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LOOKING FOR REGULAR PERTURBATIONS

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Abstract

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A method of the solution of Schwinger-Dyson equation for the generating functional, proposed recently in [1], is discussed for ϕ^4_d-theory and the Gross-Neveu model.

Аннотация

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Метод решения уравнения Швингера-Дайсона для производящего функционала, предложенный недавно в [1], обсуждается на примере ϕ^4_d-теории и модели Гросса-Невье.

Introduction

An approximate solution of differential equations is one of the basic methods for the studies of mathematical physics problems.

The differential equation of the quantum field theory is the functional-differential Schwinger-Dyson equation for the generating functional of Green functions (vacuum expectation values of T–product of quantized fields). The most closely explored method for the approximate solution of the Schwinger-Dyson equation is an expansion over degrees of the renormalized coupling constant, for which the leading approximation is a free field solution and an interaction is considered as a perturbation.

Hitherto the coupling constant perturbation theory (CCPT) is the only universal computational method of the quantum field theory. From the point of view of differential equation theory the CCPT is attributed to the type of so-called singular perturbations. This fact defines, to a considerable extent, both the poor convergence properties (the CCPT series is an asymptotic expansion at best) and the limited nature of the field of its applicability. A perturbation is named to be singular if it contains a higher derivative term. Since interaction terms correspond to the higher functional derivative terms of the Schwinger-Dyson equations, the CCPT is singular in this sense for any quantum field model with an interaction.

It is well-known that the sum of any finite number of the CCPT series cannot describe many important physical phenomena such as bound states, dynamical symmetry breaking etc. Moreover, the perturbative solution (the sum of all expansion terms) for the singularly perturbed system can have nothing in common with the true solution of a given problem. (A classical example of such situation is the problem of the flowing of viscous liquid near a boundary.)

These circumstances motivate a search for other schemes of approximate solution of quantum field equations. In this report a method for the solution of Schwinger-Dyson equations proposed recently in [1] is represented. In contrast to the CCPT this method is based on regular perturbations of the Schwinger-Dyson equation for the generating functional, since the term neglected in the leading approximation does not include higher

functional derivatives. This fact allows one to hope for better convergence properties of the expansion in comparison with the CCPT. Still more essential feature is that even the first terms of the expansion describe nonperturbative effects such as trivialization of ϕ_4^4-theory and spontaneous breaking of chiral invariance in the Gross-Neveu model. A systematic character of the proposed expansion allows one to carry out the renormalization program for ultraviolet divergences removing. (This problem often becomes a stumbling-block for nonperturbative approaches based on the truncation of Schwinger-Dyson equations.)

1. General consideration

To elucidate the general idea of the method let us consider an elementary example, namely, the problem of an approximate calculation of a solution $y(x)$ of the ordinary differential equation

$$\lambda y'' + (a^2 - 2x)y' - y = 0 \tag{1}$$

near the point $x = 0$ with the initial condition (normalization condition) $y(0) = 1$.

This problem is really the problem of calculation of Green functions for a zero-dimensional theory with the quadric interaction (see below). In this equation (1) plays part of the Schwinger-Dyson equation for the "generating functional" y of this toy model. The perturbation theory over λ is singular in the above sense, since the leading approximation consists in the neglecting of the higher derivative term. The leading approximation equation is

$$(a^2 - 2x)y_0' - y_0 = 0, \tag{2}$$

and the iteration scheme of the perturbation theory over λ consists in step-to-step solutions of the equations

$$(a^2 - 2x)y_n' - y_n = -\lambda y_{n-1}''. \tag{3}$$

The series of the perturbation theory over λ is the divergent asymptotic expansion. An important effect of the singularity of the perturbation theory over λ is a drastic contraction of the number of solutions under consideration. Really, eq. (1) is the second order equation, and in addition to the normalization condition $y(0) = 1$ a subsidiary initial condition is necessary for the definition of the solution (for example, the condition of $y'(0) = \Delta$ type). But leading approximation equation (2) is the *first* order equation, and its solution is fixed uniquely by the normalization condition. The subsidiary condition is fulfilled for the only partial value $\Delta = \Delta^{pert} = y_0'(0) + y_1'(0) + \cdots$.

