STATE RESEARCH CENTER OF RUSSIA

INSTITUTE FOR HIGH ENERGY PHYSICS

IHEP 96-95

G.P.Pron'ko

THE RELATIVISTIC GAMOV VECTORS AND POINCARE SEMIGROUP

Talk given at the XXI International Conference on Group Theoretical Methods in Physics, Goslar, 1996

Protvino 1996

UDK 539.1.01 M–24

Abstract

Pron'ko G.P. The Relativistic Gamov Vectors and Poincare Semigroup: IHEP Preprint 96-95. – Protvino, 1996. – p. 17, refs.: 11.

The main points of the nonrelativistic rigged Hilbert space formulation of unstable states are generalized for the relativistic case. The consideration is made in the framework which may be interpreted as a relativistic Friedrichs model admitting the explicit construction of Gamov vectors. Instead of a nonrelativistic semigroup of time translation, here naturally arises the Poincare semigroup.

Аннотация

Пронько Г.П. Релятивистские векторы Гамова и полугруппы Пуанкаре: Препринт ИФВЭ 96-95. – Протвино, 1996. – 17 с., библиогр.: 11.

В работе строится релятивистское обобщение нерелятивистской теории нестабильных состояний в оснащенном гильбертовом пространстве. Рассмотрение проводится в рамках модели, которая может рассматриваться как релятивистская модель Фридрихса, допускающая явное построение векторов Гамова. Взамен полугруппы трансляций по времени, релятивистская формулировка естественным образом приводит к полугруппе Пуанкаре.

The complex energy eigenstates were introduced long ago by George Gamov for the description of α-decay [1]. Many years had passed since that time until heuristic consideration of Gamov found a rigorous explanation in the new formulation of quantum mechanics based on the Rigged Hilbert Space (RHS). In the long list of references on this subject we shall mention here only a series of papers of Austin-Brussels group [2]-[5] and Arno Bohm's textbook on Quantum Mechanics [6]. This new formulation of quantum mechanics, of course, coincides with the usual one in the common domain but in addition opens a possibility to describe unstable states and irreversible processes, leading to the invention of semi-group of time translation related to a microphysical arrow of time. The main features of this theory could be explicitly demonstrated for the Friedrichs model that serves as an universal example with resonances and which is exactly solvable in non-analytic way [7]. In this model the eigenstates of free hamiltonian — the discrete state $|1>$ and continuum $|\omega>$, $0 \leq \omega \leq \infty$, interact with intensity λ:

$$\begin{aligned} H = H_0 + \lambda V \quad &= \\ = \omega_1 |1><1| + \int_0^\infty d\omega |\omega><\omega| \quad &+ \quad \lambda \int_0^\infty d\omega \left\{ V(\omega)|\omega><1| + V^*(\omega)|1><\omega| \right\}, \end{aligned} \tag{1}$$

where the formfactor $V(\omega)$ is smooth, square integrable function. The solution of eigenvalue problem for (1)

$$(H - E)\Psi = 0 \tag{2}$$

could be represented in the following form:

$$\Psi = f(E)|1> + \int_0^\infty d\omega f(E,\omega)|\omega> . \tag{3}$$

Problem (2) gives two equation for functions $f(E)$ and $f(E,\omega)$:

$$\begin{aligned} (\omega_1 - E) f(E) + \lambda \int_0^\infty d\omega V^*(\omega) f(E,\omega) &= 0, \\ (\omega - E) f(E,\omega) + \lambda f(E) V(E) &= 0. \end{aligned} \tag{4}$$

The solution of (4) and its properties were discussed in numerous publications and we will not dwell on details showing only the straight way to the subject of the present talk. The second equation of (4) permits us to express function $f(E,\omega)$ via $f(E)$:

$$f(E,\omega) = A\delta(\omega - E) - \lambda \frac{V(\omega)}{\omega - E} f(E), \tag{5}$$

where A is an arbitrary constant. Substitution of (5) into (4) immediately gives us the key equation:

$$\left[(\omega_1 - E) - \lambda^2 \int\limits_0^\infty d\omega' \frac{|V(\omega')|^2}{\omega' - E} \right] f(E) = -\lambda A V^*(E). \tag{6}$$

