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# IMPEDANCE TREATMENT OF LONGITUDINAL COUPLED-BUNCH FEEDBACKS IN A PROTON SYNCHROTRON

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## Abstract

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Characteristic equation of coupled-bunch motion of a beam governed by a feedback (FB) is given to find the beam FB stabilizing effect against coherent instabilities or, say, injection error damping rates. Quite a general FB using filter method is involved: (i) it has two paths, the in-phase and quadrature (or amplitude and phase in a small-signal approach), with unequal gains; (ii) may employ distinct RF-bands to pick-up beam data and feed correction back to the beam. To account for cross-talk between various field and beam current harmonics inflicted by frequency down- and up-mixing inside the FB circuit, an impedance matrix (with, at most, three non-trivial elements per row) is introduced as a natural concept to gain insight into 'FB & beam' dynamics.

## Аннотация

Иванов С.В. Импедансный подход к расчету продольных систем обратной связи по пучку в протонном синхротроне: Препринт ИФВЭ 96-8. – Протвино, 1996. – 19 с., 4 рис., библиогр.: 6.

Получено характеристическое уравнение для сгруппированного пучка, контролируемого цепью обратной связи (ОС). Оно позволяет оценить ее стабилизирующее влияние по отношению к когерентным неустойчивостям сгустков, а также скорость демпфирования начальных ошибок инжекции. Рассмотрена цепь ОС самого общего вида, которая (i) имеет два канала контроля — синфазный и квадратурный (или амплитудный и фазовый в малосигнальном приближении) с разными частотными характеристиками; (ii) может использовать различные ВЧ-диапазоны для измерения пучка и внесения коррекции. Для учета взаимодействия между различными гармониками тока и поля, вызванными преобразованиями частотного спектра сигнала в цепи ОС, вводится матрица импедансов, имеющая не более трех ненулевых элементов в строке.

## Introduction

The paper expounds a consistent frequency-domain approach to linear longitudinal coupled-bunch (CB) beam feedbacks (FB) in a proton synchrotron. It provides a formal quantitative basis for a commonly used intuitive notion that a FB is seen by the beam as an artificial coupling impedance controlled from the outside.

The impedance approach to a beam FB has at least two plain advantages:

(i) The FB effect is readily mounted into the well-established theory of coherent instabilities. It allows a straightforward application of the advanced techniques developed there by now: beam transfer functions, threshold maps, Landau damping rates, handling of coupled-bunch motion, etc.

(ii) Destabilizing effect of beam environment is commonly available in terms of the coupling impedances and thus may be naturally taken into account during the FB design.

The frequency-domain treatment of the beam FB does not necessarily imply that this FB is a narrow-band one. The wider the FB bandwidth, the less sensitive the FB impact to the particular azimuthal CB mode processed is, the coupled-bunch FB gradually turning into a bunch-by-bunch FB. However, the frequency-domain formulae easily account for the adverse effect of the finite response times of a real pick-up, an acting device and electronics, which may not be a simple matter to deal with when the beam FB is viewed entirely from the time domain.

In **Section 1**, to account for the cross-talk between various field and beam current harmonics inflicted by frequency down- and up-mixing inside the FB circuit, an impedance matrix is introduced as a natural concept to gain insight into the 'FB & beam' dynamics. This matrix has, at most, three non-trivial elements per row, these being expressed through the FB's path transfer functions and its set-point parameters.

**Section 2** interprets an important class of RF FBs around the final power amplifier as degenerate beam FBs with pick-up and acting device merged into a single unit — an accelerating cavity, and extends the impedance treatment to these widespread circuits as well.

**Section 3** yields a characteristic equation of the CB motion of beam governed by the FB, which allows one to find the FB stabilizing effect against coherent instabilities and injection error damping rates.

**Section 4** presents two examples of practical application of the impedance treatment of the RF and beam FBs used to outline their technical contours for the UNK Project.

# 1. Beam Feedback

### Coordinates and Fourier Transforms

Let $\vartheta = \Theta - \omega_0 t$ be azimuth in a co-rotating frame, where $\Theta$ is azimuth around the ring in the laboratory frame, $\omega_0$ is the angular velocity of a reference particle, $t$ is time. The co-moving frame origin $\vartheta = 0$ traverses the origin $\Theta = 0$ of the lab-frame at $t = 0$, and after each revolution period $2\pi/\omega_0$ further on.

The beam current $J(\vartheta, t)$ and longitudinal electric field $E(\vartheta, t)$ are decomposed into travelling waves

$$\sum_k (J, E)_k (\Omega)\, e^{ik\vartheta - i\Omega t} \tag{1}$$

with $\Omega$ being the frequency of Fourier transform in time w.r.t. the co-rotating frame. In the laboratory frame, $\Omega$ is seen as a side-band near each revolution frequency line, $\omega = k\omega_0 + \Omega$.

### CB Feedback Layout and Set-Point

Quite a general CB beam FB circuit employing filter methods is shown in Fig.1, [1]. The circuitry extracts beam data from a PU as a band-pass signal at $\omega \simeq \pm\overline{h}\omega_0$, processes it at the intermediate frequency $\omega = 0$ after frequency down-mixing, and then feeds an up-mixed band-pass correction back to the beam through an AD at $\omega \simeq \pm h'\omega_0$.

Here, harmonic numbers $\overline{h}, h'$ are integers. Take the general case of $\overline{h} \neq h'$ and $\overline{h}, h' \neq h$, where $h$ is the main RF harmonic number.

Generally, the FB has the inphase ($c$) and quadrature ($s$) paths with unequal gains, $H^{(c)} \neq H^{(s)}$. Treated in a small-signal approach near the FB set-point, the former one controls an amplitude, while the latter — a phase of the accelerating voltage seen by the beam. Either of the paths may be switched off altogether, say, $H^{(c)} = 0$ for an injection error damping system, or in case of a dedicated phase control loop.

To simplify the matters, let a PU and an AD of the FB in question be closed-volume resonant objects which excite longitudinal electric field

$$E^{(a)}(\Theta, t) = L^{-1} G^{(a)}(\Theta)\, u_a(t); \quad a = \mathrm{PU}, \mathrm{AD}, \tag{2}$$

where $u_a(t)$ is voltage across the gap, $L$ is the orbit length. Function $G^{(a)}(\Theta) = G^{(a)}(\Theta + 2\pi)$ specifies the field localization and is normalized as

$$\frac{1}{2\pi}\int_0^{2\pi} |G^{(a)}(\Theta)| d\Theta = 1.$$

Its decomposition into $\sum_k G_k^{(a)} \mathrm{e}^{ik\Theta}$ provides $G_k^{(a)}$, the complex transit-time factors at $\omega = k\omega_0$ with $|G_k^{(a)}| \leq 1$ and $\arg G_k^{(a)}$ being proportional to $\Theta^{(a)}$, the object's coordinate along the ring.

To set the FB operating point is (i) to provide an external reference to an adder node behind the PU, if required, and (ii) adjust carrier phases $\overline{\phi}$, $\phi'$ of the frequency down- and up-mixing w.r.t. to the beam and net accelerating voltage so as to comply with the FB's purpose and layout along the ring.

Assume the CB beam FB be idle when it observes an ideal beam configuration — a steady-state closed train of $M$ identical equispaced bunches. Put the origin points $\vartheta = 0$ (mod $2\pi/M$) of the co-moving frame at the reference beam bunch centers, and let any of the harmonic numbers $h, \overline{h}, h'$ be integer multiples of $M$.

**Down-Mixing Carrier.** Let $-W'(\omega)$ with $\mathrm{Re}\, W'(\omega) \geq 0$ denote a transfer function from the beam current to the PU gap voltage, $S(\omega)$ be the admittance of the front-end band-pass electronics behind the PU. To simplify notations, use the concatenation agreement

$$SW'(\omega) = S(\omega)\, W'(\omega). \tag{3}$$

For the time being, let the half-bandwidth at base of $SW'(\omega)$ be $\Delta\omega_{SW'} < M\omega_0$, the bunch-to-bunch recur frequency. Then, the reference beam would have been seen in the PU branch as a cosinusoidal current

$$\overline{J}(t) \propto \ \cos(\overline{h}\omega_0 t - \overline{\phi}), \tag{4}$$

its phase being

$$\overline{\phi} \ = \ \pi + \arg\left(SW'(\overline{h}\omega_0)\, G_{-\overline{h}}^{(\mathrm{PU})}\, J_{\overline{h}}^{(0)}\right). \tag{5}$$

Here $J_k^{(0)}$ is a Fourier series component of the reference steady-state beam current $J^{(0)}(\vartheta, t) = J^{(0)}(\vartheta)$. Due to the convention used on the co-moving frame origin and the time lock-in, $\arg J_k^{(0)} = 0$.

