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ON SCALAR PRODUCT
IN RELATIVISTIC QUANTUM MECHANICS

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Abstract

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It is shown that the scalar product in the physical subspace of a relativistic quantum mechanical system could be constructed making use of the Faddeev-Popov procedure with respect to gauge group of time reparametrization.

Аннотация

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Показано, что скалярное произведение для векторов из физического подпространства релятивистских квантовых систем может быть построено с помощью процедуры Фаддеева-Попова по отношению к калибровочной группе репараметризации времени.

The quantization procedure of mechanical system with the first class constraints according to Dirac theory [1] could be performed in two ways. The former which is more close to the conventional one consists in adding to the set of the first class constraints their gauge fixing conditions and introducing independent coordinates on the reduced phase space. After that the quantization postulate is introduced in terms of the reduced symplectic structure or in other words for Dirac brackets. This way works fairly well for e.g. gauge field theories where its shortcomings are unessential. The latter originates from the fact that the independent variables on the reduced phase space often have very complicated transformation law with respect to the Lorentz group and one has to pay special attention to the ordering problem and definition of quantum Poincare generators.

Another way of quantization of gauge invariant system, as was also pointed out by Dirac, consists in quantization of coordinates of extended phase space and constructing the representation of fundamental commutation relations in extended space of states. The physical subspace of this extended space is formed by the vectors, invariant under the action of gauge group whose generators are quantum constraints. In this case we do not face the necessity to introduce gauge fixing conditions for operators. But the quantization procedure is not completed at this stage. The matter is that the scalar product that exists in the extended space of states couldn't be reduced to physical subspace because the latter contains only vectors invariant with respect to gauge group and integration in this scalar product automatically will also include integration over infinite volume of this gauge group, making each scalar product of physical vectors infinite. This situation is familiar to us from the theory of gauge fields and the correct answer consists in eliminating integration over gauge group with the help of Faddeev-Popov procedure [2]. The definition of scalar product in the physical subspace along this way was suggested in [3] for the case when the gauge group generators are linear functions of momenta. For more complicated situations, e.g. when the gauge group generators are quadratic in momenta, the corresponding Faddeev-Popov determinant does not commute with gauge fixing δ-function and we need specify some ordering of operators which define the correct scalar product for physical vectors.

The situation when the gauge group generators are a quadratic function of the momenta is common for every relativistic particle and extended relativistic object if we formulate the theory in manifestly covariant way. This quadratic constraint — the mass shell condition is the generator of time reparametrization. We consider first the example of point relativistic particle, the simplest object that is described by the minimal set of variables : particle's coordinate $x(\tau)$ and its velocity $\dot{x}(\tau)$. Here the τ is an arbitrary parameter that numerates the points on the world line. The action of the point particle should be invariant with respect to Lorentz transformation:

$$\begin{aligned} \delta x^\mu &= \omega^\mu_\nu x^\nu, \\ \omega_{\mu\nu} &= -\omega_{\nu\mu} \end{aligned} \tag{1}$$

and also it should depend only on the world line of the particle but not on its parameterization. These requirements fix the action up to a numerical factor m - the mass of the particle:

$$A = m \int d\tau \sqrt{\dot{x}^2(\tau)}. \tag{2}$$

The canonical momentum p_μ, conjugated to the coordinate x_μ is given by:

$$p_\mu = \frac{\delta A}{\delta \dot{x}^\mu(\tau)} = m \frac{\dot{x}_{mu}(\tau)}{\sqrt{\dot{x}^2(\tau)}}. \tag{3}$$

Due to degeneracy of the Lagrangian we can't express all components of velocity $\dot{x}_\mu$ via p_μ, in terms of canonical variables i.e. the components of momentum are not independent, but are subject to constraint:

$$\phi(\tau) = p^2 - m^2 = 0. \tag{4}$$

The function $\phi(\tau)$ is called the primary constraint. The canonical hamiltonian H_c vanishes because the lagrangian is the homogeneous function of velocities of the first degree, which, in turn, is the consequence of reparametrization invariance of the action:

$$H_c = p_\mu x^\mu - L = 0. \tag{5}$$

According to general theory [1], the role of the generator of evolution is played by constraint (4)

$$H = a(\tau)\phi(\tau), \tag{6}$$

where the Lagrange multiplier $a(\tau)$ is a smooth, positive function.