An alternative for the perturbation theory over λ can be other iterative scheme that is based on an approximation of eq. (1) near the point $x = 0$ by an equation with constant coefficients. Take as a leading approximation the equation

$$\lambda y_0'' + a^2 y_0' - y_0 = 0. \tag{4}$$

The term $-2xy'$ will be considered as a perturbation. The iteration scheme will consist in step-to step solutions of inhomogeneous equations with constant coefficients

$$\lambda y_n'' + a^2 y_n' - y_n = 2xy_{n-1}'. \tag{5}$$

As basic equation (1) the leading approximation equation (4) is the second order equation and obeys two linearly independent solutions $y_0^{(1)} = e^{\alpha_1 x}$ and $y_0^{(2)} = e^{\alpha_2 x}$, where $\alpha_{1,2}$ are the roots of characteristic equation for eq. (4). Thus the problem to fulfil the second initial condition is absent for any $y'(0)$. The question about a small parameter for the expansion defined by eqs. (4)-(5) arises. There is no manifest small parameter for this expansion , but it is clear intuitively, that the expansion approximates well the exact solution not only for small values of λ. To be more exact, the question about the small parameter should be replaced by the question about a convergence of the expansion. But the convergence of this iteration scheme can be easily proved. Notice, the iteration scheme defined by eqs. (4)-(5) is equivalent to iterations of the second kind Volterra equation with the continuous kernel

$$v(x) = y_0' + \int_0^x K(x,u)v(u)du,$$

where

$$v(x) = y'(x), \quad K(x,u) = \frac{2u}{\lambda(\alpha_1 - \alpha_2)}(\alpha_1 e^{\alpha_1(x-u)} - \alpha_2 e^{\alpha_2(x-u)}).$$

The convergence of the iteration of this equation is fulfilled by the textbook theorem.

So, this regular-perturbation expansion possesses two important advantages in comparison with the CCPT: a number of permitted solutions arises, and the expansion is convergent in contrast to the asymptotic expansion over λ.

Let us go to the field theory. Consider the theory of a scalar field $\phi(x)$ in the Euclidean space E_d with the action

$$\mathrm{S}(\phi) = \int dx\{\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 + \lambda\phi^4\} \tag{6}$$

and with the generating functional of Green functions (vacuum expectation values)

$$G(\eta) = c\int D\phi \exp\{-S + \phi\eta\phi\}. \tag{7}$$

Here $\eta(x,y)$ is a bilocal source. The nth derivative of G over η with the source being switched off is the $2n$-point Green function. The constant c is defined by the condition of the normalization of the generating functional $G(0) = 1$.

The Schwinger-Dyson equation for the generating functional $G(\eta)$ is a corollary from the translational invariance of the functional integration measure: from the identity

$$0 = \int D\phi \frac{\delta}{\delta\phi(x)}(\phi(y)\exp\{-S + \phi\eta\phi\}) \tag{8}$$

it is easy to get taking into account the above definitions the Schwinger-Dyson equation in functional derivatives over the source η

$$4\lambda\frac{\delta^2 G}{\delta\eta(y,x)\delta\eta(x,x)} + (m^2 - \partial^2)\frac{\delta G}{\delta\eta(y,x)} - 2\int \eta(x,u)\frac{\delta G}{\delta\eta(y,u)}du - \delta(x-y)G = 0. \tag{9}$$

At $d = 0$ (zero-dimensional theory, or "single-mode approximation") the functional derivatives transform into usual ones, and eq. (9), after obvious redesignations, reduces to the ordinary differential eq. (1). At $d = 1$ the model corresponds to the quantum-mechanical anharmonic oscillator. At $d \geq 2$ (field theory) for the cancellation of ultraviolet divergences the appropriate counterterms should be included in the action. The Schwinger-Dyson equation for the theory with counterterms has the form of eq. (9) with the substitution

$$\lambda \to \lambda + \delta\lambda, \quad m^2 \to m^2 + \delta m^2, \quad \partial^2 \to (1 + \delta z)\partial^2,$$

where $\delta\lambda, \delta m^2$ and δz are correspondingly counterterms of coupling, mass and wave function renormalizations. Let apply to Schwinger-Dyson equation (9) the same idea about the approximation by an equation with "constant" (i.e., independent from η) coefficients. As the leading approximation equation we will consider the equation

$$4\lambda \frac{\delta^2 G_0}{\delta\eta\delta\eta} + (m^2 - \partial^2)\frac{\delta G_0}{\delta\eta} - G_0 = 0, \tag{10}$$