Let Eq.5 define the down-mixing carrier phase $\overline{\phi}$.

(The dynamical CB FB under study would not follow any steady-state beam loading signals at $\omega = k\omega_0$. Thus, an explicit adder unit at the PU side to cancel $\overline{J}(t)$ is redundant: the reference current is subtracted with the periodic notch filters in the FB paths themselves.)

**Up-Mixing Carrier.** Let $T(\omega)$ denote a transfer function from an external RF drive current to the AD gap voltage, $K(\omega)$ be current-to-current gain through an RF power amplifier, a feeder and a coupler to the gap. Use the same concatenation notation as earlier, Eq.3,

$$TK(\omega) = T(\omega)\, K(\omega). \tag{6}$$

Assume, for the time being, that the RF power amplifier followed by the AD is driven by an external low-level cosinusoidal current

$$J(t) \propto \ \cos(h'\omega_0 t - \phi) \tag{7}$$

with phase

$$\phi \;=\; \varphi'_s - \arg\left(TK(h'\omega_0)\,G^{(\mathrm{AD})}_{h'}\right). \tag{8}$$

Then, it imposes a steady-state $E$-field wave in the co-moving frame,

$$E(\vartheta,t) \propto\; \cos(h'\vartheta + \varphi'_s), \tag{9}$$

whose phase $\varphi'_s$ is prescribed from dynamical considerations (a stable phase angle).

Take Eq.8 as a definition of the up-mixing carrier phase $\phi' = \phi$. Thus, Eq.9 becomes a reference wave w.r.t. which the FB produces either an inphase or a quadrature correction as seen in the co-moving frame.

The FB set-point chosen settles transit-time effects at $\omega = \pm\overline{h}\omega_0$ and $\omega = \pm h'\omega_0$ due to a finite PU–AD distance (these enter Eqs.5, 8 via arg $G^{(a)}_k$).

### Band-Pass to Low-Pass Conversion

Take a reference oscillation

$$J_*(t) = I_*\,\cos(h_*\omega_0 t - \phi_*), \tag{10}$$

where $h_*$ denotes $\overline{h}$ or $h'$ and $\phi_*$ stands for phases $\overline{\phi}$ or $\phi'$, respectively. An arbitrary signal $j(t)$ can be expanded into a sum of its inphase ($\propto$ cos) and quadrature ($\propto$ sin) amplitude modulated components w.r.t. $J_*(t)$, [2],

$$\begin{aligned} j(t) \;&=\; i^{(c)}(t)\,\cos(h_*\omega_0 t - \phi_*) \;-\; i^{(s)}(t)\,\sin(h_*\omega_0 t - \phi_*) = \qquad (11)\\ &=\; \textstyle\sum_{\xi=c,s} i^{(\xi)}(t)\,\cos(h_*\omega_0 t - \phi^{(\xi)}_*);\\ &\phantom{=}\;\; \phi^{(c)}_* = \phi_*, \qquad \phi^{(s)}_* = \phi_* - \pi/2. \qquad (12) \end{aligned}$$

In the frequency domain

$$j(\omega) = 0.5\textstyle\sum_{\xi=c,s}\left(i^{(\xi)}(\omega + h_*\omega_0)\mathrm{e}^{-i\phi^{(\xi)}_*} + i^{(\xi)}(\omega - h_*\omega_0)\mathrm{e}^{+i\phi^{(\xi)}_*}\right). \tag{13}$$

There are many formal ways to construct inphase and quadrature amplitudes $i^{(\xi)}$ so as to comply with Eqs.11,13. Only two of them are of interest.

The first one yields an ‘analytical’ decomposition. It proceeds from the complex analytical signal $\tilde{j}(t)$ whose $\mathrm{Re}\,\tilde{j}(t) = j(t)$,

$$\tilde{j}(t) = (1/\pi)\int_0^\infty j(\omega)\,\exp(-i\omega t)\,d\omega, \tag{14}$$

and defines $i^{(\xi)}_1$ as a projection of $\tilde{j}(t)$ onto a rotating phasor $\mathrm{e}^{-ih_*\omega_0 t\,+\,i\phi_*}$,

$$i^{(c)}_1(t) - i\cdot i^{(s)}_1(t) = \tilde{j}(t)\,\exp(ih_*\omega_0 t - i\phi_*). \tag{15}$$

Herefrom,

$$i_1^{(\xi)}(\omega) \;=\; j(\omega+h_*\omega_0)\,\theta(\omega+h_*\omega_0)\mathrm{e}^{-i\phi_*^{(\xi)}}\, + \qquad (16)$$
$$+ \;\; j(\omega-h_*\omega_0)\,\theta(-\omega+h_*\omega_0)\mathrm{e}^{+i\phi_*^{(\xi)}},$$

where $\theta(\omega)$ is Heaviside's step function, $\theta(\omega) = (1+\mathrm{sgn}(\omega))/2$.

On the other hand, in practice, the technique available of frequency mixing with multipliers yields a feasible 'technical' decomposition whose $i_2^{(\xi)}(t) = j(t)\times\cos(h_*\omega_0 t - \phi_*^{(\xi)})$, and

$$i_2^{(\xi)}(\omega) = 0.5\Big(j(\omega+h_*\omega_0)\mathrm{e}^{-i\phi_*^{(\xi)}} + j(\omega-h_*\omega_0)\mathrm{e}^{+i\phi_*^{(\xi)}}\Big). \qquad (17)$$

Eqs.16,17 show that

$$i_2^{(\xi)}(\omega) = 0.5\, i_1^{(\xi)}(\omega), \qquad |\omega| < h_*\omega_0. \qquad (18)$$

That is, as is well known, Ref.[2], an inphase-quadrature circuit with a limited-bandwidth low-pass signal processing (half-bandwidth at base $\Delta\omega_H < h_*\omega_0$) would handle directly the real and imaginary parts halved of the analytical input signal, Eq.15.

In a small-signal approach, $i_1^{(c)}(t)$ and $i_1^{(s)}(t)$ can be related by

$$i_1^{(c)}(t) \;\simeq\; \Delta I_*(t), \qquad i_1^{(s)}(t) \;\simeq\; -I_*\,\Delta\phi_*(t) \qquad (19)$$

to an amplitude and a phase perturbation suffered by the reference oscillation, Eq.10, due to an additive perturbation $j(t)$.

According to Eqs.18, 19, a FB operating in a 'polar' amplitude-phase parameter space, being treated in a linear approach near its set-point, is reducible locally to a 'rectangular' inphase-quadrature FB in question. This observation extends the applicability range of the results presented here to a small-signal behavior of widespread amplitude-phase FBs as well.

## Open-Loop In-Out Susceptibility

On neglecting the PU (small) impact on the beam, the net correcting voltage imposed by the CB beam FB can be put down as

$$u_{\mathrm{AD}}^{(tot)}(t) = u_{\mathrm{AD}}^{(b)}(t) - u_{\mathrm{AD}}^{(ind)}(t), \qquad (20)$$

where $(b)$ and $(ind)$ denote beam-excited and FB-induced voltages, correspondingly; $u_{\mathrm{AD}}^{(ind)}(t)$ is a linear functional of $u_{\mathrm{PU}}^{(b)}(t')$ taken at $t' \le t$ due to causality. To put down this functional, move to the frequency domain and follow the data flow through the FB, Fig.1:

