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11. [15] *In return for their foolish and wicked thoughts, which led them astray to worship irrational serpents and worthless animals, thou didst send upon them a multitude of irrational creatures and worthless animals, to punish them,* [16] *that they might learn that one is punished by the very things by which he sins.* [17] *For thy all-powerful hand, which created the world out of formless matter, did not lack the means to send upon them a of bears, or bold lions,* [18] *or newly created unknown beasts full of rage, or such as breathe out fiery breath, or belch forth a thick pall of smoke, or flash terrible sparks from their eyes;* [19] *not only could their damage exterminate men, but the mere sight of them could kill by fright.* [20] *Even apart from these men could fall at a single breathe when pursued by justice and scattered by the breath of thy power. But*
thou hast arranged all things by measure and number and weight.

The wisdom of Solomon

The great mathematician and physicist Bogoliubov considered the last line of this sentence as definition of physics. The history of quantum mechanics indicates that all great discoveries are connecting with the arrangement "all things by measure and number and weight". Neglect of this rule leads as warned wise Solomon to the false hypothesizes and replacing common sense by superstitions. The Quantum mechanics unlike of many other sciences knows the exact data of its birthday. 29 July 1925 editorial staff of *"Zeitschrift für Physik"* received the article *"Quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen".* This article of Werner Heisenberg has for physics of the XX century the same importance as the Newton's "Principia" for the XVII century.

The outward events of Heisenberg life have nothing specify life of the man who has created the theory having no predecessors. Especially his youth has passed near so outstanding persons as Bohr, Sommerfeld, Pauli. However, in the school characteristic of the ten-years pupil of Munich Maximilian classic school was spoken, that "his sight fastens to essential, he does not burden himself with details."

In 1923 Bohr visited Göttingen. He discussed very much with brilliant pupils of Sommerfeld. Eldest – Pauli – was invited in Copenhagen immediately. Conversation with Heisenberg finished such sentence:

You look still very young. Such impression is created as if you have begun from study of atomic physics and only mastered former physics and all other later. It seems to me that Sommerfeld very much early has entered you into this Adventure World of atoms... You should somehow visit us in Copenhagen and arrive to us on longer time.

When in 1924 Heisenberg went to Denmark Pauli has responded on travel his friend by the letter to Bohr:

> Recently I saw Heisenberg. I always feel myself very awkwardly with him. When I reflect on his ideas, they seem to me dreadful, and about myself I am abusing them terrible. He is very unphilosophical. He does not care of, that the basic assumptions were clearly developed and are connected to the previous theories. When I talk with him he is pleasant to me very much, and I see, that he – at least in soul – has heap of new arguments... Therefore I am very glad, that you have invited him in Copenhagen... He will acquire philosophical installation of your thinking.

Having got in Copenhagen Heisenberg quickly has joined in work. Soon Kramers and Heisenberg have published work about dispersion of light on atoms, which till now is quoted in all serious manuals under the quantum theory.

> Renewing in a summer semester 1925 the work in Göttingen... I began the scientific researches from attempt to guess the correct formulas of intensity of lines of a spectrum of hydrogen, using a method similar to what was tested in ours joint with Kramers works in Copenhagen. This attempt has not managed. I stuck in impenetrable thickets of the mathematical formulas. However as a result of this attempt at me the opinion was strengthened, that is not necessary to study the electron's orbits. The set of frequencies and amplitudes of spectral lines can serve replacement to orbits. Anyway, these quantities could be directly observed.

Heisenberg formulated the Ritz's combinatorial principle as a pure kinematic law of the composition of frequencies of radiation by atoms:

$$\omega(m,n) \quad = \quad \omega(m,r) \quad + \quad \omega(r,n).$$

In the classical physics any periodic movement can be characterized by the basic tone Ω and the set of obertones

$$\omega_n \quad = \quad n\Omega.$$

The frequency Ω depends on energy of system which can be connected with dimensionless variable m, i.e. the classical frequencies can also be designated by two indexes:

$$\omega_{cl} \quad = \quad \omega_{cl}(m,n).$$

These frequencies obey such law of composition

$$\omega_{cl}(m,n) + \omega_{cl}(m,r) \quad = \quad \omega_{cl}(m,n+r).$$

The law of a composition of frequencies ω_{cl} differs from the experimental law of a composition of frequencies of radiation of atoms.