and the term $2\eta\frac{\delta G}{\delta\eta}$ (that contains the source η manifestly) should be treated as a perturbation. Since Green functions are the derivatives of $G(\eta)$ in zero and only the behaviour of G near $\eta = 0$ is essential. Such an approximation seems to be acceptable. The iteration procedure for the generating functional

$$G = G_0 + G_1 + \cdots + G_n + \cdots$$

consists in the step-to-step solution of the equations

$$4\lambda \frac{\delta^2 G_n}{\delta\eta\delta\eta} + (m^2 - \partial^2)\frac{\delta G_n}{\delta\eta} - G_n = 2\eta\frac{\delta G_{n-1}}{\delta\eta}. \tag{11}$$

The solution of the leading approximation equation (10) is the functional

$$G_0 = \exp\{\int dxdy\eta(y,x)\,\triangle_0\,(x-y)\}, \tag{12}$$

where $\triangle_0$ is a solution of the "characteristic" equation

$$4\lambda\,\triangle_0\,(0)\,\triangle_0\,(x-y) + (m^2 - \partial^2)\,\triangle_0\,(x-y) = \delta(x-y). \tag{13}$$

At $d \geq 2$ the quantity $\triangle_0(0)$ must be considered as some regularization.

Equation (13) looks as self-consistency equation, but differs in the coefficient at λ: in the self-consistency equation the coefficient is three times greater. In this sense equation (13) is more similar to the equation for the propagator in the leading approximation of the $1/N$-expansion. Certainly, the similarity is completely superficial, since the principle of the construction of the approximation scheme is different.

The solution of equation (13) is the free propagator $\triangle_0 = (\mu^2 - \partial^2)^{-1}$ with the renormalized mass $\mu^2 = m^2 + 4\lambda \triangle_0(0)$. The quantity $\triangle_0(0)$ is defined from the self-consistency condition. The propagator is the first derivative of $G(\eta)$ over the source η : $\triangle = \frac{\delta G}{\delta \eta} |_{\eta=0}$. As can be easily seen, it is simply $\triangle_0$ for the leading approximation.

Notice, that all the higher Green functions of the leading approximation starting with the four-point function $G^4 = \frac{\delta^2 G}{\delta \eta^2} |_{\eta=0}$ do not possess the correct connected structure and, correspondingly, the complete bose-symmetry. The correct connected structure and other consequences of bose-symmetry (e.g., crossing etc.) will be restored in consecutive order at following steps of the iteration scheme. Such a peculiarity of the iteration scheme is originated by the bilocal source and is not something exceptional : as is well-known, the similar phenomenon appears also in constructing the $1/N$-expansion in the bilocal source formalism.

In the general case, the solution of equation for the n-th step of the iteration scheme is the functional $G_n = P_n(\eta)G_0$, where P_n is a polynomial in η of a degree $2n$. Therefore at the n-th step the computation of Green functions reduces to solving a system of $2n$ linear integral equations.

A solution of the first step equation is $G_1 = P_1(\eta)G_0$ where $P_1 = \frac{1}{2}F\eta^2 + \triangle_1\eta$. Eq.(11) at $n = 1$ gives us a system of equations for F and $\triangle_1$. Equations for F and $\triangle_1$ are simple linear integral equations. The exact form of solutions of these equations see in [1]. At $\lambda \to 0$ the first step propagator reproduces correctly the first term of the usual CCPT.

2. ϕ^4_d-theory and Gross–Neveu model

At $d = 1$ the model with action (6) describes the quantum-mechanical anharmonic oscillator. Ultraviolet divergences are absent, quantities of $\triangle_0(0)$ type are finite and the above formulae are applied directly for the computation of Green functions.

To calculate a ground state energy E one can use the well-known formula

$$\frac{dE}{d\lambda} = G^4(0,0,0,0),$$

where G^4 is the four-point (or two-particle) function. Integrating the formula with a boundary condition $E(\lambda = 0) = m/2$ taken into account, one can calculate the ground state energy for all values of the coupling (see [1]).

