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NEWTON RELATIONS FOR QUANTUM MATRIX ALGEBRAS OF RTT-TYPE

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Abstract

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A quantum version of Newton relations is found for the matrix T of the generators of the RTT-algebra.

Аннотация

Пятов П.Н., Сапонов П.А. Соотношения Ньютона для квантовых алгебр RTT-типа: Препринт ИФВЭ 96-76. – Протвино, 1996. – 5 с., библиогр.: 8.

Получены квантовые аналоги соотношений Ньютона для матрицы T генераторов алгебры RTT-типа.

The two kinds of quantum matrix algebras are known in the quantum group theory. One of them is the *Reflection Equation Algebra* (REA) (see [1] and references therein). The other is the algebra defined by the famous RTT-relations (1) (see [2]), and we will refer to it as the RTT-algebra hereafter. In recent papers [3] there were considered the sets $\{s_k(L)\}$ and $\{\sigma_k(L)\}$ of elements of the REA where L denotes the matrix of the REA generators. In the classical limit ($R =$ permutation matrix) the sets $\{s_k\}$ and $\{\sigma_k\}$ turn, respectively, into the so-called power sums and basic symmetric polynomials in the eigenvalues λ_i of matrix L. Found in [3] was a system of quantum Newton relations among the elements of the sets $\{s_k(L)\}$ and $\{\sigma_k(L)\}$ and a polynomial identity for the quantum matrix L. These generalize, respectively, the iterative Newton relations and the Cayley-Hamilton theorem of classical matrix analysis [8]. In the present paper we give a definition of the two sets $\{s_k(T)\}$ and $\{\sigma_k(T)\}$ and find the generalized Newton relations among their elements for the case of quantum matrix algebra of the RTT-type.

According to [2] the RTT-algebra is generated by N^2 operators T_i^j obeying the commutation relations

$$R_{12}T_1T_2 = T_1T_2R_{12} \ , \tag{1}$$

where the compact matrix notations of [2] are employed. The nondegenerate $N^2 \times N^2$ matrix R is a solution of the Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \ , \tag{2}$$

and besides this usual requirement we will assume it to be *the closed Hecke symmetry of finite rank* p [4]. It means that in addition to (2) R satisfies the Hecke condition

$$R^2 = \mathbf{I} + \lambda R \qquad \lambda = q - q^{-1} \ , \tag{3}$$

while the closure condition requires the matrix $(R_{12})^{t_1}$ be invertible. The meaning of the finite rank condition will be explained below.

As was shown in [4] with such a choice of the matrix R algebra (1) can be given the structure of the Hopf algebra and one can define q-analogues of Young (anti)symmetrizing projectors and quantum Levi-Civita tensors associated with R. These projectors will be a usefull technical tool for us and we will write down part of their properties relevant for our consideration. For the proofs and more detailed treatment the reader is referred to [4].

The Young q-antisymmetrizer of the rank k is defined by the inductive relations:

$$\begin{aligned} P^{(1)} &= \mathbf{I} \\ P^{(k)} &= \frac{1}{k_q}\,\left(q^{k-1}\mathbf{I} - q^{k-2}R_{k-1} + \ldots + (-1)^{k-1}R_1 \cdot \ldots \cdot R_{k-1}\right)\,P^{(k-1)}\,. \end{aligned} \tag{4}$$

Here the abbreviated notations for the matrices $R_i \equiv R_{i,i+1}$ (acting nontrivially in the tensor product of i-th and $(i+1)$-th matrix spaces) are used, and the symbol k_q stands for the q-number $k_q \equiv (q^k - q^{-k})/\lambda$. Note, that definition (4) is correct provided that $k_q \neq 0$, $k \geq 1$, therefore q *is not a root of unity.* Later on we always assume that this is the case.