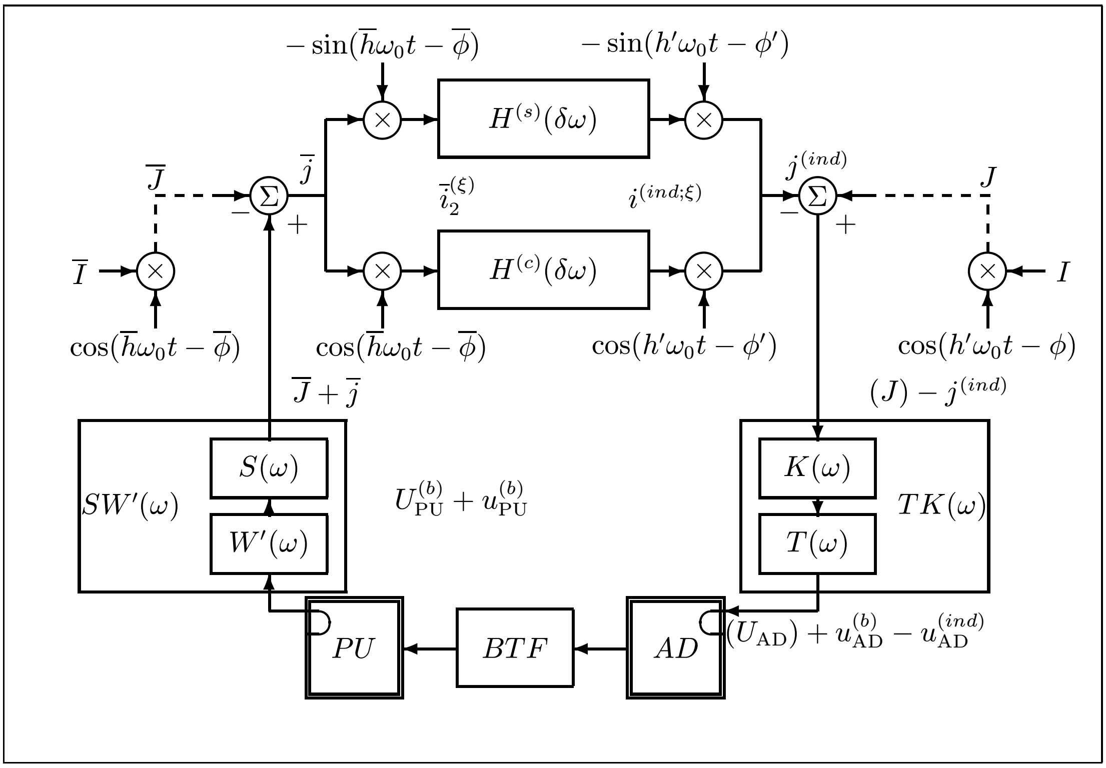


Fig. 1. CB beam FB layout.

(a) transform $u_{\mathrm{PU}}^{(b)}(\omega)$ through band-pass electronics sensitivity to get the measured current $\overline{j}(\omega)=S(\omega)\,u_{\mathrm{PU}}^{(b)}(\omega)$;

(b) extract the inphase and quadrature content $\overline{i}_2^{(\xi)}(\omega)$ of $\overline{j}(\omega)$ w.r.t. the carrier $(\overline{h},\overline{\phi})$ with Eq.17;

(c) filter each $\overline{i}_2^{(\xi)}(\omega)$ individually through two low-pass paths available to get $i^{(ind,\xi)}(\omega)=H^{(\xi)}(\omega)\,\overline{i}_2^{(\xi)}(\omega)$, $\xi=c,s$;

(d) up-mix these $i^{(ind,\xi)}(\omega)$ with carrier $(h',\phi')$ to build up induced current $j^{(ind)}(\omega)$ according to Eq.13;

(e) and, finally, pass $j^{(ind)}(\omega)$ through a power amplifier and AD to get the correction $u_{\mathrm{AD}}^{(ind)}(\omega)=TK(\omega)\,j^{(ind)}(\omega)$.

The result is a lengthy four-summand linear involvement from cause $u_{\mathrm{PU}}^{(b)}(\omega)$ to effect

$$
\begin{aligned}
u_{\mathrm{AD}}^{(ind)}(\omega) \;=\;& 0.25\,TK(\omega)\times\Big[\;\Big(H^{(c)}(\omega+h'\omega_0)-H^{(s)}(\omega+h'\omega_0)\Big)\times \\
&\times\exp\Big(-i(\phi'+\overline{\phi})\Big)\,S(\omega+h'\omega_0+\overline{h}\omega_0)\;u_{\mathrm{PU}}^{(b)}(\omega+h'\omega_0+\overline{h}\omega_0)+ \\
&+\Big(H^{(c)}(\omega+h'\omega_0)+H^{(s)}(\omega+h'\omega_0)\Big)\,\exp\Big(-i(\phi'-\overline{\phi})\Big)\times \\
&\times\,S(\omega+h'\omega_0-\overline{h}\omega_0)\;u_{\mathrm{PU}}^{(b)}(\omega+h'\omega_0-\overline{h}\omega_0)+ \\
&+\quad\cdots\quad(h'\to-h',\;\overline{h}\to-\overline{h},\;\phi'\to-\phi',\;\overline{\phi}\to-\overline{\phi})\;\Big].
\end{aligned}
\tag{21}
$$

Let $\delta\omega$ denote a frequency deviation with $|\delta\omega| < \overline{h}\omega_0, h'\omega_0$. Assume half-bandwidths (at base) of $H^{(c,s)}(\omega)$ be small enough

$$\Delta\omega_H < \overline{h}\omega_0, h'\omega_0, \tag{22}$$

to ensure $H^{(\xi)}(\pm 2h'\omega_0 + \delta\omega) = 0$ and $H^{(\xi)}(\pm 2\overline{h}\omega_0 + \delta\omega) = 0$. Insert $\omega = h'\omega_0 + \delta\omega$, and then $\omega = -h'\omega_0 + \delta\omega$, into Eq.21 to see that in this case the state of the system is given by 2-D column-vectors

$$\vec{u}_{\rm PU}(\delta\omega) = \Big(u(\overline{h}\omega_0 + \delta\omega);\ u(-\overline{h}\omega_0 + \delta\omega)\Big)^T_{\rm PU}, \tag{23}$$

$$\vec{u}_{\rm AD}(\delta\omega) = \left(u(h'\omega_0 + \delta\omega);\ u(-h'\omega_0 + \delta\omega)\right)^T_{\rm AD}. \tag{24}$$

The in-out gain through the open FB loop becomes

$$\vec{u}^{(ind)}_{\rm AD}(\delta\omega) = \widehat{\chi}(\delta\omega)\ \vec{u}^{(b)}_{\rm PU}(\delta\omega), \tag{25}$$

where $\widehat{\chi}(\delta\omega)$ is a $2\times 2$ FB 'susceptibility' matrix. Its elements are

$$\begin{aligned}
\chi_{11}(\delta\omega) &= 0.25\ TK(h'\omega_0 + \delta\omega)\, S(\overline{h}\omega_0 + \delta\omega)\, \times \qquad (26)\\
&\quad \times\ \Big(H^{(c)}(\delta\omega) + H^{(s)}(\delta\omega)\Big)\ {\rm e}^{i(\phi' - \overline{\phi})};\\
\chi_{12}(\delta\omega) &= 0.25\ TK(h'\omega_0 + \delta\omega)\, S(-\overline{h}\omega_0 + \delta\omega)\, \times \qquad (27)\\
&\quad \times\ \Big(H^{(c)}(\delta\omega) - H^{(s)}(\delta\omega)\Big)\ {\rm e}^{i(\phi' + \overline{\phi})};\\
\chi_{21}(\delta\omega) &= \chi_{12}(-\delta\omega^*)^*;\ \ \chi_{22}(\delta\omega) = \chi_{11}(-\delta\omega^*)^*.
\end{aligned}$$

Eq.20 can now be rewritten in a vector notation,

$$\vec{u}^{(tot)}_{\rm AD}(\delta\omega) = \vec{u}^{(b)}_{\rm AD}(\delta\omega) - \widehat{\chi}(\delta\omega)\ \vec{u}^{(b)}_{\rm PU}(\delta\omega). \tag{28}$$

The beam-excited voltages across PU and AD (the closed-volume cavities) to enter this equation are

$$u^{(b)}_a(\omega) = -\begin{pmatrix} W'(\omega) \\ T'(\omega) \end{pmatrix} \sum_{k=-\infty}^{\infty} G^{(a)}_{-k}\, J_k(\omega - k\omega_0), \tag{29}$$

where $-T'(\omega)$ with ${\rm Re}\, T'(\omega) \geq 0$ denotes a transfer function from the beam current to the AD gap voltage. Generally, the response function of AD to external RF-drive $T(\omega)$ is other than $T'(\omega)$.

Eqs.28, 29 describe the open-loop beam-to-correction response of the FB. Still, they do contain much surplus data which the beam itself, as a subject to this correction, would not recall. One should extract the dynamically essential content of these equations.

