})$ means normal derivative of $u_{st}(x^{'})$ in the surface Σ.

We choose some functions $N_{st}^{k}(x^{'})$ (k is the number of group parameters).

Using this condition one can obtain equations, which define group variables as functionals of $\gamma_{st}(x)$ and $\gamma_{n}^{st}(x)$ on the Σ in the differential form

$$\frac{\delta\tau^{a}}{\delta\gamma_{st}(x)} = -\epsilon Q_{b}^{a}\tilde{N_{n}^{st}}^{b}(x^{'}), \qquad \frac{\delta\tau^{a}}{\delta\gamma_{n}^{st}(x)} = \epsilon Q_{b}^{a}\tilde{N}_{st}^{b}(x^{'}),$$

where Q_{b}^{a} are the solution of the equation:

$$Q_{b}^{a} = \delta_{b}^{a} - \epsilon R_{c}^{a} Q_{b}^{c}.$$

Here $\left(\tilde{N}^{st}\right)^{a}$ is a linear combination of $(N^{st})^{a}$, such that the equations $\omega\left(\left(\tilde{N}^{st}\right)^{a}, v_{stb}\right) = \delta_{b}^{a}$ holds true; and R_{c}^{a} is a c-number, calculated with help of $v_{st}(x^{'})$ and $u_{st}(x^{'})$.

It is possible to define on Σ operators $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x)$

$$\hat{p}^{st}(x) = \frac{1}{\sqrt{2}}\Big(f_{n}^{st}(x) + i\frac{\delta}{\delta f_{st}(x)}\Big),$$

$$\hat{q}_{st}(x) = \frac{1}{\sqrt{2}}\Big(f_{st}(x) - i\frac{\delta}{\delta f_{n}^{st}(x)}\Big).$$

They are hermitian in the appropriately chosen space and satisfy the formal commutation relation

$$[\hat{q}_{st}(x), \hat{p}^{s^{'}t^{'}}(x^{'})] = i\delta_{st}^{s^{'}t^{'}}\delta(x - x^{'}).$$

So we can treat $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x)$ as operators of coordinate and momentum of oscillators of field and we can develop the secondary quantization scheme. However there is another pair of selfconjugated operators which satisfies the same commutation relations. So the number of possible field states turns out to be doubled, so we have to reduce the number of possible field states.

The following scheme is proposed:
we use Bogoliubov transformation and, in spite of appearance of extra states, we will develop scheme of perturbation theory. After that reduction of number of states is made, so it will depend on dynamic system equations.

Now we can quantize and substitute $\gamma_{st}(x^{'})$, and $\pi^{st}(x^{'})$ as follows

$$\gamma_{st}(x^{'}) \longrightarrow \hat{q}_{st}(x), \qquad \pi^{st}(x^{'}) \longrightarrow \hat{p}^{st}(x).$$

In terms of new variables $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x)$ are the series with respect to inverse powers of the coupling constant. Hence integrals of motion of the system can be represented as series with respect to inverse powers of the coupling constant

$$O = O_0 + \epsilon O_1 + \epsilon^2 O_1 + ...$$

In this series operators O_0 are C-numbers and operators O_1 are linear with respect to $u_{st}(x')$, $u_n^{st}(x')$, $\frac{\partial}{\partial u_{st}(x')}$, $\frac{\partial}{\partial u_n^{st}(x')}$. There are unnormalizable eigenvectors of these operators, so it is required to set them to zero for perturbation theory construction. Let's explore if it is possible. We define some functions $F_{st}(x')$ that proportional to the classical component $v_{st}(x')$. In terms of values $F_{st}(x')$ operators O_{-1} are equal to zero if some boundary conditions are accomplished, and the following equation holds true

$$F_n = \frac{2a}{\sqrt{\gamma}}\left(F_{n_{st}} - \frac{1}{2}F_{st}F_n\right),$$

$$F_n^{st} = -a\sqrt{F}\left(R^{st} - \frac{1}{2}F^{st}R\right) + \frac{a}{2\sqrt{F}}\left(F_{n_{st}}F_n^{st} - \frac{1}{2}F_n^2\right)F^{st}$$

$$-\frac{a}{2\sqrt{F}}\left(F_{n_l}^{s}F_n^{lt} - \frac{1}{2}F_n^{st}F_n\right) + \sqrt{F}\left(F^{sl}c_{;l}^{t} - \gamma^{st}c_{;l}^{l}\right),$$

$$+\left(F_n^{st}b^l\right)_{;l} - F_n^{sl}b_{;l}^{t} - F_n^{lt}b_{;l}^{s}$$

$$\frac{1}{\sqrt{F}}\left(F_{n_{st}}F_n^{st} - \frac{1}{2}F_n^2\right) - \sqrt{F}R = 0.$$

We can treat the 1st and 2nd equations as equations of evolution and 3rd and 4th as constraint equations, so we can state that for the perturbation theory application one should demand for Einstein equation to hold.