The Hecke symmetry R is of finite rank p if $P^{(p)} \neq 0$ and $P^{(p+1)} = 0$. Together with the closure condition it allows one to prove that the projector $P^{(p)}$ is one dimensional and therefore can be presented in the form

$$P^{(p)}{}_{i_1\ldots i_p}^{\;\;j_1\ldots j_p} = u_{|i_1\ldots i_p\rangle}v^{\langle j_1\ldots j_p|} \equiv u_{|12\ldots p\rangle}v^{\langle 12\ldots p|}\,. \tag{5}$$

Tensors $u_{|\,\rangle}$ and $v^{\langle\,|}$ are the left and right q-antisymmetric Levi-Chivita tensors and their defining property reads as follows

$$R_i u_{|12..p\rangle} = v^{\langle 12..p|}R_i = -\frac{1}{q}\,R_i \quad \forall\, i \leq p-1\,. \tag{6}$$

The full contraction of these tensors is normalised to be unity

$$\sum_{\{i\}} v^{\langle i_1\ldots i_p|}u_{|i_1\ldots i_p\rangle} \equiv v^{\langle 12..p|}u_{|12..p\rangle} = 1\,.$$

We will also need the relation connecting projectors of different ranks

$$P^{(k)} = q^{p(p-k)}\binom{p}{k}_q Tr_{q(k+1..p)}P^{(p)}\,, \qquad \binom{p}{k}_q \equiv \frac{p_q!}{k_q!\,(p-k)_q!}\,. \tag{7}$$

The symbol $Tr_{q(\ldots)}$ stands for the quantum trace operation [5,2] over several matrix spaces and q-factorials are defined as $0_q! = 1_q! = 1$, $k_q! = k_q(k-1)_q!$. The definition of the quantum trace in our case reads:

$$Tr_q X = Tr\,\mathcal{C}\cdot X \qquad \mathcal{C} \stackrel{\mathrm{def}}{=} Tr_{(1)}\left((R_{12}^{t_1})^{-1})^{t_1}P_{12}\right)\,, \tag{8}$$

where X is an arbitrary matrix, and $\mathcal{C}$ is correctly defined provided that R matrix is closed. Important properties of $\mathcal{C}$ consist in the following:

$$R_{12}\mathcal{C}_1\mathcal{C}_2 = \mathcal{C}_1\mathcal{C}_2R_{12} \tag{9}$$

$$v^{\langle 12..p|}\mathcal{C}_1\ldots\mathcal{C}_p = q^{-p^2}v^{\langle 12..p|} \quad , \quad \mathcal{C}_1\ldots\mathcal{C}_p u_{|12..p\rangle} = q^{-p^2}u_{|12..p\rangle}\,. \tag{10}$$

Let us introduce now the two commuting subsets of algebra (1). The first of them was defined in [6] for an arbitrary solution R of the Yang-Baxter equation (2)

$$s_k = Tr_{(1\dots k)}\left(R_{k-1}\dots R_1 T_1 \dots T_k\right) \ .$$

In the classical limit (R = permutation matrix), where the matrix T becomes the usual one with commuting entries, the generators s_k turn into the power sums of eigenvalues $\{\tau_i\}$ of the matrix T:

$$s_k \to Tr\, T^k \equiv \sum_{i=1}^{N} (\tau_i)^k \ .$$

For our consideration it is more convenient to use a slightly modified definition of s_k. This modification is based on the existence of one parameter family of automorphisms of algebra (1)

$$T \to \mathcal{C}^\alpha T \ , \tag{11}$$

which, in turn, is a direct consequence of (9). Now, fixing $\alpha = 1$ we get another possible set of $s_k(T)$ which will be more appropriate for our purposes

$$s_k(T) = Tr_{q(1\dots k)}\left(R_{k-1}\dots R_1 T_1 \dots T_k\right) \ . \tag{12}$$

With the help of (1) (2) and (9) one can prove by direct calculation that $s_k(T)$ from (12) commute with each other.

Define now another set of commuting elements

$$\sigma_k(T) = q^{k(1-p)} \binom{p}{k}_q v^{\langle 12\dots p|} T_1 \dots T_k u_{|12\dots p\rangle} \ , \tag{13}$$

where the normalizing factor is taken for the future convenience. In the classical limit the elements $\sigma_k(T)$ turn into the basic symmetric sums of the eigenvalues τ_i of matrix T. One can prove the commutativity of σ_k by straightforward calculations too. However it is more convenient to establish a q-analogue of the Newton relations among $\{s_k\}$ and $\{\sigma_k\}$ first. These relations allow us to express the set $\{\sigma_k\}$ in terms of $\{s_k\}$. Then the commutativity of $\{s_k\}$ will directly lead to that of $\{\sigma_k\}$.