It is known that any periodic function $F(t)$ can be presented in the form

$$F(t) \quad = \quad \sum_{n=-\infty}^{\infty} F_n exp(in\Omega t).$$

The coefficients F_n and frequencies $n\Omega$ are one-to one connect with the function $F(t)$. To distinguish one function from others, it is possible to take advantage of a continuous index m. Passing on the physical language consider the set

$$\Big\{A(m,n), \omega_{cl}(m,n) = n\Omega(m)\Big\}$$

as the representative of the variable $A(t)$.

The study of interaction of atoms with an electromagnetic field supplies us with sets $\left\{A(m,n),\omega_{mn}\right\}$, in which frequency satisfy to Ritz principle. Which variable is represented by such series? To answer this question it is necessary to solve such task: let variable $A(t)$ and $B(t)$ correspond sets $\left\{A(m,n),\omega_{mn}\right\}$ and $\left\{B(m,n),\omega_{mn}\right\}$. What should be the representative of the variable $f(A,B)$?

Let in classical system exists periodic dynamic variable $F(t)$:

$$F(t) \quad = \quad \sum_{n=-\infty}^{\infty} F(m,n)exp(i\omega_{cl}(m,n)t).$$

If to define $F^2(t)$ as product of corresponding series

$$F^2(t) \quad = \quad \sum_{n,r} F(m,n)F(m,r)exp(i(\omega_{cl}(m,n)t + i\omega_{cl}(m,r)t)).$$

then

$$F^2(t) \quad = \quad \sum_{n=-\infty}^{\infty} F^{(2)}(m,n)exp(i\omega_{cl}(m,n)t),$$

where

$$F^{(2)}(m,n) \quad = \quad \sum_{r+p=n} F(m,r)F(m,p).$$

The received formulas are a special case of the general law: if at the initial moment of time the dynamic variables ϕ and F are connected by formula $\phi = f(F)$, $\phi(t)$ and $F(t)$ are connected by the same ratio: $\phi(t) = f(F(t))$.

The experimental analysis of emission and absorption of an electromagnetic field by atoms supplies physicists with a set $\{F_{mn},\omega_{mn}\}$. If to consider this set as representative of variable $F(t)$, opportunity to define F by Fourier series is mutually connected to the classical law of composition of frequencies. For example, for electron movement in atom if mechanical frequencies coincide with frequencies of electromagnetic radiation the representative of coordinate of electron is the set $\{x_{mn},\omega_q(m,n)\}$ and it is impossible to define function

$$x(t) \quad = \quad \sum_{n=-\infty}^{\infty} x_{mn}exp(i\omega_q(m,n)t).$$

It has allowed Heisenberg to state in 1925 the stunning statement:

> in the theory coordinated with a combinatorial principle of Ritz there can not be a classical concept about a trajectory of electron inside the atom (in der Quantentheorie nict möglich war, dem Elektron einen Punkt im Raum als Funktion der Zeit mittels beobachtbarer Größen zuzuordnen).

Thus the quantum theory should search for the way of definition of coordinate. What should be this definition? Heisenberg answer was ingeniously simple: if frequencies of radiation of atom are numbered by pairs of numbers all dynamic variables connected with atom should be listed by pairs of integers .

So let quantities A and B aredefined by their representatives

$$A \leftrightarrow \{A_{mn},\omega_m - \omega_n\}; B \leftrightarrow \{B_{mn},\omega_m - \omega_n\},$$

and the set $C_{mn}(t)$ is calculated under the formula $C_{mn} = \sum_r A_{mr}B_{rn}$, then

$$C_{mn}(t) = \sum_r A_{mr}(t)B_{rn}(t) = e^{i\omega_m t}\sum_r A_{mr}B_{rn}e^{-i\omega_n t}.$$

In all cases a set of frequencies are differences $\omega_m - \omega_n$. It is obvious that it is possible to consider quantity C as product A and B:

$$C \leftrightarrow \{(AB)_{mn}, \omega_m - \omega_n\}.$$

Heisenberg has paid attention that in his theory multiplication is noncommutative. However now it is not too essential. The direct physical consequence of this definition is much more important. The way of obtaining of functions of dynamical quantities shows how Heisenberg has removed from the theory the concept of a trajectory of a particle: as the representatives of variable (including the representatives of coordinate) contain two indexes and at calculation of the representatives of functions both indexes are essential (in a classical case it not so), it is possible to say that in the description of movement are considered at once all possible classical trajectories (i.e. any is not considered separately).