At $\lambda \to 0$ the first step calculation reproduces the perturbation theory up to the second order. At $\lambda \to \infty$: $E = \epsilon_0 \lambda^{1/3} + O(\lambda^{-1/3})$, and $\epsilon_0 = 0.756$. The coefficient ϵ_0 differs by 13% from the exact numerical one $\epsilon_0^{exact} = 0.668$. At $\lambda/m^3 = 0.1$ the result of the calculation differs from the exact numerical one by 0.8% and at $\lambda/m^3 = 1$ differs by 6.3%. Therefore, the first step calculations approximate the ground state energy for all values of λ with the accuracy that varies smoothly from 0 (at $\lambda \to 0$) to 13% (at $\lambda \to \infty$).

At $d \geq 2$ action (6) should be added by counterterms for the elimination of ultraviolet divergences. There is no need to add a counterterm of wave function renormalization for

the leading approximation, and the equation of the leading approximation will be

$$4(\lambda+\delta\lambda_0)\frac{\delta^2 G_0}{\delta\eta\delta\eta}+(\delta m_0^2+m^2-\partial^{\,2})\frac{\delta G_0}{\delta\eta}-G_0=0. \tag{14}$$

At $n\geq 1$ the counterterms $\delta\lambda_n$, δm_n^2 and δz_n should be considered as perturbations. Therefore, the corresponding terms should be added to the r.h.s. of equation (11). So, the first step equation will be

$$\begin{aligned}4(\lambda+\delta\lambda_0)\frac{\delta^2 G_1}{\delta\eta\delta\eta}+(\delta m_0^2+m^2-\partial^{\,2})\frac{\delta G_1}{\delta\eta}-G_1=\\ =2\eta\frac{\delta G_0}{\delta\eta}-\delta m_1^2\frac{\delta G_0}{\delta\eta}+\delta z_1\partial^2\frac{\delta G_0}{\delta\eta}-4\delta\lambda_1\frac{\delta^2 G_0}{\delta\eta\delta\eta}.\end{aligned} \tag{15}$$

For the super-renormalizable theory ($d=2$ and $d=3$) it is sufficient to add counterterms of mass renormalization and wave function renormalization, i.e. $\delta\lambda_n=0$ for all n. The normalization condition on the physical renormalized mass μ^2 gives us a counterterm of the mass renormalization in the leading approximation. This counterterm diverges logarithmically at $d=2$ and linearly at $d=3$. The counterterm δz_1 is finite at $d=2,3$. The counterterm δm_1^2 diverges as that of the leading approximation does, namely, logarithmically at $d=2$ and linearly at $d=3$.

At $d=4$ besides the renormalizations of the mass and the wave function a coupling renormalization is necessary. Due to the presence of the counterterm $\delta\lambda$ the normalization condition on the renormalized mass μ^2 for the leading approximation becomes the connection between counterterms δm_0^2 and $\delta\lambda_0$. Counterterm $\delta\lambda_0$ (and, consequently, δm_0^2) will be fixed at the *following* step of the iteration scheme.

A solution of the equation for the four-point function F at $d=4$ diverges logarithmically, and a renormalization of the coupling is necessary. The equation for F contains the counterterm $\delta\lambda_0$ only. Therefore by defining a renormalized coupling λ_r as a value of the amplitude in a normalization point we obtain the counterterm of the coupling renormalization $\delta\lambda_0$ and the renormalized amplitude. Taking the renormalization of the two-particle amplitude in such a manner, one can solve the equation for $\triangle_1$ and renormalize the mass operator in correspondence with the general principle of normalization on the physical mass. But in four-dimensional case one gets an essential obstacle. At the regularization removing, $\delta\lambda_0\to-\lambda$, and the coefficient $\lambda+\delta\lambda_0$ in the leading approximation equation (14) vanishes. The same is true for all the subsequent iterations. The theory is trivialized. One can object that an expression

$$(\lambda+\delta\lambda_0)\cdot\frac{\delta^2 G}{\delta\eta(y,x)\delta\eta(x,x)} \tag{16}$$

is really an indefinite quantity of $0\cdot\infty$ type, and the renormalization is, in the essence, a definition of the quantity. But it does not save a situation *in this case* since the renormalized amplitude possesses a nonphysical singularity in a deep-euclidean region (it is a well-known Landau pole). The unique noncontradictory possibility is a choice $\lambda_r\to 0$ at

the regularization removed. This is the trivialization of the theory again. This trivialization appears almost inevitably in an investigation of ϕ_4^4-theory beyond the CCPT and is a practically rigorous result. Notice, that contrary to the CCPT which is absolutely nonsensitive to the triviality of the theory, the method proposed leads to the trivialization already at the first step.