Quantization of this system is straightforward. The canonical Poisson brackets

$$\{x_\mu, p_\nu\} = g_{\mu\nu} \tag{7}$$

become commutation relations for operators x_μ and p_μ

$$[x_\mu, p_\nu] = -i g_{\mu\nu}, \tag{8}$$

which act in the extended space of states $\mathcal{H}$. The physical subspace $\mathcal{H}_{ph} \subset \mathcal{H}$ consists of the vectors, satisfying the quantum constraint condition:

$$\phi\Psi_{ph} = (p^2 - m^2)\Psi_{ph} = 0, \tag{9}$$

which is apparently the Klein-Gordon equation.

It is a general property of the manifestly covariant formulation of quantum theory of any relativistic mechanical system that the set of the first class constraints included the mass shell condition in quantum theory defines a physical subspace in an extended space of state, where we construct the representation of fundamental commutation relations. This extended space of states is yet a linear vector space. Now we have to endow it with some scalar product, with respect to which the operators, that represent the fundamental variables will be formally hermitian. Apparently, for our case this scalar product in $x-$ representation is given by

$$< \Psi_1\Psi_2 >= \int d^4x\Psi_1^*(x)\Psi_2(x). \tag{10}$$

With respect to this scalar product the operators x_μ and p_μ are self-adjoint in the extended Hilbert space $\mathcal{H}$. Further we must consider the reduction of this scalar product for the vectors belonging to the physical subspace. This procedure sheds light on the origin of the so called " Klein-Gordon scalar product" and that will be discussed now is very important for each quantum relativistic system. The matter is that scalar product (10) in the extended Hilbert space does not exist for the vectors which belong to the physical subspace $\mathcal{H}_{ph}$. The reason for that is the invariance of physical vectors under the gauge transformation, generated by ϕ. In other words, in integral (10) the integration is performed over the whole $x-$ space, including automatically the orbits of the gauge group, that has an infinite volume. Indeed, the general solution of (9) has the following form:

$$\Psi_{ph}(x) = \int d^4pe^{ipx}\Psi_{ph}(p) = \int d^4pe^{ipx}\delta(p^2 - m^2)\psi(p) \tag{11}$$

Note that the factor $\delta(p^2 - m^2)$ in the p-representation of state $\Psi_{ph}(p)$ arises due to the invariance of the physical states under transformations, generated by ϕ. Taking two states from the physical subspace and substituting it into the scalar product (10) we shall get

$$< \Psi_{1ph}, \Psi_{2ph} >= (2\pi)^4 \int d^p\Psi_{1ph}^*(p)\Psi_{2ph}(p), \tag{12}$$

so in the r.h.s. the integrand contains two factors $\delta(p^2 - m^{2)}$, which make the scalar product infinite. It is exactly the same situation which arises in the functional integral approach in the gauge fields theory and the cure for this decease is the Faddeev-Popov procedure [2] , which reduces the integration over the whole region to the integration over the equivalence classes with respect to the action of the gauge group. Intuitively it is clear that this process should cancel one of the factor $\delta(p^2 - m^2)$ out of the integrand. That is very simple in the $p-$ representation, but we must derive the general formula for scalar product, independent of the representation of the wave functions. To do that according to Faddeev-Popov we must insert into the integral noninvariant operator which will remove the integration over the infinite gauge group orbit, the operator

$$A = \Delta\delta(nx - t), \tag{13}$$

where n_μ is c-number vector (though we can consider also the case when n_μ depends on the dynamical variable e.g. p_μ), $n^2 \geq 0$. The reader, familiar with the gauge field theory recognized in (13) the δ function of the gauge fixing condition and the Faddeev-Popov determinant Δ. Usually the latter is determined by the equation

$$\Delta \int_{-\infty}^{\infty} d\alpha \delta(nx^\alpha - t) = 1, \tag{14}$$

where the x_μ^α is the gauge transformation of operator x_μ:

$$x_\mu^\alpha = e^{-i\alpha\phi/2} x_\mu e^{i\alpha\phi/2} = x_\mu + \alpha p_\mu. \tag{15}$$