**Impedance Matrix**

A conventional theory of longitudinal coherent beam instabilities employs an adequate tool to describe beam-environment interaction. It assumes that, while rotating near passive components inside the vacuum chamber, the beam drives $E$-field waves, Eq.1 with amplitude

$$E_k(\Omega) = -L^{-1} Z_{kk}(\omega)\, J_k(\Omega), \quad \omega = k\omega_0 + \Omega, \tag{30}$$

where $Z_{kk}(\omega)$ with $\mathrm{Re}\, Z_{kk}(\omega) \geq 0$ is the conventional longitudinal impedance. Its main-diagonal element is cut from the entire matrix $Z_{kk'}(\omega)$ (it describes the lumped nature of the beam environment) due to a narrow-band response appropriate to, as a matter of fact, slowly perturbed bunched beams,

$$J_{k'}((k-k')\omega_0 + \Omega) \simeq J_k(\Omega)\, \delta_{kk'}, \quad |\Omega| \ll \omega_0, \tag{31}$$

with $\delta_{kk'}$ being Kronecker's delta-symbol.

Insert Eq.29 into Eq.25 and pull out synchronous-to-beam $E$-field waves from Eq.2. Use Eq.31 to truncate $\sum_k$. Then, to generalize the commonly used impedance concept introduced by Eq.30, the beam FB can be thought of as imposing the $E$-field waves with amplitudes

$$\begin{aligned} E_k^{(fb)}(\Omega) &= -L^{-1}\Big( Z_{kk}(\omega)\, J_k(\Omega) + \\ &\quad + Z^{(fb)}_{k,k-h'+\overline{h}}(\omega)\, J_{k-h'+\overline{h}}(\Omega) + Z^{(fb)}_{k,k-h'-\overline{h}}(\omega)\, J_{k-h'-\overline{h}}(\Omega)\Big) \end{aligned} \tag{32}$$

through mediation of coupling impedances

$$Z_{kk}(\omega) = T'(\omega) |G_k^{(\mathrm{AD})}|^2, \tag{33}$$

$$\begin{aligned} Z^{(fb)}_{k,k-h'+\overline{h}}(\omega) &= -\chi_{11}(\omega - h'\omega_0) \times \\ &\times\ W'(\omega - h'\omega_0 + \overline{h}\omega_0)\, G_k^{(\mathrm{AD})} G^{(\mathrm{PU})}_{-k+h'-\overline{h}}, \end{aligned} \tag{34}$$

$$\begin{aligned} Z^{(fb)}_{k,k-h'-\overline{h}}(\omega) &= -\chi_{12}(\omega - h'\omega_0) \times \\ &\times\ W'(\omega - h'\omega_0 - \overline{h}\omega_0)\, G_k^{(\mathrm{AD})} G^{(\mathrm{PU})}_{-k+h'+\overline{h}}. \end{aligned} \tag{35}$$

Here $\omega = k\omega_0 + \Omega$, $k \sim h' > 0$, $|\Omega| \ll \omega_0$. The negative-frequency domain of $k \sim -h' < 0$ is arrived at with the reflection property $Z_{-k,-k'}(-\omega^*)^* = Z_{kk'}(\omega)$.

Eq.33 yields coupling impedance of the AD itself treated as a passive structure in line with Eq.30. Eqs.34, 35 represent an active response of the FB and account for the cross-talk between harmonics $E_k$, $J_{k'}$ with $k \neq k'$ caused by down- and up-mixing of frequencies. As expected, the PU and front-end electronics enter these equations through their joint transfer function $SW'(\omega)$. Impedances $Z^{(fb)}_{kk'}(\omega)$ are no longer subject to a passive-structure restriction $\mathrm{Re}\, Z^{(fb)}_{kk'}(\omega) \geq 0$, which is to introduce damping into the beam motion. The balance $H^{(c)}(\delta\omega) = H^{(s)}(\delta\omega)$ of the FB path gains results in matrix $\widehat{\chi}$ becoming diagonal, and in $Z^{(fb)}_{kk'}(\omega)$ with $|k-k'| = h' + \overline{h}$ vanishing. In injection error damping systems, the FB path gains and, hence, $Z_{kk'}^{(fb)}(\omega)$ may be scaled reciprocally to, say, the average beam current $J_0$.

In practice, a CB beam FB alone is unable to override destabilizing effect of the numerous accelerating cavities unless coupling impedance of their fundamental mode is depressed with the RF FB to be discussed in short in the next Section.

## 2. RF Feedback

It is a pure RF-engineering circuit around the final power amplifier to stabilize the net accelerating field against any periodic (with $2\pi/\omega_0$) interference [3]. The system handles a pair of adverse effects of beam dynamics inflicted by accelerating cavity's high-Q fundamental mode: (i) heavy periodic transient beam loading, and (ii) strong CB instabilities at lower-order within-bunch multipole modes.

The RF FB can be viewed as a degenerate case of a CB beam FB, Fig.1, where not only PU and AD do employ the same RF-band, but are merged into a single device AC, an accelerating cavity. Hence, first take $h', \overline{h} = h$ and $W'(\omega) = T'(\omega)$.

### RF Feedback Set-Point

The accelerating system must impose a steady-state $E$-field wave in the co-moving frame,

$$E(\vartheta, t) = L^{-1} V_0 \cos(h\vartheta + \varphi_s), \qquad \eta\varphi_s < 0, \tag{36}$$

where $V_0$ is the accelerating voltage amplitude per turn, $\varphi_s$ is the stable phase angle, $\eta = \alpha - \gamma^{-2}$, $\alpha$ is the momentum compaction factor, $\gamma$ is the relativistic factor.

**Down-Mixing Carrier.** Let the RF FB be idle when it observes a cosinusoidal voltage across the AC gap,

$$U(t) = V\ \cos(h\omega_0 t - \varphi), \tag{37}$$

whose amplitude and phase

$$V = V_0 / \left( N_{\mathrm{AC}} |G_h^{(\mathrm{AC})}| \right), \qquad \varphi = \varphi_s - \arg G_h^{(\mathrm{AC})}, \tag{38}$$

are such as to yield the wave from Eq.36. $N_{\mathrm{AC}}$ is the number of (identical and inphase) ACs in the ring.

Reference voltage, Eq.37 is seen behind the AC field probe and front-end electronics as a cosinusoidal current

$$\overline{J}(t) = \overline{I}\ \cos(h\omega_0 t - \overline{\phi}) \tag{39}$$

with amplitude and phase

$$\overline{I} = |S(h\omega_0)|\ V, \qquad \overline{\phi} = \varphi + \arg S(h\omega_0). \tag{40}$$

This current must be cancelled by the equal external reference $\overline{J}(t)$ entering an adder node.

Let the second of Eqs.38, 40 define the down-mixing carrier phase $\overline{\phi}$.

**Up-Mixing Carrier.** To provide voltage $U(t)$ (without beam), the RF power amplifier followed by the AC must be driven by an external low-level cosinusoidal current

$$J(t) = I\,\cos(h\omega_0 t - \phi) \tag{41}$$

with amplitude and phase

$$I = V/\left|TK(h\omega_0)\right|, \qquad \phi = \varphi - \arg TK(h\omega_0). \tag{42}$$

One may have adopted $\phi$ as the up-mixing carrier phase, $\phi' = \phi$, thus getting an all-real negative FB signal at $\omega = \pm h\omega_0$. However, this straightforward choice may not be optimal for damping beam instabilities due to the AC fundamental mode. The phase shift $(\phi' - \phi)$ is the RF FB operational parameter which is subject to tuning so as to provide a rational trade between beam loading compensation and beam stabilization against coherent instabilities.

**Impedance Matrix**

In case of the RF FB, Eq.20 is kept intact while the field probe pickups both, the beam-imposed and correction signals. Therefore, Eq.25 have to undergo an essential modification

$$\vec{u}^{(ind)}_{\rm AC}(\delta\omega) = \widehat{\chi}(\delta\omega)\;\vec{u}^{(tot)}_{\rm AC}(\delta\omega), \tag{43}$$

Eq.28 thus turning into

$$\vec{u}^{(tot)}_{\rm AC}(\delta\omega) = \vec{u}^{(b)}_{\rm AC}(\delta\omega) - \widehat{\chi}(\delta\omega)\;\vec{u}^{(tot)}_{\rm AC}(\delta\omega) \tag{44}$$

due to which the coupling impedances to enter Eq.32 acquire the form other than that given by Eqs.33–35

$$Z_{kk}(\omega) + Z^{(fb)}_{kk}(\omega) = \varepsilon^{-1}_{11}(\omega - h\omega_0)\;T'(\omega)\;|G^{(\rm AC)}_k|^2, \tag{45}$$

$$Z^{(fb)}_{k,k-2h}(\omega) = \varepsilon^{-1}_{12}(\omega - h\omega_0)\;T'(\omega - 2h\omega_0)\;G^{(\rm AC)}_k G^{(\rm AC)}_{-k+2h}, \tag{46}$$

where $\omega = k\omega_0 + \Omega$, $k \sim h > 0$, $|\Omega| \ll \omega_0$ and

$$\widehat{\varepsilon}(\delta\omega) = \widehat{I} + \widehat{\chi}(\delta\omega), \tag{47}$$

$$\widehat{\varepsilon}^{-1}(\delta\omega) = \frac{1}{\mathrm{Det}\,\widehat{\varepsilon}(\delta\omega)} \begin{pmatrix} 1 + \chi_{22}(\delta\omega) & -\chi_{12}(\delta\omega) \\ -\chi_{21}(\delta\omega) & 1 + \chi_{11}(\delta\omega) \end{pmatrix}. \tag{48}$$

Here $\widehat{I}$, $\widehat{\varepsilon}(\delta\omega)$ and $\widehat{\varepsilon}^{-1}(\delta\omega)$ are the $2 \times 2$ matrix unit, RF FB 'permittivity' matrix and its inverse, correspondingly.