As is well-known the best remedy for the triviality is an asymptotic freedom. In the asymptotically free theory these nonphysical singularities do not appear. As an example let us consider the Gross–Neveu (GN) model with the proposed method. The GN model is the model of spinor field $\psi(x)$ in the two-dimensional Minkovski space M_2 with the action

$$\mathrm{S}(\psi) = \int d^2x(\bar{\psi}_j i\hat{\partial}\psi_j + \frac{\lambda}{2N}(\bar{\psi}_j\psi_j)^2). \tag{17}$$

Here $j = 1, \ldots, N$ is a flavor index.

GN model (17) is asymptotically free and possesses a self-consistent ultraviolet behaviour. In addition, the model obeys the discrete chiral symmetry and is an excellent theoretical laboratory for the study of the phenomenon of spontaneous symmetry breaking.

The scheme considered above is based essentially on the bilocality of the source. Since the bilocal source is connected with $2n$-point functions only, the method cannot be applied in its present form to a *scalar* theory with spontaneous symmetry breaking, when $< 0 \mid \phi \mid 0 > \neq 0$. For a description of the spontaneous symmetry breaking in the scalar theory it is necessary to modify the scheme by switching on a single source (see [1]). However, to investigate the spontaneous breaking of the chiral symmetry in a *spinor* theory with four-fermion interaction this scheme is quite applicable since the bilocal source corresponds to a necessary type of fluctuations in the case.

The Schwinger-Dyson equation for the generating functional G of the GN model has the form [1]

$$\frac{(\lambda+\delta\lambda)}{N}\frac{\delta}{\delta\eta}\mathrm{tr}\frac{\delta G}{\delta\eta} + (1+\delta z)i\hat{\partial}\,\frac{\delta G}{\delta\eta} - \eta\frac{\delta G}{\delta\eta} + G = 0. \tag{18}$$

Here $\eta \equiv \eta_{\alpha\beta}^{jj'}(x,y)$ is a bilocal fermion source with two spinor indices α, β and two flavor indices j, j'; $\delta\lambda$ and δz are counterterms of the renormalization of the coupling and wave function. As above, let us approximate equation (18) by an equation with "constant" coefficients, i.e. by the equation without next-to-last term $\eta\frac{\delta G}{\delta\eta}$, which will be considered as a perturbation.

The leading approximation equation has the solution

$$G_0 = \exp\mathrm{tr}(\eta S_0), \tag{19}$$

where the leading approximation propagator S_0 is the free propagator $S_0^{jj'} = \delta^{jj'}(\mu - \hat{p})^{-1}$ with the renormalized mass

$$\mu = (\lambda + \delta\lambda_0)\frac{1}{iN}\mathrm{tr}\, S_0(0). \tag{20}$$

[1] The results below have been obtained in collaboration with P.A. Saponov.

The expression in r.h.s. of eq. (20) as that of eq. (16) is an indefinite quantity of $0 \cdot \infty$ type. To definite this quantity the renormalization of the two-particle amplitude is necessary. The two-particle amplitude is defined at the solution of the first step equations. These equations as those of the scalar theory are simple linear integral equations. The renormalization of the first step amplitude defines a behaviour of the counterterm $\delta\lambda_0$. At the regularization removing $\delta\lambda_0 \to -\lambda$ as in the scalar case, nevertheless nonphysical singularities do not appear. As a result, the r.h.s. of eq. (20) has a finite limit at the regularization removed, and eq. (20) becomes an equation for the dynamical fermion mass. This equation has the trivial solution $\mu = 0$ corresponding to the infrared-unstable symmetrical phase, and a nontrivial solution corresponding to the dynamical breaking of the chiral symmetry of the GN model. The nontrivial solution at small λ is

$$\mu \sim \exp(-\frac{\pi}{\lambda}), \tag{21}$$

which exactly corresponds to the result of $1/N$-expansion.

Thus the presented method of approximate solution of the Schwinger-Dyson equations describes successfully the nonperturbative effects in the models considered. The method is quite universal in the following sense: the general idea of the method about the approximation of the Schwinger-Dyson equation by an exactly soluble equation with "constant" coefficients can be applied to any quantum field model with any types of sources. Of course concrete calculations only can give an answer if the method is able to provide with nontrivial nonperturbative information for more complicated models. However, the very first results of its application are quite optimistic ones.

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