The factor in square brackets in l.h.s. of (6) is nothing else but inverse resolvent $1/\eta(E)$ of H, whose properties are completely defined by the formfactor $V(\omega)$. Under certain conditions [7] this resolvent does not have poles for real, positive E and therefore the general solution of (3) takes the form:

$$\Psi = |E> -\lambda \eta(E) V^* \left[|1> -\lambda \int\limits_0^\infty d\omega \frac{V(\omega)}{\omega - E} |\omega> \right], \tag{7}$$

where we put the constant $A = 1$. The resolvent $\eta(E)$ in (7) is defined for $E \in C$ by the equation:

$$\frac{1}{\eta(E)} = \omega_1 - E - \lambda^2 \int\limits_0^\infty d\omega \frac{|V(\omega)|^2}{\omega - E}. \tag{8}$$

Apparently, the $1/\eta(\omega)$ is the analytic function without singularities on the first sheet of the complex E-plane. The physical solutions of eigenvalue problem (3) corresponding to in- and out- going asymptotic conditions are defined by boundary values of the r.h.s. of (7), when $E \to E \pm i\epsilon$. A close inspection of (8) gets us convinced us that on the second sheet of E- plane the $\eta(\omega)$ may have a set of poles which correspond to unstable states. Their eigenvalues E_c are the solutions of equation:

$$\frac{1}{\eta(E)} = 0. \tag{9}$$

Clearly, each solution of (9) will have a complex conjugated partner on the second sheet as well. The states $\Psi(E)$ with proper continuation to points E_c are the Gamov vectors — the eigenstates with complex eigenvalues. We refer here the reader to the comprehensive discussion of the definition and properties of the Gamov vectors to the above mentioned literature. The question which we may now formulate is the following: if there is a relativistic generalization of the Friedrichs model which possesses the simplicity of non-relativistic example, admits the exact solution and exhibits the universality of description of relativistic unstable states. The answer is positive and the model has been formulated and solved in our joint paper with Professors I.Antoniou and M.Gadella [8]. Here I present the review of this work.

Certainly the obvious example of the relativistic system with unstable states is provided by the field theory of two interacting scalar fields with masses M and m , which satisfy the condition:

$$M > 2m, \tag{10}$$

i.e. the mass of one field lays upper the threshold of two particles state of the second field. Unfortunately if we consider the interaction

$$S_{int} = \lambda \int d^4x \varphi(x) \psi^2(x), \tag{11}$$

where φ-field has mass M and ψ-field has mass m, then the model becomes an ordinary field theory with all difficulties which leave no hope for the exact solution of the eigenvalue problem for its hamiltonian and constructing true asymptotic states. To find the solvable model we must somehow simplify the field theory (11), leaving nonetheless the possibility of creating unstable states. As we have learned from the nonrelativistic Friedrichs model, the physical reason for transforming the stable state (particle) into unstable one is the interaction with the system whose spectrum is continuous and the transition (decay) is allowed energetically. The simplest example of such a system are two particle states without interaction.

In the relativistic case the state vector of two free particles is subject to two constraints (equations):

$$\begin{aligned} (\; p_1^2 - m^2)\psi(x_1, x_2) = 0, \\ (\; p_2^2 - m^2)\psi(x_1, x_2) = 0, \end{aligned} \tag{12}$$

where $p_\mu^{1,2}$ is usual notation for operator of translation of coordinates $x_\mu^{1,2}$. Instead of x_μ^i and p_μ^i we can introduce total and relative coordinates and the corresponding momenta

$$\begin{aligned} x_{1\mu} &= X_\mu + \frac{1}{2}q_\mu; \; p_{1\mu} = \frac{P_\mu}{2} + p_\mu, \\ x_{2\mu} &= X_\mu - \frac{1}{2}q_\mu; \; p_{2\mu} = \frac{P_\mu}{2} - p_\mu. \end{aligned} \tag{13}$$

With commutation relations

$$[X_\mu, P_\nu] = [q_\mu, p_\nu] = -ig_{\mu\nu}, \tag{14}$$

all other commutators vanish.

Equations (12) in these variables become

$$\begin{aligned} [P^2 - (4m^2 \;\; - \;\; p_\perp^2)]\psi(x_\mu, q_\mu) = 0, \\ Pp \;\;\; \psi \;\;\; (x_\mu, q_\mu) = 0. \end{aligned} \tag{15}$$

The first of eq.(15) has a meaning of mass shell condition with squared mass operator μ^2 defined by:

$$\mu^2 = 4m^2 - p_\perp^2; \; p_\perp^\mu = p^\mu - P^\mu \frac{Pp}{p^2}. \tag{16}$$

