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DETECTION OF HOMOCLINIC POINTS IN DISSIPATIVE DYNAMICAL SYSTEMS

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Abstract

Klimenko S.V., Smirnova V.V. Detection of Homoclinic Points in Dissipative Dynamical Systems: IHEP Preprint 96-101. – Protvino, 1996. – p. 5, figs. 2, tables 1, refs.: 14.

We consider some methods for finding homoclinic points of dissipative dynamical systems. The existence of these homoclinic points implies the existence of invariant hyperbolic sets. It is shown for the Lorenz, Rikitake, Rössler attractor and for a simple attractor.

Аннотация

Клименко С.В., Смирнова В.В. Определение гомоклинических точек в диссипативных динамических системах: Препринт ИФВЭ 96-101. – Протвино, 1996. – 5 с., 2 рис., 1 табл., библиогр.: 14.

В данной работе рассмотрены методы нахождения гомоклинических точек для диссипативных динамических систем. Существование гомоклинических точек указывает на наличие в таких системах инвариантных гиперболических множеств.

Most of the natural phenomena exhibit chaotic behavior. Its presence can be defined as the existence of intersections of stable and unstable manifolds of hyperbolic fixed point or as the existence of homoclinic point or homoclinic trajectories. The existence of the latter in a dynamical system enables us to discuss some of its properties. The existence of hyperbolic set follows from existence of transversal homoclinic trajectories.

There has not been any rigorous proof, found at the level of a theorem, for the problem of existence of homoclinic trajectories in a common dynamical system. In some cases homoclinic trajectories appear at small periodic disturbances of autonomous Hamiltonian systems with one degree of freedom, having closed separatrix of loop. To determine the presence of homoclinic trajectories, it's possible to use Melnikov's method [1] (or Palmer's method for n-dimensional cases [2]). But only if the dissipation is low and the equations for manifolds with zero dissipation are known this method is applicable.

This criterion does not imply anything about the appearance of a strange attractor in dissipative dynamical system showing a stable chaotic behavior in a large area of the phase space. It should be noted that an attractor is not a manifold. Let us assume that the dissipative dynamical systems are given in which the presence of homoclinic trajectories or homoclinic points should be tested, but the equations for the manifolds of hyperbolic fixed point are unknown.

We consider three-dimension dissipative dynamical systems with a strange attractor. All the notions and definitions used here are listed in Appendix 1.

Let us assume that for the given nonlinear differential equations there exists an intersection of stable and unstable manifolds for the hyperbolic fixed point and the Poincare map. In this case the unstable manifold returns to the crossing plane. In this case the Poincare map is not determined in the intersection of the stable manifold and crossing plane, and is not continous in its neighbourhood.

There are several ways of searching for homoclinic points.

Method 1. One constricts the stable and unstable manifolds for hyperbolic fixed point and finds the intersection of these manifolds. While building the stable manifold for a hyperbolic fixed point x_0 with real eigenvalues, the latter manifold can be approximated with a plane stretched over eigenvectors belonging to negative eigenvalues. An approximation of curved manifolds W^s and W^u with planes in some neighborhood of the hyperbolic fixed point induces an error in the computation of invariant manifolds. The error can be estimated using quadratic asymptotics of the manifolds mentioned [8]. Some other difficulties in constructing the stable manifolds are discussed in [7]. For example, let us consider the construction of the intersection of the stable manifold with the crossing plane, and of the Poincare map for the Lorenz attractor (Fig. 1). Because of the exponential instability of trajectories on strange attractors,the probability of getting the intersection of stable manifold and the crossing plane at the Poincare map is small [3].

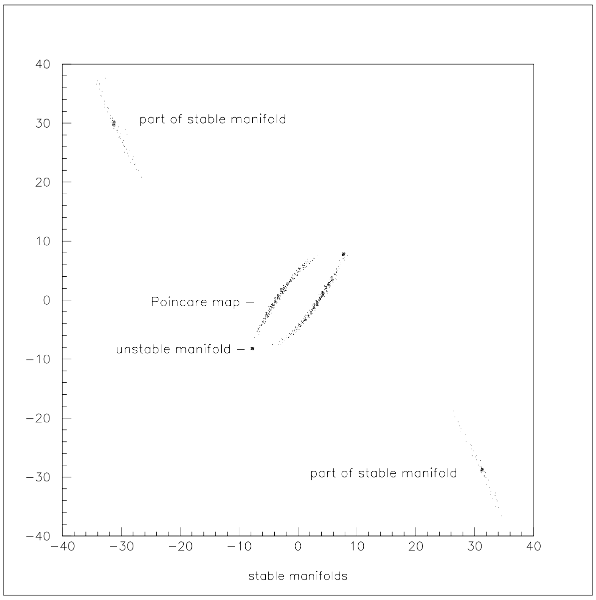

Fig. 1. Intersection of the stable and unstable manifolds of the orign with the crossing plane, and the Poincare map for Lorenz attractor.

