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ON SOLVING SCHWINGER-DYSON EQUATIONS FOR NON-ABELIAN GAUGE THEORY

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Abstract

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A method of solving Schwinger-Dyson equations for the Green function generating functional of non-Abelian gauge theory is proposed. The method is based on an approximation of Schwinger-Dyson equations by exactly soluble equations. For the $SU(2)$ model the first step equations of the iteration scheme are solved which define a gauge field propagator. Except the usual perturbative solution, a non-perturbative solution is found which corresponds to the spontaneous symmetry breaking and obeys non-singular ("constant") behaviour of the propagator in the infrared region

Аннотация

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Предложен метод решения уравнений Швингера-Дайсона для производящего функционала функций Грина неабелевой калибровочной теории. Метод основан на аппроксимации уравнений Швингера-Дайсона системой точно решаемых уравнений. Для $SU(2)$-модели решены уравнения первого шага итерационной схемы, определяющие пропагатор калибровочного поля. Помимо обычной теории возмущений, которая всегда является одним из решений, найдено непертурбативное решение со спонтанным нарушением симметрии и несингулярным ("константным") поведением пропагатора калибровочного поля в инфракрасной области.

1. The Schwinger-Dyson equations (SDE) method is one of the basic tools for the investigations of the Green functions of the quantum theory. Hitherto the unique universal method of the SDE solution is the coupling constant perturbation theory (below named simply the perturbation theory). A field of applicability of other methods (for example, $1/N$-expansion) is limited by a narrow class of models. In particular the $1/N$-expansion method cannot be applied to the investigation of non-Abelian gauge theories due to a complicated structure of the leading approximation.

On the other hand, the applicability of the perturbation theory to the investigation of non-Abelian gauge theories is limited by a deep-euclidean region. In the non-perturbative region of small momenta the physical vacuum of non-Abelian gauge theories obeys the nontrivial structure that is beyond the framework of perturbation theory. In the SDE terms this fact can be understood if one takes into account a radical difference in the properties among the leading approximation equations of the perturbation theory and the original exact equations. The SDE for the generating functional of Green functions are equations in functional derivatives. The leading approximation of the perturbation theory consists in the neglecting of terms with the higher derivatives in this equations (just such terms correspond to an interaction). The leading approximation equations of the perturbation theory have a lower order in comparison with the exact ones, therefore a class of described solutions contracts drastically, and non-perturbative solutions which correspond to the nontrivial physical vacuum practically fall out of the consideration. This feature of SDE in non-perturbative region is noted repeatedly for the simple models (see, for example, [1,2,3]).

In this work a method for the SDE solution of non-Abelian gauge theory which takes into account the terms with higher derivatives (i.e. self-interaction of the non-Abelian fields) *ab ovo* in the leading approximation is proposed. Though we limit ourselves by the simplest extension of the class of SDE solutions, the results are non-trivial: the non-perturbative solution which corresponds to the spontaneous symmetry breaking (with non-Higgs mechanism) is found. It obeys a non-singular behaviour of the propagator in non-perturbative infrared region of small momenta.

An idea of the method consists in the approximation of SDE for the generating functional by equations with "constant" (i.e. independent of sources) coefficients. These

equations have a simple solution as an exponential, which is a foundation for the linear iteration scheme. The method is universal as the perturbation theory is, i.e. it is applicable practically to any model of quantum field theory. For the scalar ϕ^4 theory it has been shown in [4] that the method describes such non-perturbative phenomena as the spontaneous symmetry breaking and the trivialization of scalar theory at $d = 4$.

2. A system of SDE for the generating functional $G(J, \eta)$ of Green functions of non-Abelian gauge theory has the form

$$\mathcal{D}^{ab}_{\nu}(\frac{\delta}{i\delta J})F^{b}_{\nu\mu}(\frac{\delta}{i\delta J})G + \frac{1}{\alpha}\partial_\mu\partial_\nu\frac{\delta G}{i\delta J^a_\nu} + gf^{abc}\frac{\delta}{\delta\bar\eta^c}\partial_\mu\frac{\delta G}{\delta\eta^b} + J^a_\mu G = 0, \tag{1}$$

$$i\partial_\mu\mathcal{D}^{ab}_{\mu}(\frac{\delta}{i\delta J})\frac{\delta G}{\delta\bar\eta^b} + \eta^a G = 0. \tag{2}$$

Here $F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ is a gauge field tensor, $\mathcal{D}^{ab}_\mu(A) = \delta^{ab}\partial_\mu - gf^{abc}A^c_\mu$ is a covariant derivative, f^{abc} are structure constants of a gauge group, $J^a_\mu(x)$ is a source of the gauge field, $\eta^a(x)$ is a source of a ghost field, α is a gauge parameter, g is the coupling constant. We work in the Minkowski space with a metric (1, -1, -1, -1), and $x_\mu y_\mu \equiv g_{\mu\nu}x^\mu y^\nu$ by definition.