So, the way of definition of functions from variable of the system in the terms of their representatives is received. But how to define these variable? Heisenberg has come to definition of momentum and coordinate from evident physical reasons.

In 1924 Kramers has received the formula for electric dipole moment of atom induced by a periodic electrical field: if the intensity of a field is equal $\vec{E} cos 2\pi\nu t$, induced dipole moment $\vec{M}$ is equal

$$\vec{M} = \vec{E}\frac{2}{h}\sum_{\alpha}\Big\{\frac{\Gamma_a(n+\alpha,n)\nu_a(n+\alpha,n)}{\nu_a(n+\alpha,n)^2-\nu^2} - \frac{\Gamma_e(n,n-\alpha)\nu_e(n,n-\alpha)}{\nu_e(n,n-\alpha)^2-\nu^2}\Big\}.$$

In this formula the processes of two types are taken into account: photon absorption with frequency $\nu_a(n+\alpha,n)$ with its subsequent radiation and radiation and subsequent absorption of photon of frequency $\nu_e(n,n-\alpha)$. The quantities Γ_a and Γ_e are appropriate Einstein factors. As at the large photon frequencies this expression should coincide with the Thomson formula

$$\vec{M} = -\frac{e^2\vec{E}}{4\pi^2 m\nu^2},$$

and the rules of the sums by Reiche - Kuhn should be carried out

$$\frac{8\pi^2 m}{e^2 h}\sum\Big(\Gamma_a\nu_a - \Gamma_e\nu_e\Big) = 1.$$

Using the correspondence principle and the define Einstein factors in the terms of Fourier coefficients of the electron trajectories (certainly these constructions are quasiclassical yet) Heisenberg postulated such rule:

$$h = 4\pi m\sum_{\alpha=0}^{\infty}\Big\{|a(n,n+\alpha)|^2\omega(n,n+\alpha) - |a(n,n-\alpha)|^2\omega(n,n-\alpha)\Big\}.$$

The origin of numbers $a(l,k)$ as was already told is connected with Fourier coefficients of a trajectory selected with number l. But for Heisenberg $a(l,k)$ are representatives of coordinate of electron as dynamical variable — about trajectories it is possible to forget now. These numbers can be differentiated by a rule:

$$\frac{da(k,l)}{dt} = i\omega(k,l)a(k,l).$$

After that it is possible to remove the frequencies from the Heisenberg's sumand to receive just kinematic relation

$$\sum_k (q(n,k)p(k,n) - p(n,k)q(k,n)) = i\hbar.$$

Correct determined momentum and coordinate should satisfy with it. Heisenberg has applied his method to nonlinear oscillator. Let's result here calculations for more simple case - harmonic oscillator.

$$\frac{d^2x}{dt^2} + {\omega_0}^2 x \quad = \quad 0$$

result in such relations between coefficients $a(m,n)$ and frequencies $\omega(m,n)$:

$$(-\omega(m,n)^2 + {\omega_0}^2)a(m,n) = 0,$$

that

$$a(m,n) = 0, \quad if \quad \omega(m,n) \neq \pm\omega_0.$$

Always it is possible to choose such numbering that

$$\omega(n, n \pm 1) \quad = \quad \pm\omega_0.$$

Now only factors $a(n, n \pm 1)$ differ from zero and the rule of Heisenbergsums accepts a kind

$$|a(n+1,n)|^2 - |a(n,n-1)|^2 \quad = \quad \frac{\hbar}{2m\omega_0}.$$

These relations are represented by the finite difference equations. If the numbers n accept values $n = 0, 1...$, the coefficients $a(m,m)$ are real and initial condition is $a(0,-1) = 0$ the nonzero representatives of coordinate are equal

$$a(n, n-1) = a(n-1, n) = \sqrt{\frac{n\hbar}{2m\omega_0}}.$$

The oscillator's energies should be

$$H(k,n) = \frac{m}{2}\sum_{l=0}^{\infty}\Big\{\frac{da(k,l)}{dt}\frac{da(l,n)}{dt} + {\omega_0}^2 a(k,l)a(l,n)\Big\}.$$