This FB may turn self-excited, which is avoided technically by putting zeros of $\mathrm{Det}\,\widehat{\varepsilon}(\delta\omega)$ into the lower half-plane $\mathrm{Im}\,\delta\omega < 0$ through a proper tailoring of $H^{(c,s)}(\delta\omega)$.

Summation of $Z_{kk'}(\omega)$, Eqs.45, 46 with the steady-state beam spectrum

$$J_k^{(0)}(\omega) = 2\pi\delta(\omega - k\omega_0) J_k^{(0)}$$

yields a residual departure of the AC gap voltage w.r.t. its nominal, Eq.37. It is a quantitative measure of a sustained beam loading compensation with the RF FB which must have nonzero gains at $\omega = k\omega_0$.

Option $|H^{(s)}| > |H^{(c)}|$ (it is included into the above equations) allows one to direct more feedback power to counteract dominant quadrature voltage excursions and more dangerous dipole beam instabilities while ensuring the same safety margin against self--excitation.

It is evident hereof that, on substituting Eqs.45–46 for Eqs.33–35, the formulae to follow can be extended to treat the stabilizing impact of the RF FB as well. It allows one to find the residual instability driving impedance of the ACs at the RF to be embraced into the ultimate estimates of the feasible beam damping rates.

# 3. CHARACTERISTIC EQUATION

## A General Case

The total $E$-field at the orbit that governs the beam motion is a sum of two terms

$$E_k^{(tot)}(\Omega) = E_k^{(ext)}(\Omega) + E_k^{(fb)}(\Omega). \tag{49}$$

The former one, $(ext)$, is imposed from the outside, say, by an external RF drive. The latter, $(fb)$, is the induced response of the environment to the coherent motion of the beam. Indeed, its perturbed current harmonics $J_k(\Omega)$ drive the FBs, both unintentional (Eq.30) and issued (Eq.32 with its negative-frequency counterpart), to yield

$$E_k^{(fb)}(\Omega) = -L^{-1} \sum\nolimits_{k'=-\infty}^{\infty} z_{kk'}(k\omega_0 + \Omega)\, J_{k'}(\Omega). \tag{50}$$

Matrix $z_{kk'}(\omega)$ has, at most, three non-trivial elements per row,

$$z_{kk'}(\omega) = Z_{kk}(\omega)\, \delta_{k'k} + Z_{kk'}^{(fb)}(\omega) \left(\delta_{k',k-(h'-\bar{h})\,\mathrm{sgn}k} + \delta_{k',k-(h'+\bar{h})\,\mathrm{sgn}k}\right). \tag{51}$$

The first member in r.h.s. of this equation incorporates effect of all the passive devices available.

From now on, one enters a standard route of the coherent instability analysis, and via Vlasov's linearized equation finds

$$J_k(\Omega) = L \sum\nolimits_{k'=-\infty}^{\infty} y_{kk'}(\Omega)\, E_{k'}^{(tot)}(\Omega), \tag{52}$$

where $y_{kk'}(\Omega)$ is the beam 'admittance' matrix which, say, for the beam of average current $J_0$ in $M \le h$ ($h/M$ is an integer) identical and equispaced bunches is equal to

$$y_{kk'}(\Omega) = C J_0\ (Y_{kk'}(\Omega)/k') \sum\nolimits_{l=-\infty}^{\infty} \delta_{k-k',lM}. \tag{53}$$

In this equation, $Y_{kk'}(\Omega)$ denotes a dispersion integral. In terms of a multipole decomposition, it reads

$$
\begin{aligned}
Y_{kk'}(\Omega) &= -i \sum_{m=-\infty}^{\infty} m \int_0^\infty \frac{\partial F_0(\mathcal{J})/\partial \mathcal{J}}{\Omega - m\Omega_s(\mathcal{J})} I_{mk}(\mathcal{J})\, I^*_{mk'}(\mathcal{J})\, d\mathcal{J} + \\
&\quad + \theta\,(-\mathrm{Im}\,\Omega)\cdot \Delta Y_{kk'}(\Omega).
\end{aligned} \tag{54}
$$

Here $(\psi, \mathcal{J})$ are the longitudinal angle-action variables introduced in the phase-plane $(\vartheta, \vartheta' \equiv d\vartheta/dt)$, $\Omega_s(\mathcal{J}) = d\psi/dt$ is the non-linear synchrotron frequency, $F_0(\mathcal{J})$ is unperturbed bunch distribution normalized to unit. Functions $I^*_{mk}(\mathcal{J})$ are the coefficients of Fourier series which expand a plane wave into a sum over multipoles

$$
I^*_{mk}(\mathcal{J}) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{ik\vartheta(\mathcal{J},\psi) - im\psi}\, d\psi. \tag{55}
$$

Additive term from the second line of Eq.54 ensures analytical continuation of $Y_{kk'}(\Omega)$ into the lower half-plane $\mathrm{Im}\,\Omega < 0$,

$$
\Delta Y_{kk'}(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \mathrm{sgn}\,(\mathrm{Im}\,\mathcal{J}_m) \left( \frac{\partial F_0(\mathcal{J})/\partial \mathcal{J}}{\partial \Omega_s(\mathcal{J})/\partial \mathcal{J}} I_{mk}(\mathcal{J})\, I^*_{mk'}(\mathcal{J}) \right)_{\mathcal{J}=\mathcal{J}_m}, \tag{56}
$$

where $\mathcal{J}_m$ is a complex root of $\Omega = m\Omega_s(\mathcal{J})$ with $\mathrm{Re}\,\mathcal{J}_m > 0$ and $\mathrm{Im}\,\mathcal{J}_m \to 0$ as $\mathrm{Im}\,\Omega \to 0$. $\mathcal{J}_m$ is assumed to be a first-order pole of integrand in Eq.54, complex plane $\mathcal{J}$ being cut along the negative real half-axis.

The leftmost factor $C$ in Eq.53 is

$$
C = \Omega_0^2 / \left( hV_0 \sin\varphi_s \right), \tag{57}
$$

where $\Omega_0 \equiv \Omega_s(0)$ is the small-amplitude synchrotron frequency (circular). In the limit of point bunches,

$$
Y_{kk'}(\Omega) = ik\,k' / \left(\Omega^2 - \Omega_0^2\right), \qquad m = \pm 1. \tag{58}
$$

Now, insert Eq.52 into Eq.50 and use Eq.49 to get 'beam & FB' medium attributes, its 'dielectric susceptibility' matrix $\chi'_{kk'}(\Omega)$,

$$
E_k^{(fb)}(\Omega) = -\sum\nolimits_{k'=-\infty}^{\infty} \chi'_{kk'}(\Omega)\; E_{k'}^{(tot)}(\Omega), \tag{59}
$$

$$
\chi'_{kk'}(\Omega) = \sum\nolimits_{k''=-\infty}^{\infty} z_{kk''}(k\omega_0 + \Omega)\, y_{k''k'}(\Omega) \tag{60}
$$

and 'dielectric permittivity' matrix $\epsilon_{kk'}(\Omega)$,

$$
E_k^{(ext)}(\Omega) = \sum\nolimits_{k'=-\infty}^{\infty} \epsilon_{kk'}(\Omega)\; E_{k'}^{(tot)}(\Omega), \tag{61}
$$

$$
\epsilon_{kk'}(\Omega) = \delta_{kk'} + \chi'_{kk'}(\Omega). \tag{62}
$$

The general characteristic equation is

$$
\mathrm{Det}\,\hat{\epsilon}(\Omega) = 0. \tag{63}
$$

Its roots, the eigen-frequencies of beam coherent oscillations, must be located in the lower half-plane $\mathrm{Im}\,\Omega \le -1/\tau_\epsilon < 0$. Here $\tau_\epsilon$ is the sought-for damping time of beam coherent oscillations which, as well, determines duration of beam injection transients under the FB showing themselves up, mainly, at the dipole side-bands $\Omega \simeq \pm\Omega_0$.