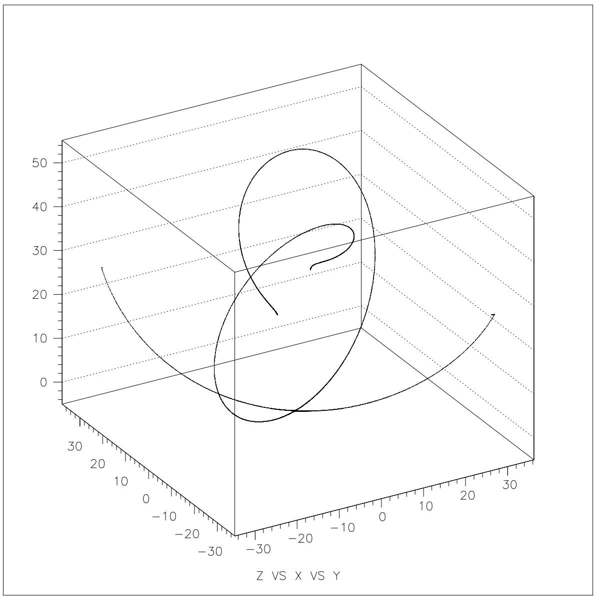

Fig. 2. Stable and unstable manifolds of the origin for the Lorenz attractor.

Method 2. For building the stable and unstable manifolds of hyperbolic fixed point the λ-lemma [4] can be used. The lemma can be applied to a local diffeomorphism and even to a C^1-mapping in the Banach space in some neigborhood of the hyperbolic fixed point. It demands that the partial derivatives be uniformly continuous, and that both the stable and unstable manifolds be of finite dimensions. So, the local stable (unstable) manifold of the hyperbolic fixed point of the diffeomorphism should always be considered in the neighborhood of the fixed point in the stable (unstable) subspace of the linear part of mapping f.

As an example we show the construction of the stable and unstable manifolds for the Lorenz attractor (Fig. 2) by means of the λ-lemma.

Method 3. Now, we find the homoclinic points in the strange attractor without requiring the calculation of the manifolds of the fixed point and the construction of the intersection of the stable manifold and crossing plane W of the Poincare map.

Let $f(x)$ be the first point where, the unstable manifold originating at $x \in W$ intersects the plane W. This defines the map $f : W\backslash S \to W$. Here S is a line of discontinuity, it divides W into two parts, W_1 and W_2. Then there is a unique limit p_1 of the images of the points from W_1, approaching S on one side, and there is a unique limit p_2 of the images of the points from W_2, approaching S on the other side.

Let us consider the process of searching for homoclinic points. To find a homoclinic point we use the fact that the Poincare map is not determined in the intersection of manifolds and is not continuous in its neighborhood. First, we test the Poincare map for discontinuity in the neighborhood of some point. As soon as the neighborhood is found, we search for a point from the crossing plane where the Poincare map is not defined. Let us denode this point by x on the plane. Then $f^k(x)$ the images of point x obtained by letting the mapping f act k times are situated along some curves (or curve) on the plane. Let these curves be called the right (RB) and the left (LB) branches, respectively. Let P_{12} be a segment of the curve LB. We select points $p_1, p_2 \in P_{12} \subset LB$ such that $p_1 \in LB \cap W_1$, $p_2 \in LB \cap W_2$, and $f(p_1) \in LB \cap W_2$, $f(p_2) \in RB \cap (W_1 \cup W_2)$. Then we find sequential points $p_1^j, p_2^j \in P_{12}^j \subset P_{12}^{j-1} \cdots \subset LB$ such that $f(p_1^j) \in LB \cap W_1$ and $f(p_2) \in RB \cap (W_1 \cup W_2)$. It is clear that if as we have homoclinic points, we have $\lim_{j\to\infty} \bigcap P_{12}^j = p^* \in S$, where point p^* is a point for which $f(p^*)$ is ill-defined. If such a point cannot be found, there are no strange attractors in the dynamical system.

Modifications of this method for attractors with one branch of the Poincare map is obvious. This algorithm was used for finding homoclinic points of the Lorenz and Rikitake attractors, as well as the Rösler attractor and a simple attractor.The results obtained for the coordinates of the homoclinic points are shown on Table 1. It was made clear that the Cantor structure exists in the neighborhoods of the homoclinic points of these attractors [13].

Table 1. The coordinates of the homoclinic points.

	X	Y	Z
Lorenz attractor *	-2.98317	1.01683	27.
Rikitake attractor *	-0.93793	0.33621	4.
simple attractor	8.6103	-2.9687	1.
Rössler attractor	-4.1892	-3.401	1.

*The coordinates of the homoclinic points are given only for the left branch of the Poincare map.

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References

[1] Mel'nikov V.K. Trans. Moscow Math. Soc. 12 (1963) 3-52.

[2] Palmer K.J. *Exponential Dychotomies and Transversal Homoclinic Points*, J. different. equations, 1984, 55, 3, p. 225-256.