It is clear that due to second equation (15) the system admits a one-time description and therefore this equation is very important [9]. System (15) is the simplest one among the so called Komar-Todorov systems [10]. In the general case the mass operator may have a more complicated form

$$\mu^2 = \mu^2(p_\mu, q_\mu),$$

but should commutes with Pp-operator. That means that μ^2 could be a function of the following form:

$$\mu^2 = \mu^2(p_\perp^2, q_\perp^2, p_\perp q_\perp).$$

For our purpose even system (15) is too complicated. We'll use a simpler one whose internal degrees of freedom are described by scalar q, instead of 4-vector q_μ. Correspondingly, the momentum conjugated to q is scalar p. In this case there is no need for second equation (15), and mass squared operator is simply

$$\mu^2 = 4m^2 + p^2 = (4m^2 - \frac{\partial^2}{\partial q^2}). \tag{17}$$

The wave function $\psi(X_\mu, q)$ corresponds to only S-wave states contained in $\psi(x_\mu, q_\mu)$ and the reduction of the first to the latter produces $\psi(x_\mu, q)$ even with respect to q :

$$\psi(x_\mu, q) = \psi(x_\mu, -q). \tag{18}$$

The solutions of the generalized Klein-Gordon equation

$$(P^2 - \mu^2)\psi(x_\mu, q) = 0, \tag{19}$$

with μ^2 given in (17) could be written in the following form:

$$\psi(x_\mu, q) = \int d^4k_\mu \int\limits_{-\infty}^{\infty} d\kappa \frac{\cos q\kappa}{(2\pi)^4} e^{-ik_\mu x^\mu} \delta(k^2 - 4m^2 - \kappa^2) B(k_\mu, \kappa), \tag{20}$$

where we took into account (18). The amplitude $B(k_\mu, \kappa)$ is also an even function of κ. Integrating r.h.s. of (20) over k_0 gives:

$$\psi(x_\mu, q) = \int\limits_{-\infty}^{\infty} d\kappa \frac{d^3k \cos \kappa q}{(2\pi)^4 2\epsilon(k, \kappa)} (B^\star(\vec{k}, \kappa) e^{ikx} + B(\vec{k}, \kappa) e^{-ikx}), \tag{21}$$

where $k_\mu = (\epsilon, \vec{k})$ and

$$\epsilon(k, \kappa) = [4m^2 + \kappa^2 + \vec{k}^2]^{1/2}. \tag{22}$$

We can change the variables in (21) making ϵ independent instead of κ

$$\kappa = (\epsilon^2 - k^2 - 4m^2)^{1/2}, \frac{d\kappa}{\epsilon} = \frac{d\epsilon}{\kappa}, \tag{23}$$

then

$$\psi(x_\mu, q) = \int\limits_0^\infty \frac{d\epsilon d^3k \cos\kappa(k)q}{(3\pi)^4\kappa(k_\mu)}(B(\vec{k},\epsilon)e^{ikx} + B^\star(\vec{k},\epsilon)e^{-ikx}). \tag{24}$$

Following the usual way of the second quantization of scalar field *mutatis muduntis* one may construct the quantum field $\psi(x_\mu, q)$. The action functional for field $\psi(x,q)$ could be written as follows:

$$A = \int d^4x dq \frac{1}{2}\left[(\partial_\mu\psi)^2 - \psi\left(4m^2 - \frac{\partial^2}{\partial q^2}\right)\psi\right]. \tag{25}$$

Defining canonical commutation relation for $\psi(x,q)$

$$[\psi(x,q), \pi(x', q'] = i\delta^4(x-x')\frac{\delta(q+q') + \delta(q-q)}{2}, \tag{26}$$

we'll finally obtain the quantum field $\psi(x,q)$

$$\psi(x,q) = \int\limits_{E_0}^\infty \frac{d\epsilon d^3k \cos\kappa(k_\mu)q}{(2\pi)^4\kappa(k_\mu)}[B^+(\vec{k},\epsilon)e^{ikx} + B(\vec{k},\epsilon)e^{-ikx}] \tag{27}$$

with the following commutation relation for creation and annihilation operators B^+ and B

$$[B(\vec{k},\epsilon), B^+(\vec{k}',\epsilon')] = (2\pi)^4\kappa(k_\mu)\delta^4(k_\mu - k'_\mu). \tag{28}$$

