# PARTICLES WITH SPIN IN A GRAVITATIONAL FIELD

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The problem of the Equivalence Principle (EP) for particle with spin 1/2 is considered. The gravitational formfactor in the lowest order of perturbation theory is found. The formfactor depends on the space curvature — this can be interpreted as a violation of EP.

## Introduction

The equivalence principle states: in a gravitational field all particles irrespective to their masses move in the same way. One can find statement that EP is not valid for a particle with spin [1]. We study this problem for an electron in a gravitational field.

The essential feature of EP — *locality*. Real bodies are extended objects (say, of length $L$). The dimension of curvature $R$ is $L^{-2}$, so EP in standard formulation is correct if

$$RL^2 \ll 1. \tag{1}$$

In the case of a non-zero spin particle it is assumed (tacitly) that the latter is pointlike. The only dimensional parameter here is its mass $m$. We will see later that one can neglect the curvature term if $R/m \ll m$, i.e. if

$$Rm^{-2} \ll 1 \qquad (R\lambda_c^2 \ll 1); \tag{2}$$

where $\lambda_c$ is the Compton length $\hbar/mc$. New important aspect here — *quantum mechanics*. So, considering particles with spin, one has to formulate EP in quantum mechanics. We calculate the gravitational formfactor of a particle with spin 1/2 in the lowest order of perturbation theory.

## 1. The Dirac equation on a manifold

According to Fock [2] the Dirac equation for electron in the Riemann space reads

$$(i\hat{D} - m)\psi = 0, \tag{3}$$

where $\hat{D} = \tilde{\gamma}^\mu D_\mu, \qquad [\tilde{\gamma}^\mu, \tilde{\gamma}^\nu]_+ = 2g^{\mu\nu}(x)I,$

$$D_\mu = \partial_\mu - \Gamma_\mu, \tag{4}$$

and $\Gamma_\mu$ is given by

$$\Gamma_\mu = \frac{1}{4} g_{\lambda\rho}(e^\rho_\alpha \partial_\mu \tilde{e}^\alpha_\nu - \Gamma^\rho_{\mu\nu})\tilde{\sigma}^{\lambda\nu}. \tag{5}$$

In (5) $\Gamma^\rho_{\mu\nu}$ are the Christoffel symbols, $e^\mu_\alpha(x)$ — the tetrade: $g^{\mu\nu}(x) = \eta^{\alpha\beta}e^\mu_\alpha e^\nu_\beta, e^\mu_\alpha \tilde{e}^\beta_\mu = \delta^\beta_\alpha$, $\eta^{\alpha\beta} = \eta^{\alpha\beta}(+---)$, and $\tilde{\sigma}^{\alpha\beta} = \frac{1}{2}[\tilde{\gamma}^\alpha, \tilde{\gamma}^\beta]_-$.
$\Gamma_\mu$ satisfy the identity

$$[\tilde{\gamma^\rho}, \Gamma_\mu] + \nabla_\mu \tilde{\gamma^\rho}(x) = 0, \tag{6}$$

where $\nabla_\mu$ is the covariant derivative. Then the curvature tensor is given by

$$\mathbf{R}_{\mu\nu} = [D_\mu, D_\nu]_- = \partial_\nu\Gamma_\mu - \partial_\mu\Gamma_\nu + [\Gamma_\mu, \Gamma_\nu]_-. \tag{7}$$

It can be defined by the Riemann-Christoffel tensor:

$$\mathbf{R}_{\mu\nu} = \frac{1}{4} g_{\lambda\eta} R^{\eta}_{\rho,\mu\nu} \tilde{\sigma}^{\lambda\rho}. \tag{8}$$

Indeed, we have

$$\nabla_\alpha(\overline{\psi}\tilde{\gamma}^\mu\psi) = \partial_\alpha(\overline{\psi}\tilde{\gamma}^\mu\psi) + \Gamma^\mu_{\alpha\beta}\overline{\psi}\tilde{\gamma}^\beta\psi \tag{9}$$

since $\overline{\psi}\tilde{\gamma}^\mu\psi$ is a vector. Using relation (4) and its conjugate $D_\mu\overline{\psi} = \partial_\mu\psi + \overline{\psi}\Gamma_\mu$, we obtain

$$\nabla_\alpha(\overline{\psi}\tilde{\gamma}^\mu\psi) = (D_\alpha\overline{\psi})\tilde{\gamma}^\mu\psi + \overline{\psi}\tilde{\gamma}^\mu(D_\alpha\psi) + \overline{\psi}(\nabla_\alpha\tilde{\gamma}^\mu + [\tilde{\gamma}^\mu, \Gamma_\alpha])\psi. \tag{10}$$