**A Narrow-Band Case**

Label the normal coupled-bunch modes by $n = 0, 1, \ldots, M-1$, phase shift between adjacent bunches being $2\pi n/M$. Introduce the FB half-bandwidth (at base) as $\Delta\omega^{(fb)} = \min(\Delta\omega_{SW'}, \Delta\omega_H, \Delta\omega_{TK})$, and assume $\Delta\omega^{(fb)} < M\omega_0/2$. Hence, there would be only four resonant harmonics $J_k(\Omega)$ of beam current perturbation which belong to the given mode $n$ and cross-talk through the FB. Their subscripts are

$$k'_{1,2} = n + l'_{1,2}M \simeq \pm h', \qquad \overline{k}_{1,2} = n + \overline{l}_{1,2}M \simeq \pm\overline{h} \tag{64}$$

with $l'_{1,2}, \overline{l}_{1,2}$ the integers. The essential $E$-field harmonics $E_k(\Omega)$ to occur within $\Delta\omega^{(fb)}$ are the two with $k = k'_{1,2}$. In this $2\times 2$ case $\mathrm{Det}\,\widehat{\epsilon}(\Omega)$ can be found, which results in characteristic equation to follow

$$1 + \chi'_{k'_1k'_1}(\Omega) + \chi'_{k'_2k'_2}(\Omega) \; + \left[\chi'_{k'_1k'_1}(\Omega)\,\chi'_{k'_2k'_2}(\Omega) - \chi'_{k'_1k'_2}(\Omega)\,\chi'_{k'_2k'_1}(\Omega)\right] \simeq 0. \tag{65}$$

Two observations allow further simplification of this equation:

(i) L.h.s. of Eq.65 involves, through Eq.60, the dispersion integrals $Y_{kk'}$ whose first and second subscripts are $k = k'_{1,2}, \overline{k}_{1,2}$ and $k' = k'_{1,2}$, respectively. Given $\Delta\omega^{(fb)}\Delta\vartheta_0/\omega_0 \ll \pi$, where $\Delta\vartheta_0$ is bunch half-length, $Y_{kk'}$ become slow functions of arguments $k, k'$, which allow substitutions $k'_{1,2} \simeq \pm h'$, $\overline{k}_{1,2} \simeq \pm\overline{h}$ to be performed in subscripts of all the essential $Y_{kk'}$ that enter the characteristic Eq.65.

(ii) Usually, at $\Omega \simeq m\Omega_0 + i0$ a single resonant term $Y^{(m)}_{kk'}$ dominates in $\sum_m$ of Eq.54. Hereof, one arrives at the reflection properties of $Y_{kk'} \simeq Y^{(m)}_{kk'}$,

$$Y_{-k,k'} \simeq Y_{k,-k'} \simeq (-1)^m Y_{kk'}, \qquad Y_{-k,-k'} \simeq Y_{kk'} \tag{66}$$

displayed in the vicinity of a side-band $\Omega \simeq m\Omega_0$.

Up to these two assumptions (also, immediately on adopting the point-bunch approximation, Eq.58), expression in square brackets of Eq.65 vanishes, while the characteristic equation itself reduces to much a simpler form

$$1 + CJ_0\left(\zeta_n(\Omega)\,Y_{h'h'}(\Omega) + \zeta^{(fb)}_n(\Omega)\,Y_{\overline{h}h'}(\Omega)\right) \simeq 0, \tag{67}$$

being put down in terms of the effective, or instability driving, impedances at side-bands $\Omega \simeq m\Omega_0$ of a coupled-bunch mode $n$,

$$\zeta_n(\Omega) \simeq Z_{k'_1k'_1}(k'_1\omega_0 + \Omega)/k'_1 + \;\ldots\quad k'_1 \to k'_2, \tag{68}$$

$$\begin{aligned}\zeta^{(fb)}_n(\Omega) \simeq\; & Z^{(fb)}_{k'_1,k'_1-h'+\overline{h}}(k'_1\omega_0 + \Omega)/k'_1 + \\ & + (-1)^m Z^{(fb)}_{k'_1,k'_1-h'-\overline{h}}(k'_1\omega_0 + \Omega)/k'_1 + \\ & + \quad \ldots \quad k'_1 \to k'_2, \quad h' \to -h', \quad \overline{h} \to -\overline{h}.\end{aligned} \tag{69}$$

The two-term reduced impedance $\zeta_n(\Omega)$ is a standard destabilizing contribution from passive structures at $\omega \simeq \pm h'\omega_0$ which is used, say, in the conventional threshold map technique.

The four-term reduced impedance $\zeta_n^{(fb)}(\Omega)$ accounts for stabilizing effect of the FBs. It may be scaled $\propto 1/J_0$ by electronics. Items with $(-1)^m$, if any, are responsible for the intrinsic asymmetry in damping of within-bunch multipole modes $m$ with opposite parity inherent in FBs with the unbalanced path gains, $H^{(c)} \neq H^{(s)}$.

In RF FBs where $\overline{h}, h' = h$, terms with $Z_{kk}$ and $Z_{kk}^{(fb)}$ from $\zeta_n(\Omega)$ and $\zeta_n^{(fb)}(\Omega)$ play together according to Eqs.45, 46.

There exists a plain physical analogue of a system described by Eq.67.

Indeed, for bunches without incoherent longitudinal tune spread and frequencies $\Omega \simeq m\Omega_0$ use can be made of the approximation

$$Y_{kk'}(\Omega) \simeq im^2\, F_{kk'}^{(m)} / \left(\Omega^2 - (m\Omega_0)^2\right), \tag{70}$$

where $F_{kk'}^{(m)}$ is the bunch formfactor ($F_{kk'}^{(m)} \to kk'\delta_{|m|,1}$ as $\Delta\vartheta_0 \to 0$),

$$F_{kk'}^{(m)} = 2\Omega_0 \int_0^\infty (-\partial F_0(\mathcal{J})/\partial\mathcal{J})\; I_{mk}(\mathcal{J})\, I_{mk'}^*(\mathcal{J})\, d\mathcal{J}. \tag{71}$$

In this case, characteristic Eq.67 to describe a coherent beam mode $(n, m)$ becomes that of a plain oscillator with a retarded feedback drive,

$$d^2x/dt^2 + (m\Omega_0)^2\, x = \int_0^\infty \mathcal{G}_{nm}(t')\, x(t - t')\, dt', \tag{72}$$

whose Green's function $\mathcal{G}_{nm}(t)$ is given through the Fourier transform

$$\mathcal{G}_{nm}(\Omega) = im^2 C J_0 \left(\zeta_n(\Omega)\, F_{h'h'}^{(m)} + \zeta_n^{(fb)}(\Omega)\, F_{\overline{h}h'}^{(m)}\right), \tag{73}$$

$\mathcal{G}_{nm}(t)$ being real only for modes $n = 0$ and $M/2$ (for even $M$).

## 4. Example of Application

As an example of practical application of the impedance approach to FBs, presented below are the results of routine calculations performed during design studies inside the UNK Project [4]. These were to outline technical contours of FB circuits foreseen, both the RF [5] and the beam ones. The design studies of the latter system were performed jointly with A. Malovitsky [6].

### RF Feedback

Take the injection flat-top of the UNK 1st Stage [4], where accelerating voltage is $V_0 = 4.5$ MV per turn; stable phase angle is $\varphi_s = -\pi/2$; revolution frequency is $\omega_0/2\pi = 14.43$ KHz; RF harmonic number is $h = 13{,}860$; radio-frequency is $h\omega_0/2\pi \simeq 200$ MHz; average beam current is $J_0 = 1.6$ A in $M = h$ bunches; bunch half-length is $h\Delta\vartheta_0/\pi = 0.54$.

The accelerating field is to be imposed by $N_{\rm AC} = 6 \times 2 = 12$ conventional copper 200-MHz cavities whose unloaded quality factor and shunt resistance are $Q_T^{(0)} = 49,000$ and $R_T^{(0)} = 7.94$ MOhm. Loading with couplers results in $Q_T = 3{,}100$ and $R_T = 0.5$ MOhm. These define the transfer functions

$$T(\omega) = T'(\omega) = R_T \times \left(1 - iQ_T \frac{\omega^2 - \omega_T^2}{\omega\omega_T}\right)^{-1}. \tag{74}$$

Let the operational detuning from the RF be $(h\omega_0 - \omega_T) = 0.375\omega_T/Q_T$.