This definition as well as formula (14) does work in the case when the constraint, that generates the gauge transformation is a linear function of the momenta. In our case ϕ is the quadratic function of the momenta and this naive prescription is not valid. The reason for that is noncommutativity of x_μ and x_μ^α. The naive answer for the Faddeev-Popov determinant which one can obtain omitting noncommutativity gives $\Delta = np$, but this determinant does not commute with $\delta(nx - t)$, the situation was not encountered in the gauge fields theory and we have to solve the ordering problem. The correct answer we obtain only with symmetric ordering:

$$\frac{1}{2}\Big[\Delta \int_{-\infty}^{\infty} d\alpha \delta(nx^\alpha - t) + \int_{-\infty}^{\infty} d\alpha \delta(nx^\alpha - t)\Delta\Big] = 1, \tag{16}$$

with $\Delta = np$. So the correct form of noninvariant operator (13) is the following:

$$A = \frac{1}{2}\Big[np\delta(nx - t) + \delta(nx - t)np\Big]. \tag{17}$$

One can get convinced in that through the following simple consideration. The δ–function of an operator O is defined by the Fourier representation:

$$\delta(O) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy e^{iyO}, \tag{18}$$

so the gauge transformed A–operator is given by

$$\begin{aligned} A^\alpha &= \frac{1}{2}\Big[np\delta(nx - t + \alpha np) \quad + \quad \delta(nx - t + \alpha np)np\Big] = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \frac{1}{2}\Big[npe^{iy(nx-t+\alpha np)} \quad + \quad e^{iy(nx-t+\alpha np)}np\Big]. \end{aligned} \tag{19}$$

Using commutation relations (11) one can prove that

$$e^{iy(nx-t+\alpha np)} = e^{iy(nx-t)} e^{i\alpha(ynp - \frac{1}{2}y^2 n^2)}, \tag{20}$$

and

$$np e^{iy(nx-t+\alpha np)} = e^{iy(nx-t+\alpha np)}(np - yn^2). \tag{21}$$

Substitution of (20) and (21) into (19) gives

$$A^\alpha = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dy e^{iy(nx-t)} \Big(np - \frac{1}{2} yn^2\Big) e^{i\alpha(ynp - \frac{1}{2}y^2 n^2)}. \tag{22}$$

Now we can fulfill the integration of A^α over α:

$$\begin{aligned} \int\limits_{-\infty}^{\infty} d\alpha A^\alpha &= \int\limits_{-\infty}^{\infty} dy e^{iy(nx-t)}(np - \frac{1}{2}yn^2)\delta\Big(y(np - \frac{1}{2}yn^2)\Big) = \\ &= \int\limits_{-\infty}^{\infty} dy e^{iy(nx-t)} \delta(y) = 1 \end{aligned} \tag{23}$$

The role of the symmetric ordering of operators in (17) now becomes evident — it leads to cancellation of the second zero of argument of $\delta-$ function in (26).

Inserting (17) into scalar product (10) we finally get the scalar product in the physical subspace, valid in any representation:

$$\Big(\Psi_{1ph}\Psi_{2ph}\Big) \equiv < \Psi_{1ph} A \Psi_{2ph} > . \tag{24}$$

In particular, choosing $n_\mu = (1, \vec{0})$ we obtain the known " Klein-Gordon scalar product"

$$\Big(\Psi_{1ph}\Psi_{2ph}\Big) = -i \int d^3x \Psi_{1ph}(x) \overset{\leftrightarrow}{\partial_0} \Psi_{2ph}(x)|_{x^0=t}. \tag{25}$$

Note, that the operator $\overset{\leftrightarrow}{\partial_0}$ in this scalar product is the heir of the Faddeev-Popov determinant. Now we can check that in p-representation this A operator does cancel one $\delta(p^2 - m^2)$ in integral (11).