[3] Sparrow C. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Appl. Math. Sc., Springer-Verlag, 41 (1982).

[4] Palis J., de Melo Jr. W. *"Geometric Theory of Dynamical System"*, Springer-Verlag, 1982.

[5] Rabinovich M.I. Usp. Ph. Nauk, 125 (1978) 123-168.

[6] Lorenz E.N. *"Deterministic Nonperiodic Flows"* S. Atmos. Sci., 20 (1963) 130-141.

[7] Parker T.S., Chua L.O. *"INSITE - A Software Toolkit for the Analysis of Nonlinear Dynamical Systems"* Proc. IEEE 75, (1987), №8.

[8] Hassard B.D. *Computation of Invariant Manifolds*: In: New Approaches to Nonlinear Problems in Dynamics, Ed. by Ph.J. Holms, Philadelphia, SIAM, 1980, 27-42.

[9] Willams R.F. *Structure of Lorenz Attractor* Publications Mathematics I.H.E.S., 50 (1980) 59-72.

[10] Cook A.E., Roberts P.H. *The Rikitake Two-Disc Dynamo System.* Proc. of Cambridge Philosophical Society, 68 (1979) 547-569.

[11] Rössler O.E. Z.Naturforsch. 57A, (1976), 397.

[12] Hirsch M., Pugh C. *Stable Manifolds and Hyperbolic Sets.* In: Global Analysis.- Proc. Symp. in Pure Math. American Math. Soc., 14 (1970) 133-163.

[13] Klimenko S.V., Smirnova V.V. IHEP Preprint 91-131, Protvino, 1991.

[14] Plykin R.V., Zhirov A.Ju. *Some Problem of Attractors of Dynamical Systems.* Topology and its Applications. North-Holland 54 (1993) 19-46.

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Appendix 1

Let us introduce some general terminology.
Let $f : M \to M$ be a diffeomorphism of smooth C^m-manifold M, $1 \leq m \leq \infty$ of dimension not less than 2.

An f-invariant set $\Lambda \subset M$ is said to be hyperbolic if for every $x \in \Lambda$ the tangent space T_xM can be written as a direct sum of spaces E^u_x, E^s_x, $T_xM = E^U_x \oplus E^s_x$, where $dimE^u_x = u$, $dimE^s_x = u$, $u + s = dimM$ and

1. $df(E^s_x) = E^s_f(x)$, $df(E^u_x) = E^u_f(x)$,
2. there exist constants $c > 0$ and $\lambda \in (0, 1)$ such that

    a) $\|df^n v\| \leq c\lambda\|v\|$, $v \in E^s_x$,
    b) $\|df_{-n} v\| \leq c\lambda_n\|v\|$, $v \in E^u_x$, $n \geq 0$,

3. E^s_x and E^u_x depend continuously on $x \in \Lambda$.

We denote by $B_r(x)$ the ball of radius r with a center at $x \in \Lambda$

Definition 1. A submanifold $W \subset M$ is called *locally stable of radius r passing through $x \in M$* if $W = \cap_{n\geq 0} f^{-n}(f^n(x))$. We write $W = W_r^s(x)$. $\tilde{W} = W_r^u(x)$ *locally unstable of radius r passing through x* if it is a locally stable manifold of radius r of the diffeomorphism f^1 passing through x.

Definition 2. If Λ is a compact hyperbolic set of diffeomorphism $f : M \to M$, then the set

$$W^s(x) = \cup_{n\geq 0} f^{-n}[W_r^s(f^n(x))]$$

is called *the stable manifold* of point $x \in \Lambda$.

$$W^u(x) = \cup_{n\geq 0} f^n[W_r^u(f^{-n}))]$$

is called *the unstable manifold* of the point x. Let p be a hyperbolic fixed point of diffeomorphism f. The homoclinic point is called the point of intersection of stable manbifold $W^s(p)$ and unstable manifold $W^u(p)$.

Appendix 2

Lorenz attractor

$$\begin{aligned} \dot{x} &= \sigma y - \sigma x \\ \dot{y} &= rx - y - xz, \qquad \text{for} \quad \sigma = 10, \quad r = 28, \quad b = 8/3 \\ \dot{z} &= xy - bz \end{aligned}$$

Rikitake attractor [10]

$$\begin{aligned} \dot{x} &= -\mu x + zy \\ \dot{y} &= -\mu y + (z - A)x, \quad \text{for} \quad \mu = 1, \quad A = 3.75 \\ \dot{z} &= 1 - xy \end{aligned}$$

Rössler attractor [11]

$$\begin{aligned} \dot{x} &= -(y + z) \\ \dot{y} &= x + Ay, \qquad \text{for} \quad A = 0.2, \quad B = -0.4, \quad C = 5.7 \\ \dot{z} &= B + xz - Cz \end{aligned}$$

Simple attractor [5]

$$\begin{aligned} \dot{x} &= z + y + 0.2x \\ \dot{y} &= x + 0.2y \\ \dot{z} &= -2z(x + 3) + 1 \end{aligned}$$

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