Hereinafter we assume $F_{st}(x)$ to be a solution of the Cauchy problem with given data on Σ and we treat $F_{st}(x)$ as a 3D classical background metric.

As we have mentioned above the number of possible field states is doubled. That is why state number reduction is necessary.

Primarily let's analyze the number of independent variables. Original number of independent variables was ∞. After defining of Bogoliubov group variables (they are considered to be independent) the number became $\infty + 4$. This number was doubled due to determination of coordinate-momentum operators: $(\infty + 4)) * 2 = 2 * \infty + 8$. Additional conditions reduced the number of independent variables on 4, that is at present time the number of possible field states is $2 * \infty + 4$.

Let's separate from field variables $u_{st}(x')$ four variables r_a which has no any physical sense and are connected with the method of perturbation scheme realization. Then the state number is $2*\infty$, and the field is described via $w_{st}(x')$ variables which are determined as follows:

$$u_{st}(x') = w_{st}(x') + \tilde{N}_{st}^{a}(x')r_a, \quad u_n^{st}(x') = w_n^{st}(x') + \tilde{N_n^{st}}^{a}(x')r_a.$$

Necessary reduction of the state number can be made by the following way: let's suppose that the field condition is defined by functionals of $w_{st}(x')$ and $w_n^{st}(x')$, in which $\frac{\delta}{\delta w_{st}(x')}$ and $\frac{\delta}{\delta w_n^{st}(x')}$ become

$$\frac{\delta}{\delta w_{st}(x')} \longrightarrow \frac{\delta}{\delta w_{st}(x')} - i w_n^{st}(x'),$$

$$\frac{\delta}{\delta w_n^{st}(x')} \longrightarrow -i w_{st}(x').$$

After reduction the independent variables become:

(4 group parameters)+(4 variables r_a)+
($\infty - 4$–dimensioned function w_{st} space)

The variables r_a have no physical sense. They have appeared as a rest of the state space reduction in the terms of Bogoliubov group variables. Separation of these variables is connected with integrals of motion structure in the zero-point order, so it is dynamic by nature.

As a result of our investigation we have got the expression for the field operator $\psi_{st}(x)$:

$$\psi_{st}(x) = F_{st}(x^{'}) + \epsilon \left(\hat{\Phi}_{st}(x^{'}) + \hat{\phi_{sta}} \frac{\partial}{\partial r_a} \right) + \epsilon^2 A(x^{'}, \tau)$$

(here $\hat{\Phi}_{st}(x^{'})$ is the solution of the wave equations:

$$\Phi_{n_{st}} = \frac{2a}{\sqrt{F}} \left(\Phi_{n_{st}} - \frac{1}{2} F_{st} \Phi_n \right),$$

$$\Phi^{st}_{nn} = 2\sqrt{F} U \Phi^{st},$$

with a known boundary condition on the Σ, U depends on space coordinates; and the state vector is

$$f = \phi(w) e^{-w w_n} * e^{A^{\alpha\beta} r_\alpha r_\beta},$$

here we know explicit expressions for the coefficients $A^{\alpha\beta}$, that depend on a concrete choice of symmetry type.

We applied Bogoliubov transformation to the quantization of gravitational field in the neighbourhood of a nontrivial classical component, that permitted us to avoid zero-mode problem.

Einstein equations for the classical component has been obtained as a necessary condition for the perturbation theory to be applicable, not as a sequence of variational principle.

We obtained expression for quantum corrections of the field operator and explicit form of the state vector, that permits us to calculate quantum corrections to the observables like effective mass, energy spectrum and so on.

Such kind of calculations for the physically interesting cases like Kerr, Schwartzshild and others exact solutions of Einstein equation are the nearest future project research; however those calculations demands only high level mathematical techniques, while the main principles of our approach are represented in the paper above.

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