Action (25) gives also the energy-momentum operators as well as the Lorentz group generators which in terms of creation and annihilation operators look like

$$P_\mu = \int \frac{d^4k}{(2\pi)^3\kappa(k)} k_\mu B^+(\vec{k},\epsilon)B(\vec{k},\epsilon), \tag{29}$$

$$M_{\mu\nu} = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4\kappa(k)}\left[\left(k_\mu\frac{\partial}{\partial k^\nu} - k_\nu\frac{\partial}{\partial k^\mu}\right)B^+(\vec{k},\epsilon)B(\vec{k},\epsilon) - B^+(\vec{k},\epsilon)\times\right.$$
$$\left.\times\left(k_\mu\frac{\partial}{\partial k^\mu} - k_\nu\frac{\partial}{\partial k^\mu}\right)B(\vec{k},\epsilon)\right]. \tag{30}$$

One may get convinced that commutation relation (28) produces the usual Poincaré algebra of P_μ and $M_{\mu\nu}$ defined in (29) and (30). If we defined the eigenvalue of the operator $\vec{P}$ as $\vec{k}$, the operator P_0 will have the continuous spectrum which lies above the point $E_0 = (4m^2 + \vec{k})^{1/2}$. In what follows we will suppose that every integral over energy will start at this point, otherwise we will explicitly write the integration limits.

The states of the field $\psi(x,q)$ in our generalization of the Friedrichs model will play the role of the system with continuous spectrum, the role of state with discrete eigenvalue will play the states of usual real scalar field $\varphi(x)$:

$$\varphi(\vec{x},t) = \int d\tilde{k}[a^+(\vec{k})e^{ikx} + a(\vec{k})e^{-ikx}], \tag{31}$$

where

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3 2\omega(k)},\ \omega(\vec{k}) = (\vec{k}^2 + M^2)^{1/2}. \tag{32}$$

The creation and annihilation operators in (31) satisfy the usual commutation relations:

$$[a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3 2\omega(k)\delta(\vec{k} - \vec{k}'). \tag{33}$$

The formulation of relativistic generalization of the Friedrichs model will be completed if we switch on the interaction between two subsystems – $\psi(x, q)$ and φ. The latter we'll introduce in the following form:

$$A_{int} = \lambda \int d^4x \int\limits_{-\infty}^{\infty} dq \psi(x, q) f(q) \varphi(x), \tag{34}$$

where the even function $f(q)$ is the Lorentz scalar and its Fourie transform is:

$$f(q) = \int dx \cos xq \alpha(x). \tag{35}$$

Clearly, the $f(q)$ plays the role of formfactor and to avoid the divergencies in our theory we can choose it to be as decreasing as we like.

The total hamiltonian of our system is given now by the equation:

$$\begin{aligned} P_0 &= \int \frac{d^3k dE E}{(2\pi)^4 \kappa(k, E)} B^+(\vec{k}, E) B(\vec{k}, E) + \\ &+ \int \frac{d^3k}{(2\pi)^3 2\omega} \omega a^+(\vec{k}) a(\vec{k}) + \int \frac{d^3k dE}{(2\pi)^3 2\omega} \frac{\lambda \alpha(\kappa(k, E))}{\kappa(k, E)} \times \\ &\times (a(\vec{k}) + a^+(-\vec{k}))(B^+(\vec{k}, E) + B(-\vec{k}, E)). \end{aligned} \tag{36}$$

The generator of three dimensional translations and rotations is the sum of two terms

$$\vec{P} = \int \frac{d^3k dE}{(2\pi)^4 \kappa(k, E)} \vec{k} B^+(\vec{k}, E) B(\vec{k}, E) + \int \frac{d^3k \vec{k}}{(2\pi)^3 2\omega} (\vec{k}) a(\vec{k}), \tag{37}$$

$$\begin{aligned} M_{ij} &= -i \int \frac{d^3k}{(2\pi)^4 \kappa(k, E)} B^+(\vec{k}, E) \left(k_j \frac{\partial}{\partial k_j} - k_j \frac{\partial}{\partial k_i} \right) B(\vec{k}, E) - \\ &- i \int \frac{d^3k}{(2\pi)^3 2\omega} a^+(\vec{k}) \left(k_i \frac{\partial}{\partial k_j} - k_j \frac{\partial}{\partial k_i} \right) a(\vec{k}), \end{aligned} \tag{38}$$

but Lorentz-boost generators acquire an additional term due to the interaction:

$$M_{oi} = i \int \frac{d^3k dE}{(2\pi)^4 \kappa(k, E)} B^+(\vec{k}, E) \left(E \frac{\partial}{\partial k_i} + k_i \frac{\partial}{\partial E} \right) B(\vec{k}, E) +$$

$$
\begin{aligned}
&+ \quad i\int \frac{d^3k}{(2\pi)^3 2\omega} a^+ \left(\omega \frac{\partial}{\partial k_i}\right) a(\vec{k}) + \\
&+ \quad \frac{i}{2}\int \frac{d^3k dE}{(2\pi)^3} \frac{\lambda\alpha(\kappa(k,E))}{\kappa(k,E)} \left(E\frac{\partial}{\partial k_i} + k_i \frac{\partial}{\partial E}\right) \frac{B(\vec{k},E) + B^+(-\vec{k},E)}{E} \times \\
&\times \quad \frac{a(-\vec{k}) + a^+(k)}{w}. \qquad (39)
\end{aligned}
$$