The iteration scheme is formulated as follows: the leading approximation is a system of equations with all terms containing the sources J and η omitted. (For the system of eqs. (1) and (2) there are the last terms.) Such system has a simple exponential solution $G_0 = \exp i\{ J * V + \bar\eta * C + \bar C * \eta \}$, where $J * V \equiv \int d\,x\, J^a_\mu V^a_\mu$, etc. When constructing the iteration scheme for the generating functional $G = G_0 + G_1 + \cdots + G_n + \cdots$ the omitted terms JG and ηG should be considered as perturbations, i.e. equations of the iteration scheme are

$$\{\mathcal{D}_\nu(\frac{\delta}{i\delta J})F_{\nu\mu}(\frac{\delta}{i\delta J}) + \frac{1}{\alpha}\partial_\mu\partial_\nu\frac{\delta}{i\delta J_\nu} + gf\frac{\delta}{\delta\bar\eta}\partial_\mu\frac{\delta}{\delta\eta}\}G_n = -J_\mu G_{n-1}, \tag{3}$$

$$i\partial_\mu\mathcal{D}_\mu(\frac{\delta}{i\delta J})\frac{\delta G_n}{\delta\bar\eta} = -\eta G_{n-1}. \tag{4}$$

A solution of eqs. (3)-(4) has the form $G_n = P_n G_0$, where P_n is a polynomial over J and η. Therefore at each step of the iterations we obtain a closed system of equations for coefficient functions of the polynomial P_n, which defines completely the Green functions of the given step. There is no manifest small parameter in the usual sense in this scheme: "smallness" is defined by the circumstance that Green functions are derivatives of the generating functional at $J = \eta = 0$, and it is sufficient for us to know $G(J, \eta)$ near zero, i.e. in the region where the terms neglected are small. At each step of the iteration we approximate the functional G/G_0 by a sum of the polynomials P_n, and a degree of the polynomial arises with each step. As is known, for the ordinary differential equations the scheme of this type is equivalent to the iterations of Volterra-type integral equations and gives a well-convergent expansion. That is why we may hope that this scheme obeys good convergence properties. In any case it is clear that the convergence of this scheme is not worse in comparison with the perturbation theory. The perturbation theory is singular

in the sense of the differential equation theory since the higher derivatives are omitted in the leading approximation. In contrast to the singular perturbation theory the scheme proposed is regular in the sense above. This circumstance gives us a hope to improve of the convergence properties.

For the ultraviolet divergences removing there it is necessary to supplement SDE (1)-(2) and the iteration scheme equations (3)-(4) with the corresponding counterterms. The counterterms are also defined by the iteration procedure: $\delta z = \delta z_0 + \delta z_1 + \cdots$, i.e. it is necessary at each step to take into account the counterterms of the corresponding order.

Let us consider the leading approximation in more detail. As it has been noted above the solution of the leading approximation equations is the linear exponential in the sources. We limit ourselves to the case $C = 0$ and choose the leading approximation in the form

$$G_0 = \exp i\{J * V\}, \tag{5}$$

where V_μ^a is a solution of the "characteristic equation"

$$\mathcal{D}_\nu^{ab}(V)F_{\nu\mu}^b(V) + \frac{1}{\alpha}\partial_\mu\partial_\nu V_\nu^a = 0. \tag{6}$$

The simplest class of solutions of eq. (6) is constant (i.e., independent in the space-time variable) vectors V_μ^a whicn satisfy the condition

$$f^{abc}f^{cdh}V_\nu^b V_\nu^d V_\mu^h = 0. \tag{7}$$

Below we shall consider this class of the leading approximation solutions only. It is convenient to introduce the matrix quantity

$$W_\mu^{ab} = igf^{abc}V_\mu^c. \tag{8}$$

Then the iteration scheme equations have the form of equations for the polynomials P_n

$$\{[\mathcal{D}_\nu(V) + igf\frac{\delta}{\delta J_\nu}][F_{\nu\mu}(\frac{\delta}{i\delta J}) + W_\nu\frac{\delta}{\delta J_\mu} - W_\mu\frac{\delta}{\delta J_\nu}] + \tag{9}$$
$$i[W_\mu, W_\nu]\frac{\delta}{\delta J_\nu} + \frac{1}{\alpha}\partial_\mu\partial_\nu\frac{\delta}{i\delta J_\nu} + gf\frac{\delta}{\delta\bar{\eta}}\partial_\mu\frac{\delta}{\delta\eta}\}P_n = -J_\mu P_{n-1},$$

$$\partial_\mu[\mathcal{D}_\mu(V) + igf\frac{\delta}{\delta J_\mu}]\frac{\delta P_n}{\delta\bar{\eta}} = i\eta P_{n-1}. \tag{10}$$