where the last term is zero according to Eq. (6). In the same way

$$\nabla_\rho\nabla_\alpha(\overline{\psi}\tilde{\gamma}^\mu\psi) = (D_\rho D_\alpha\overline{\psi})\tilde{\gamma}^\mu\psi + \overline{\psi}\tilde{\gamma}^\mu(D_\rho D_\alpha\psi), \tag{11}$$

and taking commutator we find

$$[\mathbf{R}_{\rho\alpha}, \tilde{\gamma}^\mu] + R^\mu_{\varphi,\rho\alpha}\tilde{\gamma}^\varphi = 0. \tag{12}$$

By solving these relations and using definition (7) we receive (8). Thus, tensor (7) contains the same information as the curvature tensor.

Now we are ready to solve problem of electron scattering by an external gravitational field.

## 2. Electron in an external gravitational field

Taking the standard definition of gravitational field $h_{\mu\nu}$: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\kappa = 8\pi G$, $G$ - the Newton constant, we observe that the proper interaction term is (since we are interesting in the formfactor in the lowest (i.e in the first) order of the perturbation theory we can change $\tilde{\gamma}^\mu \to \gamma^\mu$):

$$i\gamma^\mu\Gamma_\mu = i\gamma^\rho A_{\rho,\alpha\beta}\sigma^{\alpha\beta} \tag{13}$$

(it corresponds to $e\gamma^\mu A_\mu$ in electrodynamics). Using the identity

$$2m\gamma^\rho\sigma^{\mu\nu} = P^\rho\sigma^{\mu\nu} + P^\nu\sigma^{\rho\mu} + P^\mu\sigma^{\nu\rho} + \Sigma^{\mu\nu,\rho\lambda}P_\lambda - \Sigma^{\mu\nu,\rho\lambda}q_\lambda -$$
$$-\eta_{\alpha\beta}\sigma^{\rho\alpha}\Sigma^{\mu\nu,\beta\lambda}q_\lambda - \sigma^{\rho\lambda}\sigma^{\mu\nu}q_\lambda - (\hat{p}' - m)\gamma^\rho\sigma^{\mu\nu} - \gamma^\rho\sigma^{\mu\nu}(\hat{p} - m), \tag{14}$$

where $P^\mu = p^\mu + p'^\mu$, $q^\mu = p'^\mu - p^\mu$, $p(p')$ is momentum of incoming (outgoing) electron, $\Sigma^{\mu\nu,\rho\lambda} = \eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\nu\rho}\eta^{\mu\lambda}$, we find the "gravitational formfactor"

$$\overline{u}(p')(\gamma^\rho\sigma^{\mu\nu})u(p) = \frac{1}{2m}\overline{u}(p')[(P^\rho\sigma^{\mu\nu} + P^\nu\sigma^{\rho\mu} + P^\mu\sigma^{\nu\rho} + \Sigma^{\mu\nu,\rho\lambda}P_\lambda) -$$
$$-(\Sigma^{\mu\nu,\rho\lambda} + \eta_{\alpha\beta}\sigma^{\rho\alpha}\Sigma^{\mu\nu,\beta\lambda} + \sigma^{\rho\lambda}\sigma^{\mu\nu})q_\lambda]u(p). \tag{15}$$

The Fourier transform of connection $\tilde{A}_{\rho,\mu\nu}(q)$ depends only on $q$, so terms $P\tilde{A}(q)$ cannot be derivatives of $A_{\rho,\alpha\beta}(x)$ and can be discarded. The derivatives can be converted into curvature tensor

$$\Sigma^{\mu\nu,\rho\lambda}q_\lambda\tilde{A}_{\rho,\mu\nu} \to \frac{1}{4}R \tag{16}$$

$$\sigma^{\rho\lambda}\sigma^{\mu\nu}q_\lambda\tilde{A}_{\rho,\mu\nu} \to \frac{1}{2}\sigma^{\rho\lambda}R_{\lambda\rho,\alpha\beta}\sigma^{\alpha\beta}. \tag{17}$$

The last term in (15) is of order $O(\kappa^2)$ and can be omitted.

We conclude that motion of a particle with spin 1/2 in an external gravitational field depends on the curvature tensor and cannot be excluded by choice of coordinates.

## References


[1] A.A. Logunov, M.A. Mestvirishvili, Yu.V. Chugreev. Usp. Fiz. Nauk, **166**, 81 (1996).

[2] V.A. Fock. Zs. f. Phys., **57**, 261 (1929).