The fundamental cavity mode is $E_{010}$. Thus $G^{(a)}(\Theta)$ is the piece-wise constant and nonzero at $|\Theta - \Theta^{(a)}| \leq 0.5\,\Delta\Theta^{(a)}$, where $\Delta\Theta^{(a)} = 1.512{\cdot}10^{-4}$ rad (for cavity of 0.5 m in length). Therefore,

$$G_k^{(a)} = \frac{\sin(0.5\,k\Delta\Theta^{(a)})}{0.5\,k\Delta\Theta^{(a)}} \exp(-ik\Theta^{(a)}), \quad a = \mathrm{AC}. \tag{75}$$

Let electronics after a field probe and RF power amplifier be circuit sections that are wide-band in the scale of $T, T'(\omega)$, i.e. $\Delta\omega_S, \Delta\omega_K \gg \omega_T/Q_T$. Take

$$S(\omega) = R_T^{-1}, \qquad K(\omega) = 1 \tag{76}$$

so as to provide $\max|TKS(\omega)| = 1$, and lump the time delay and real in-out voltage scale gain through joint band-pass transfer function $TKS(\omega)$ inside the inphase and quadrature paths $H^{(\xi)}(\delta\omega)$.

Insert into the latter a pair of identical ideal first-order IIR comb filters. Adjust the net time delay to one turn to get

$$H^{(\xi)}(\delta\omega) = A^{(\xi)} \exp\left(2\pi i\delta\omega/\omega_0\right) \frac{1 - B}{1 - B\exp\left(2\pi i\delta\omega/\omega_0\right)} \tag{77}$$

with $H^{(\xi)}(k\omega_0) \neq 0$. A low-pass filter to limit half-bandwidth (at base) of $H^{(\xi)}(\delta\omega)$ to $\Delta\omega_H < h\omega_0$ is implied, but not shown in this equation.

Under these assumptions, the RF FB behavior is controlled with the following five global parameters: real in-out scale gains through each path $A^{(c)}, A^{(s)}$ (these are twice the open-loop scale gains for the analytical signal, see Eq.18); comb filter's local positive feedback gain $B$; phase shift of the up-mixing carrier $(\phi' - \phi)$ see comments to Eq.42).

To affect appreciably the dipole coherent oscillations, take $B = 0.96$ which sets the comb filter's half-bandwidths near each $\omega = k\omega_0$ (at $-3$ dB) equal to synchrotron frequency $\Omega_0/2\pi = 96$ Hz.

To bias the FB to better handling quadrature voltage excursions and dipole coherent instabilities, take $A^{(c)} = 10$, $A^{(s)} = 50$. Set $(\phi' - \phi)$ to $0.25\pi$. The Nyquist mapping of $\mathrm{Det}\,\widehat{\varepsilon}(\delta\omega)$, Eq.47 shows that in this case the following safety margins against self-excitation are ensured: a factor of two in amplitude (given $A^{(c)}/A^{(s)} = 1/5$), and from 0.0 to $0.45\pi$ in phase $(\phi' - \phi)$. Both comply with the standard RF engineering prescriptions [2].

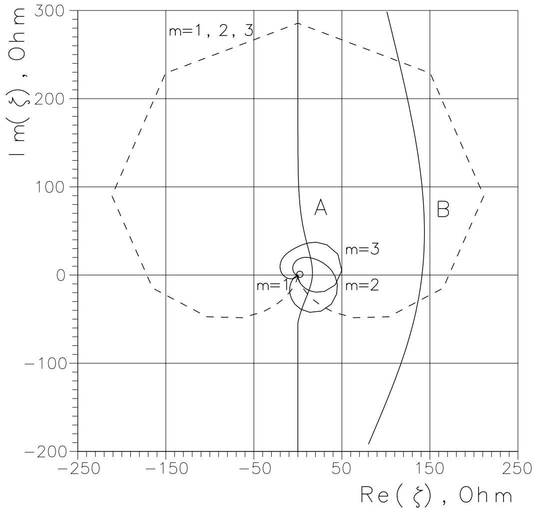


Fig. 2. Residual instability driving impedances of ACs.

Fig.2 is the longitudinal threshold map for the destabilizing effect of ACs. Curves $A$ and $B$ (the images of line $\Omega \simeq m\Omega_s(\mathcal{J}) + i0$ through mapping $(-CJ_0Y_{hh}(\Omega))^{-1}$) are the threshold curves for $m = 1, 2$, respectively. The curve for $m = 3$ runs still farther from the origin. The stable region is the entire left half-plane, and right half-plane vicinity of the origin encircled by the threshold curve.

Solid lines near the origin are drawn through the points $\zeta_n(\Omega) + \zeta_n^{(fb)}(\Omega)$ at $\Omega = m\Omega_0$ for $m = 1, 2, 3$ and various CB modes $n$ counted counter-clockwise. All $N_{\rm AC} = 12$ cavities available in the ring are taken into account. For comparison, the dashed broken line is drawn through the impedances $\zeta_n(\Omega)$ with FB off. (In this case, curves for $m = 1, 2, 3$ are practically undistinguishable.) The RF FB decreases the scale of the apparent destabilizing impedance for dipole within-bunch mode by about a factor of 35, while that for quadrupole and sextupole modes — by about (4 – 4.5) times. Their values fall below the instability threshold.

## CB Beam Feedback

This FB is to serve a dual purpose: (i) to damp longitudinal injection offsets (in phase or momentum), and (ii) to provide a better beam stability against CB lower-order (odd) multipole perturbations. The UNK Project's option [4] is to employ a pair of issued over-coupled ACs driven in quadrature to the net accelerating field as an AD of the beam FB in question.

Thus, take $H^{(c)}(\delta\omega) \equiv 0$ (no inphase, or amplitude control), $h' = h$, $\varphi'_s = \varphi_s$, $N_{\rm AD} = 2$. The AD transfer functions are given by Eq.74 with $R_T$ and $Q_T$ decreased to shorten the response time, say, by a factor of five w.r.t. those of the ACs: $Q_T = 620$, $R_T = 0.1$ MOhm. Tune AD to the RF precisely, $\omega_T = h\omega_0$.

Adopt $\overline{h} = h$. Let a short PU followed by electronics be a circuit section with bandwidth $\Delta\omega_{SW'} \gg \omega_T/Q_T$, and take

$$SW'(\omega) = 1; \qquad G^{(a)}(\Theta) = 2\pi\delta(\Theta - \Theta^{(a)}),\ a = \text{PU}. \tag{78}$$

Use the last of Eqs.76 so as to lump the time delay and real in-out current scale gain through band-pass transfer functions $SW'(\omega)$ and $K(\omega)$ into the quadrature path $H^{(s)}(\delta\omega)$ alone.

Insert into the latter an ideal three-tap periodic FIR filter with a global one-turn delay,

$$H^{(s)}(\delta\omega) = A^{(s)} \exp\left(2\pi i\delta\omega/\omega_0\right) \sum\nolimits_{q=0}^{2} w_q \exp\left(2\pi i\delta\omega d_1 q/\omega_0\right). \tag{79}$$

Here, $w_q$ are real weight coefficients, $d_1$ is an integer delay step measured in a number of turns, $A^{(s)}$ is a real scale gain from the beam quadrature current in the PU gap to RF drive current seen inside the AD gap (twice the gain for the analytical signal, Eq.18). Like earlier in Eq.77, a low-pass filter to ensure $\Delta\omega_H < h\omega_0$ is implied, but skipped in this equation.

Two conditions are imposed on $H^{(s)}(\delta\omega)$ to find weights $w_q$:

(i) For $H^{(s)}(\delta\omega)$ to become a notch filter which rejects heavy beam loading signals, require

$$H^{(s)}(k\omega_0) = A^{(s)} \sum\nolimits_{q=0}^{2} w_q = 0. \tag{80}$$

(ii) For $H^{(s)}(\delta\omega)$ to provide a prescribed phase shift $\upsilon$ at the dipole side-bands, set

$$H^{(s)}(k\omega_0 + \Omega_0) = A^{(s)} \exp(-i\upsilon). \tag{81}$$

We adopt the standard option of a quadrature shift $\upsilon = \pi/2$ in which case, given $\omega_T = h\omega_0$ and $\Omega_0 \ll \omega_T/Q_T$, the utmost all-active correction is applied to the beam coherent 'barycentric' mode $(n, m) = (0, 1)$ which sees a real negative impedance

$$\zeta_0^{(fb)}(\Omega_0) \simeq N_{\text{AD}}\, R_T \left| G_h^{(\text{AD})} G_{-h}^{(\text{PU})} \right| A^{(s)} \sin\varphi_s/h < 0, \tag{82}$$

where $|G_{-h}^{(\text{PU})}| = 1$ due to the last of Eqs.78. (In principle, the beam FB can be tuned through $\upsilon$ so as to impose the strongest active correction to another beam mode $(n, m)$.)