The solution of the first step equations is

$$P_1 = \frac{1}{2i}J * D * J + i\bar{\eta} * \Delta * \eta. \tag{11}$$

Eqs. (9) and (10) give us equations for $D_{\mu\nu}^{ab}(x-y)$ and $\Delta^{ab}(x-y)$. The equation for $D_{\mu\nu}$ can be simplified essentially by modifying a gauge condition. Instead of the usual

covariant gauge $\partial_\mu A_\mu = 0$ that have been used above it is convenient to use the following gauge condition ("V-gauge")

$$\mathcal{D}_\mu(V)A_\mu = 0. \tag{12}$$

In the case the gauge fixing term in the effective lagrangian of gauge field has the form $\mathcal{L}_{gauge} = -\frac{1}{2\alpha}(\mathcal{D}_\mu(V)A_\mu)^2$, and ghost terms should be changed correspondingly. For the transition into the gauge of eq. (12) in formulae (1)-(4), (6) and (9)-(10) it is sufficient to perform the substitution

$$\partial_\mu \to \mathcal{D}_\mu(V). \tag{13}$$

An essential circumstance is non-changing of the leading approximation condition (7) in the case.

The equations have particularly simple form in the gauge $\alpha = 1$ ("diagonal V-gauge"). Then the equation for $D_{\mu\nu}$ in the momentum space is

$$\{\mathcal{K}^2 g_{\mu\nu} + 2[\mathcal{K}_\mu, \mathcal{K}_\nu]\}\tilde{D}_{\nu\lambda}(k) = -g_{\mu\lambda}, \tag{14}$$

where k is a momentum, and the notation is introduced

$$\mathcal{K}^{ab}_\mu = k_\mu \delta^{ab} - W^{ab}_\mu. \tag{15}$$

In the region of large k eq. (14) goes in the equation for the free propagator in the diagonal gauge, i.e. at $k \to \infty$

$$\tilde{D}_{\mu\nu}(k) \approx -\frac{1}{k^2} g_{\mu\nu}. \tag{16}$$

The equation for the ghost propagator Δ is

$$\mathcal{K}^2 \tilde{\Delta}(k) = -1. \tag{17}$$

In the large k region the propagator $\tilde{\Delta}$ also goes into the free propagator. Therefore the ultraviolet behaviour of the solutions with a nontrivial vacuum vector $\mathbf{V}_\mu$ is the same as for the usual perturbation theory.

Below we restrict ourselves by the case of $SU(2)$ gauge group. It is not difficult to prove that for $SU(2)$ group the leading approximation condition (7) is equivalent to the condition

$$\epsilon^{abc} V^b_\mu V^c_\nu = 0. \tag{18}$$

Eq. (18) gives us $[\mathcal{K}_\mu, \mathcal{K}_\nu] = [W_\mu, W_\nu] = 0$, and the solution of eq.(14) is reduced to the inversion of the matrix $\mathcal{K}^2$:

$$\tilde{D}^{ab}_{\mu\nu}(k) = -g_{\mu\nu}[\frac{p}{p_1}\delta^{ab} + \frac{2(kW^{ab})}{p_1} + (\frac{1}{k^2} - \frac{p}{p_1})\frac{(V^a V^b)}{V^2}], \tag{19}$$

where the following notations are introduced

$$p(k,V) = k^2 + g^2V^2,\ p_1 = p^2 - 4g^2(kV)^2. \tag{20}$$

A solution of the ghost propagator equation (17) is also given by formula (19) (without $g_{\mu\nu}$).

Except eqs. (14) and (17) the first step equations give one more relation that contains a quantity $D_{\mu\nu}(0)$, which should be understood as some regularization. In essence this relation is a condition for the first step counterterms δz_1. (There is no need to introduce the leading approximation counterterms in the case, i.e. $\delta z_0 = 0$). Since the Green functions of the first step are finite, this condition for the counterterms is necessary for ultraviolet divergences removing (renormalization) of the Green functions of the *second step* of the scheme. This peculiarity of the given iteration scheme is displayed here by exactly the same manner as for scalar field theory (see [4]).