First of all let transform σ_k in (13). Taking into account equations (7), (10) and (5) one gets:

$$\begin{aligned}
\sigma_k &\equiv \alpha(k) v^{\langle 12\dots p|} T_1 \dots T_k u_{|12\dots p\rangle} = \alpha(k) q^{p^2} v^{\langle 12\dots p|} (\mathcal{C}T)_1 \dots (\mathcal{C}T)_k \mathcal{C}_{k+1} \dots \mathcal{C}_p u_{|12\dots p\rangle} \\
&= \alpha(k) q^{p^2} Tr_{q(1\dots p)}(P^{(p)} T_1 \dots T_k) = \alpha(k) q^{p^2} \binom{p}{k}_q^{-1} q^{-p(p-k)} Tr_{q(1\dots k)}(P^{(k)} T_1 \dots T_k) \\
&= q^k Tr_{q(1\dots k)}(P^{(k)} T_1 \dots T_k).
\end{aligned} \tag{14}$$

Here $\alpha(k)$ stands for the numerical factor in (13). Now with the help of (14) and the definition of $P^{(k)}$ we can prove the following:

Proposition. *The generators s_k and σ_k defined in (12) and (13) are connected by the generalized Newton relations of the form:*

$$\frac{n_q}{q^n}\,\sigma_n - \sigma_{n-1}s_1 + \sigma_{n-2}s_2 - \ldots + (-1)^{n-1}\sigma_1 s_{n-1} + (-1)^n s_n = 0 \quad n = 1, 2, \ldots\ p\ . \tag{15}$$

Proof. From the definition of q-antisymmetryzer $P^{(k+1)}$ (4) one can get the following equality

$$P^(k) = \frac{(k+1)_q}{q^k}\,P^{(k+1)} + \frac{1}{q^k}(q^{k-1}R_k - q^{k-2}R_{k-1}R_k + \ldots + (-1)^{k-1}R_1\ldots R_k)P^{(k)}\ .$$

Using this formula we transform identically a typical term of (15):

$$\begin{aligned}
\sigma_k s_{n-k} &= q^k Tr_{q(1\ldots n)}(P^{(k)}T_1\ldots T_k R_{n-1}\ldots R_{k+1}T_{k+1}\ldots T_n)\\
&= (k+1)_q Tr_{q(1\ldots n)}(P^{(k+1)}R_{n-1}\ldots R_{k+1}T_1\ldots T_n)\\
&+ \sum_{i=0}^{k-1} q^{k-1-2i}\ Tr_{q(1\ldots n)}(P^{(k)}R_{n-1}\ldots R_k T_1\ldots T_n)\\
&= (k+1)_q Tr_{q(1\ldots n)}(P^{(k+1)}R_{n-1}\ldots R_{k+1}\,T_1\ldots T_n)\\
&+ k_q Tr_{q(1\ldots n)}(P^{(k)}R_{n-1}\ldots R_k\,T_1\ldots T_n)
\end{aligned} \tag{16}$$

In passing to the second equality we used the cyclic property of the quantum trace, formula (1) and the following relations on the q-antisymmetrizers $P^{(k)}$ [4]:

$$P^{(k)}R_i = -\frac{1}{q}P^{(k)} \quad \text{for} \quad 1 \le i \le k-1\ .$$

Eventually one should note that the boundary terms of above relations (16) read:

$$\begin{aligned}
\sigma_{n-1}s_1 &= \frac{n_q}{q^n}\,\sigma_n + (n-1)_q Tr_{q(1\ldots n)}(P^{(n-1)}R_{n-1}T_1\ldots T_n)\\
\sigma_1 s_{n-1} &= 2_q Tr_{q(1\ldots n)}(P^{(2)}R_{n-1}\ldots R_2 T_1\ldots T_n) + s_n\ .
\end{aligned} \tag{17}$$

Now assertion (15) is a simple consequence of equations (16) and (17). ■

The quantum Newton relations (15) obtained here for the case of the RTT-algebra (1) coincide in their form with those of the reflection equation algebra [3]. However, we would like to emphasize one important difference. The corresponding elements σ_k generate *the center* of REA, whereas elements (13) are mutually commuting but *not central* in the algebra (1). This allows one to interpret T as the monodromy matrix of some finite dimensional integrable model and the generators σ_k, being connected with the spectrum of T, play the role of the involutive set of integrals of motion. Actually in order to realize the above remark one should develope a kind of representation theory for algebra (1). For example, using the specific representation of the RTT-algebra with the R matrix of $SL_q(N)$ type one can reproduce the relativistic Toda chain [7].

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