This sum is easily calculated and gives values of energy levels of harmonic oscillator:

$$H(k,n) = \delta_{kn}\Big(n + \frac{1}{2}\Big)\hbar\omega_0.$$

By using definition of derivative on time it is easy to check up that

$$\frac{dH(k,n)}{dt} \quad = \quad 0.$$

This means the energy conservation law.

That's all — the quantum mechanics was created.

On way to Göttingen Heisenberg has come to Hamburg to Pauli expecting to undergo to the strictest criticism, but has received hot support. Pauli approved the younger comrade. During two weeks they supported the brisk correspondence, while on July 9, 1925 Heisenberg was sent to the friend final variant of the manuscript. Pauli's answer was unexpected for himself: "It is Aurora (Das Morgenröte) of quantum theory". After such answer Heisenberg has shown his article to Born, with the request to inform what he thinks about. Born recollected:

> After representation of work (by Heisenberg) for printing I thought about Heisenberg's formalism and has found out that it is identical to matrix calculation well known for mathematics.

On July 19 Born and Pauli have met at station in Hanover where they have arrived on session of the German Physical Society. Born has told about matrixes, about difficulties in manipulation with them. In particular it was not clear what to do with not diagonal matrix elements. Born has offered Pauli cooperation but has heard sarcastic answer:

> your inclination to the long and artful formulas is known for me. But by useless mathematics you only spoil physical ideas of Heisenberg.

The first strict statement of ideas of the "matrix mechanics" has appeared very quickly . This article was a sample *"of Göttingen learning"* (Gelerhsamkeitsschwall der Göttingen). It was actually offered to physicists to study a university rate of linear algebra: eigenvalues and eigenvectors, canonical transformations of matrix, reduction of matrix to a diagonal kind, Hilbert infinite-dimensional square-law form, commutation relations. However, the physics was not forgotten: in this paper the way of description of electromagnetic field was offered in the new theory. The paper has made the large impression.

Heisenberg unwillingly publicly talked about his scientific occupations. Having visited Cambridge and having made there on July 28, 1925 the report under the name *"Zoology of terms and Zeeman's botany"*, he has not mentioned the opening. The name of the report does not speak about desire to joke on Englishmen. He used the especially technical terms.

So England not at once has familiarized with new opening. News about Heisenberg article has delivered to Cambridge by Fowler — the physicist well understanding the problems of the newest physics. He has acquainted with this work his post-graduate student Dirac. Heisenberg's coeval Dirac searched for the ways for explanation of the nuclear phenomena practically in loneliness. Many years after he wrote about Heisenberg's article:

> Both of us were the young people and solved the same problem simultaneously. He has achieved success where I have failed.

However soon Dirac has made up for delay on start. He proposed an idea of description of evolution of the system with the help of the differential equations for noncommuting variables (q-numbers), similar to the Hamilthon-Jakobi equations where Poisson brackets are replaced by commutators. As the commutators satisfy with all relations defining Poisson brackets the internal consistency of such equations (Heisenberg equations) does not cause doubts. Moreover as Poisson brackets in classical mechanics can be taken as definition of canonical conjugate variables so in the quantum mechanics it is possible to define dynamic variables in the terms of commutators between them. After that quantum mechanics became self-sufficient and didn't need classical mechanics for its definition.

On January 17, 1926 the magazine *"Zeitschrift für Physik"* has received the article by Pauli *"About a Spectrum of Hydrogen from the Point of View of the New Quantum Mechanics".* It is one of most beautiful and important works in this area. In essence Pauli was first who investigated connection multiplicity of degeneration of levels and existence of noncommuting integrals of movement. He has restored in rights analysis of symmetry properties of physical systems. After Pauli work it was possible to assume that creation of the new quantum mechanics was completed.