By explicit calculations one can prove that scalar product (24) does not depend on the gauge fixing parameter t:

$$\frac{d}{dt}\Big(\Psi_{1ph}\Psi_{2ph}\Big) = < \Psi_{1ph} \frac{d}{dt} A \Psi_{2ph} > = 0. \tag{26}$$

Apparently not every operator may be defined on the $\mathcal{H}_{ph}$. In addition to common concepts of the operator theory on the Hilbert space (or rigged Hilbert space) we must differentiate the operators which act in the extended space $\mathcal{H}$ and which leave $\mathcal{H}_{ph}$ invariant. In the case of the point particle, the operator p_μ commutes with quantum constraint ϕ and therefore transforms the physical vector into another physical vector:

$$\phi\Big(p_\mu \Psi_{ph}\Big) = p_\mu \phi \Psi_{ph} = 0. \tag{27}$$

We shall call the operators, that leave the physical subspaces invariant the physical operators. Another example of the physical operators provides the Lorentz angular momentum $M_{\mu\nu}$:

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \tag{28}$$

The important and nontrivial property of the physical operators is that in spite of the fact that they do not commute with operator A, that defines the scalar product in the physical subspace, the following property is valid:

$$\Big(\Psi_{1ph} F \Psi_{2ph}\Big) = \Big(F^+ \Psi_{1ph} \Psi_{2ph}\Big), \tag{29}$$

where F and F^+ denote a physical operator and its hermitean conjugation. The reason for this property is that the commutator of any physical operator with A proportional to $\frac{d}{dt}A$ and due to (26) this operator has a vanishing matrix element on the physical subspace.

A more complicated situation arises with the operator x_μ. Apparently the physical vectors belong to the domain of this operator, but acting on it, x_μ transforms it into nonphysical ones:

$$\phi\Big(x_\mu \Psi_{ph}\Big) = x_\mu \phi \Psi_{ph} + [\phi, x_\mu]\Psi_{ph} = 2ip_\mu \Psi_{ph} \neq 0, \tag{30}$$

therefore the operator x_μ is not physical itself. Here originate the difficulties with the notion of localizability of quantum relativistic particle. Indeed, if we have not the hermitian operator on the physical subspace, whose eigenvalues are x_μ or x_i, we can't speak about pure states of this operator. The extensive and very profound discussion of this subject was presented by Pryce [4] and Wigner and Newton [5].

The same procedure holds true also for the Dirac particle, whose classical theory was suggested in the paper of Berezin and Marinov [6]. We will not present here the whole discussion and details of this paper. For our purposes we need only the canonical formalism of the theory. The phase space of the Dirac particle contains, apart from usual canonical coordinate and momentum $x_{\mu,}, p_\mu$ also the Grassmannian variables $\xi_{\mu,}, \xi_5$ with Poisson brackets:

$$\Big\{\xi_\mu, \xi_\nu\Big\}^D = ig_{\mu\nu}.\Big\{\xi_5, \xi_5\Big\} = -i. \tag{31}$$

The set of first class constraints on canonical variables are the following:

$$L = p^2 - m^2, \lambda = p\xi - m\xi_5. \tag{32}$$

These constraints form the simplest nontrivial graded algebra with respect to Poisson brackets (7), (31) and are the generators of gauge transformation with usual bosonic parameter and with the Grassmannian one. Quantization of the Dirac particle consists in postulating commutation relations for bosonic and anticommutation relations for Grassmannian variables:

$$\begin{aligned} [x_\mu, p_\nu] &= -ig_{\mu\nu}, \\ [\xi_\mu, \xi_\nu]_+ &= g_{\mu\nu}, \\ [\xi_5, \xi_5]_+ &= -1, \end{aligned} \tag{33}$$

and again the constraints convert into conditions on the physical states

$$\begin{aligned} \left(p^2 - m^2\right) \quad \Psi_{ph} &= 0, \\ \left(p\xi - m\xi_5\right) \quad \Psi_{ph} &= 0. \end{aligned} \tag{34}$$