3. Let us apply now to a possible physical interpretation of the solutions. At $\mathbf{V}_\mu = 0$ $D_{\mu\nu}$ and Δ are free propagators of gauge and ghost field, and the whole iteration scheme is a reconstructed series of the perturbation theory. At $\mathbf{V}_\mu \neq 0$ the situation is more complicated. It is clear that in this case it is difficult to interpret the function $D_{\mu\nu}$ given by eq.(19) as a propagator of a particle in the Poincaré invariant theory. Let us remind, however, that we have a number $\{G(V)\}$ of solutions of SDE each corresponding some vector $\mathbf{V}_\mu$ satisfying leading approximation condition (7). In other words, a "physical" vacuum is a superposition of V-vacua. We shall exploit the fact for the construction of a Poincaré invariant solution which can be interpreted as a particle propagator, i.e. a function depending on the momentum k and the scalar quantity

$$v^2 = V^2 \equiv V^a_\mu V^a_\mu \tag{21}$$

only. (The quantity v^2 plays a part of an order parameter.) The construction is equivalent in essence to some averaging, i.e. an integration with a measure $d\mu(V)$, since we shall note it by brackets: $< G >= \sum_V G(V)$ etc. In foundation of a definition of this operation we put the following conditions

$$< V^a_\mu >= 0, \quad < V^2 >= v^2. \tag{22}$$

Its necessity for the Poincaré invariant theory is evident. It is also evident that

$$< V^a_\mu V^b_\nu >= \frac{1}{4} v^2 g_{\mu\nu} E^{ab} \tag{23}$$

where $tr\, E = 1$. For a determination of the form of the matrix E^{ab} consider the leading approximation condition (18). Geometrically the condition denotes the collinearity of the vectors $\mathbf{V}_\mu$ in the isotopic space. Consequently at $\mathbf{V}_\mu \neq 0$ there exists a selected direction in the isotopic space. This direction can be chosen as a basic vector, for example $\mathbf{n}_3$. In this basis $V^a_\mu = \delta^{a3} v_\mu$, and $E^{ab} = \delta^{a3}\delta^{b3}$. Therefore at $\mathbf{V}_\mu \neq 0$ the isotopic symmetry is spontaneously broken.

Further calculation is reduced to a definition of the function $f((kV)^2) = p/p_1$. First of all note that at $k \to 0$ $f \to 1/g^2 v^2$, and at $k \to \infty$ $f \to 1/k^2$. These Poincaré invariant properties of f should be, of course, conserved for $< f >$ too.

With eqs.(22)-(24) and its generalizations for an arbitrary monomial in $\mathbf{V}_\mu$ the formula can be proved

$$< (kV)^{2n} >= (k^2v^2)^n \frac{\Gamma(n+1/2)}{(n+1)!\Gamma(1/2)}, \tag{24}$$

which is necessary for the calculation of $< D_{\mu\nu} >$. The result of the calculation in the basis above is the following:

$$< \tilde{D}^{33}_{\mu\nu}(k) >= -g_{\mu\nu}\frac{1}{k^2}, \tag{25}$$

$$< \tilde{D}^{11}_{\mu\nu}(k) >=< \tilde{D}^{22}_{\mu\nu}(k) >= -g_{\mu\nu}\frac{k^2+g^2v^2}{2g^2v^2k^2}\left(1-\sqrt{1-\frac{4g^2v^2k^2}{(k^2+g^2v^2)^2}}\right). \tag{26}$$

(Other isotopic components are equal to zero.) Therefore along the selected isotopic direction the particle propagates as free one, but along other directions a separation of the region of momenta exists, the scale of the separation is the quantity g^2v^2. It is necessary to stress, that both limiting cases above ($k^2 \to 0$ and $k^2 \to \infty$) belong to the region of applicability of the calculations performed which is defined by the condition $\mid 4g^2v^2k^2/(k^2+g^2v^2)^2 \mid< 1$. Consider the question about an analytical continuation. Eq.(26) defines two analytical functions depending on a choice of a branch of the function $\sqrt{z^2}$, but none of them satisfies simultaneously both asymptotical conditions above and, consequently, is not a solution of the problem. Hence one should choose as a solution at large k the branch with the behaviour $1/k^2$, and at small k - another branch which is a constant $1/g^2v^2$. In the points $\pm g^2v^2$ the solution goes from one branch to another, i.e. from the "perturbative" sheet to the "non-perturbative" one. At the point $k^2 = g^2v^2$ the solution is continuous, and at the point $k^2 = -g^2v^2$ has a disconnection. (Note the first step calculations of the iteration scheme do not fix a sign of v^2.) Such unusual features of the solution near the points of the separation are likely to be connected with our limitation of the characteristic equation solutions. Probably an extension of the class of solutions will lead to smoothing the propagator behaviour near the points of separation of perturbative and non-perturbative regions.

In conclusion note that the hypothesis of non-singular infrared behaviour has been recently discussed intensively (see, for example, [5,6,7] and refs. therein).

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