Solving Eqs.80,81 for $w_q$ yields

$$\begin{aligned} w_0 &= 0.5\,(+\sin\mu - \cos\mu\, \cot 0.5\delta\mu_1)\,/\sin\delta\mu_1, && (83)\\ w_1 &= 0.5\cos\mu/\sin^2 0.5\delta\mu_1, && (84)\\ w_2 &= 0.5\,(-\sin\mu - \cos\mu\, \cot 0.5\delta\mu_1)\,/\sin\delta\mu_1, && (85) \end{aligned}$$

$$\mu = \upsilon + \delta\mu_0 + \delta\mu_1, \quad \delta\mu_0 = 2\pi\Omega_0/\omega_0, \quad \delta\mu_1 = 2\pi\Omega_0 d_1/\omega_0. \tag{86}$$

To detect reliably a slow longitudinal motion ($\Omega_0 \ll \omega_0$), one has to adopt a large enough delay step $d_1$. However, with $d_1$ increasing, phase-frequency performances of the circuit do degrade, the beam FB itself tending to destabilize higher-order odd multipole oscillations. Of these, only sextupole ones ($m = \pm 3$) might be of danger in practice. For example, coherent mode $(n, m) = (0, 3)$ experiences a correction with an active component ($\text{Re}\,\zeta_0^{(fb)}(3\Omega_0) \lesssim 0$) until $\delta\mu_1/2\pi \lesssim 1/6$, i.e. for delay steps less than 1/6 of the synchrotron oscillation period. Let us reduce this fraction to 1/10, and set $d_1$ in UNK-1 to 15 turns which entails filter summation weights $w_0 = 2.30$, $w_1 = -3.26$ and $w_2 = 0.96$ for $\upsilon = \pi/2$.

Application of a band-pass correction with large gains $A^{(s)} \gtrsim 12$ would expel off-resonance azimuthal CB modes $n$ of dipole within-bunch oscillations beyond the Landau-damping threshold. Let us adopt a conservative factor-of-two safety margin, and take $A^{(s)} = 6$. The maximal current $|K(h\omega_0)|I_{\max}$ yielded by the power amplifier, as seen in the AD gap, is roughly equal to $|J_h^{(0)}|$, the amplitude of beam RF harmonic. In this case, the value of injection error treated in a linear regime would be (in the units of an RF phase offset) $|h\delta\vartheta_{\text{inj}}| \lesssim 2/A^{(s)}$, i.e. about 20°.

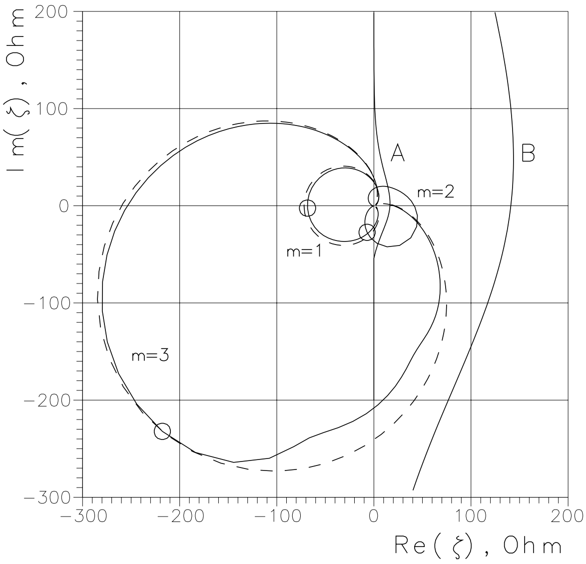


Fig. 3. Stabilizing effect of beam FB.

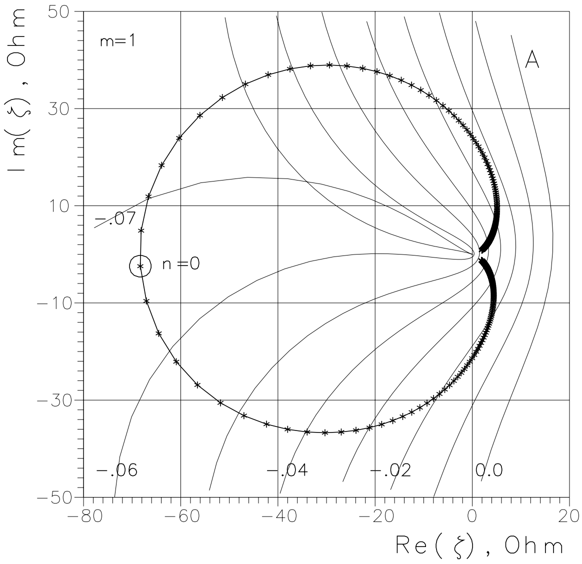


Fig. 4. Damping of dipole oscillations.

Fig.3 is the longitudinal threshold map for the beam FB stabilizing effect. Dashed lines are drawn through $\zeta_n(\Omega)+\zeta_n^{(fb)}(\Omega)$ at $\Omega = m\Omega_0$ for $m = 1, 3$, and account for action of the beam FB alone. (Within-bunch mode $m = 2$ is kept unaffected by the quadrature FB in question.) These lines are transformed into the solid ones by a residual destabilizing impact of the ACs, Fig.2. Points $n = 0$ are marked with circlets for reference, the other CB modes $n$ being counted counter-clockwise. Curves $A$ and $B$ are the threshold ones for $m = 1, 2$, respectively.

The injection transients show themselves up, mainly, as dipole coherent beam motion. Fig.4 is the detailed dipole threshold map with contour lines of constant decrement, i.e. the images of straight lines $\Omega \simeq \Omega_s(\mathcal{J}) + i\Omega_2$ from the lower half-plane $(\Omega)$ through mapping $(-CJ_0Y_{hh}(\Omega))^{-1}$ plotted for $\Omega_2/\Omega_0 = -0.07(0.01)0.0$. These represent the closed-loop system oscillation modes with the slowest decay which define the response time $\tau_\epsilon = \tau_\epsilon(n, m = 1)$ of the 'beam & FB' medium. The contours' excursions from straight lines parallel to imaginary axis are due to nonlinearity of the synchrotron motion which interferes into the corrective impact of FB through a phase-plane mixing (filamentation).

Injection into the UNK 1st Stage is to be performed in bunch trains each filling 1/14 of the orbit. Hence, one should anticipate a nonzero initial inphase beam offsets localized within a 1/14 of the orbit. Their CB mode spectrum is confined, mostly, to modes $|n| \lesssim 14$ (mod$M$). Fig.4 shows that these occur well inside the RF and beam FB bandwidths and thus experience an appreciable damping. Damping time of the injection transients depends on the CB mode index $n$ and falls into the range of

$$0.015 \lesssim 1/\left(\Omega_0\tau_\epsilon\right) \lesssim 0.075$$

or, in terms of incoherent longitudinal tune spread (at base) $\Delta\Omega_s$,

$$0.08 \lesssim 1/\left(\Delta\Omega_s\tau_\epsilon\right) \lesssim 0.42.$$

In a band-pass beam FB whose AD is a cavity and half-bandwidth (at base) is $\omega_0 \lesssim \Delta\omega^{(fb)} \lesssim M\omega_0/2$, the minimal feasible value of damping time $\tau_\epsilon$ is, roughly, an invariant for a given circuit design. It cannot be made less than a certain number of decoherence times $2\pi/\Delta\Omega_s$. Indeed, say, for shorter bunches one is forced to decrease the FB gain $A^{(s)} \propto \Delta\Omega_s$ so as to avoid beam destabilization at off-resonance CB modes $n$. Such systems are destined to operate in regimes with a strong phase-plane mixing when coherent tune shifts are comparable to the incoherent tune spread. Therefore, their study must apply to general beam transfer functions, Eqs.54, 56, rather than to plain second-order oscillatory ones, Eq.70.

**Conclusion**

The two above examples show that the frequency-domain impedance approach is a convenient practical tool to study longitudinal feedbacks in a synchrotron. It provides a deep and detailed insight into 'beam & feedback' system dynamics.

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