To satisfy algebra (33) we set

$$\begin{aligned} \xi_\mu &= \frac{1}{\sqrt{2}}\gamma_5\gamma_\mu, \\ \xi_5 &= \frac{1}{\sqrt{2}}\gamma_5, \end{aligned} \tag{35}$$

then the last equation takes the form of Dirac equation

$$\left(p^\mu\gamma_\mu - m\right)\Psi_{ph} = 0 \tag{36}$$

As in the case of the point scalar particle, we shall define first the Lorentz invariant scalar product in the extended Hilbert space $\mathcal{H}$:

$$< \Psi_1\Psi_2 >= \int d^4x\bar{\Psi}_1(x)\Psi_2(x), \tag{37}$$

where $\bar{\Psi}_1(x)$ is the usual Dirac-conjugated spinor:

$$\bar{\Psi}_1(x) = \Psi_1^+(x)\gamma^0. \tag{38}$$

With respect to this scalar product all our variables, x_μ, p_μ and γ's are hermitean or anti-hermitean. As in the case of the scalar point particle, this scalar product does not exist for the states belonging to the physical subspace $\mathcal{H}_{ph}$, and again the divergence of the scalar product of physical states arises due to their invariance under gauge group transformations. The gauge group of spin particle corresponds to the algebra:

$$\lambda^2 = L, \qquad [\lambda, L] = 0 \tag{39}$$

where λ and L were defined in (32). Apart from the usual reparametrization it contains transformations with the Grassmannian parameter generated by an odd constraint λ and we can expect that the divergence of scalar product in the case of spin particle will be "stronger" than in the case of spinless one. But the matter is that the integration over Grassmannian variable never diverges [7], therefore the divergence of the scalar product of physical vectors will be produced only by integration over the subgroup, generated by L, i.e. is the same as in spinless case and may be eliminated with the same Faddeev-Popov operator A:

$$\begin{aligned} \left(\Psi_{1ph}, \Psi_{2ph}\right) &\equiv < \Psi_{1ph}, A\Psi_{2ph} >= \\ &= \int d^4x\bar{\Psi}_{1ph}(x)\frac{1}{2}\left[np\delta(nx - t) + \delta(nx - t)np\right]\Psi_{2ph}. \end{aligned} \tag{40}$$

This formula seems to be different from the usual scalar product for Dirac particle [8], but making use of Dirac equation (36) we can express the action of operator np on physical vectors by the l.h.s. of the following equation:

$$pn\Psi_{ph} = m\gamma n\Psi_{ph} - \gamma np_{\perp}\gamma\Psi_{ph}, \qquad p_{\perp}^{\mu} = p^{\mu} - n^{\mu}pn/n^2. \tag{41}$$

Substituting (41) into (40) we arrive at

$$\begin{aligned}\Big(\Psi_{1ph}, \Psi_{2ph}\Big) &= 2m\int d^4x\delta(xn-t)\bar{\Psi}_{1ph}(x)n\gamma\Psi_{2ph}(x) - \\ &- \int d^4xi\frac{\partial}{\partial x_{\perp}^{\mu}}\Big[\bar{\Psi}_{1ph}(x)\gamma n\gamma^{\mu}\Psi_{2ph}(x)\delta(xn-t)\Big].\end{aligned} \tag{42}$$

The last term in (42) vanishes as the integral of the total derivative and we finally obtain:

$$\Big(\Psi_{1ph}, \Psi_{2ph}\Big) = 2m\int d^3x_{\perp}\bar{\Psi}_{1ph}(x)n\gamma\Psi_{2ph}(x), \tag{43}$$

which for $n_{\mu} = (1, \vec{0})$ coincides with thee familiar scalar product for the Dirac particle:

$$\Big(\Psi_{1ph}, \Psi_{2ph}\Big) = 2m\int d^3x\Psi_{1ph}^{+}(x)\Psi_{2ph}(x), \tag{44}$$

A more complicated relativistic mechanical system e.g. a relativistic oscillator contains apart from the mass shell constraint also a constraint linear in canonical momenta which involves no difficulties [3]. The same situation occurs in any system of Komar-Todorov type [9].The present approach is not applicable to the case of covariant quantization of the relativistic string because in this case the quantum constraints do not form the first class algebra due to anomaly [10]. It will be interesting to apply the formalism in the case of canonical gravity where one of constrains is also quadratic in canonical momenta and, besides, is a nonlinear function of coordinates.

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