Again, using commutation relations (28) and (33) one could obtain the Poincaré algebra for P_μ and $M_{\mu\nu}$ defined by (36)-(39).

The task we are facing now is the diagonalizing of hamiltonian (36). This could be written as the eigenvalue problem

$$
[P_\mu, b^+(E,\vec{k})] = k_\mu b^+(E,\vec{k}), \qquad (40)
$$

where $k_\mu = (E,\vec{k})$ and P_μ is given by (36), (37).

The solution of the eigenvalue problem could be obtained similar to the nonrelativistic case and we will not present it here. The result could be expressed with the help of the Green function $G(E,k)$ — the object analogous to nonrelativistic resolvent $\eta(E)$. This Green function is defined by the equation:

$$
G(E,k) = \frac{1}{\omega^2 - E^2 - \Pi(E,k)}, \qquad (41)
$$

where

$$
\Pi(E,k) = \int_{E_0}^{\infty} dE' 2E' \frac{\rho(E',k)}{E'^2 - E^2} \qquad (42)
$$

and $E_0 = (4m^2 + k^2)^{1/2}$ (see note after eq.(30)). The spectral density $\rho(E,k)$ is defined by the formfactor $f(q)$:

$$
\rho(E,k) = 2\pi \frac{\lambda^2 \alpha^2(\kappa(E,k))}{\kappa(E,k)}. \qquad (43)
$$

Equations (41) and (42) are written for the complex values E^2 and we'll specify its meaning for real E^2 later. First of all we'll suppose that integral in r.h.s. of (42) exist and moreover

$$
\Pi(E,k) \to 0, \qquad (44)
$$

when $E \to \infty$. From (42) we see that $\Pi(E,k)$ is an analytical function of E^2 with a cut on physical sheet from $E^2 = E_0^2$ to infinity. The discontinuity of $\Pi(E,k)$ on this cut is given by:

$$
\Delta\Pi(E,k) = \Pi(E + i\epsilon, k) - \Pi(E - i\epsilon, k) = 2\pi i \rho(E,k). \qquad (45)
$$

It is assumed that $\rho(E,k)$ vanishes at threshold

$$
\rho(E_0(k), k) = 0. \qquad (46)
$$

The function $G(E,k)$ also has the same analytical properties, and its discontinuity has the following form:

$$\begin{aligned}\Delta \quad & G(E,k) = G(E+i\epsilon,k) - G(E-i\epsilon,k) = \\ = \quad & \rho(E,k)G(E+i\epsilon,k)G(E-i\epsilon,k) = \rho(E,k)|G(E,k)|^2,\end{aligned} \tag{47}$$

therefore $G(E,k)$ satisfies the following dispersion relation:

$$G(E,k) = \int dE'^2 \frac{1}{E'^2 - E^2} \rho(E',k)|G(E',k)|^2, \tag{48}$$

which is really the Kallen-Lehman representation for the propagator. Under assumption (44), it is evident that

$$G(E,k) \to -\frac{1}{E^2}, \tag{49}$$

when $E \to \infty$, and comparing (49) and (48) we conclude that

$$\int\limits_{E_0^2}^{\infty} dE'^2 \rho(E',k)|G(E',k)|^2 = 1. \tag{50}$$

Also the following statement, based on (44) is valid:

$$G(E,k) \to \frac{1}{\omega^2 - E^2} + o(\frac{1}{\omega^2 - E^2}), \tag{51}$$

$$|\omega^2 - E^2| \to \infty$$

wherefrom we get

$$\int\limits_{E_0^2}^{\infty} dE'^2 (\omega^2 - E'^2) \rho(E',k)|G(E',k)|^2 = 0. \tag{52}$$

These sum rules (50) and (52) will be used in what follows.

The unstable case which we are going to consider here corresponds to the Green function $G(E,k)$ (41) without poles on real axis in the complex plane E, its only singularity on physical sheet is the cut. Two unphysical sheets of the function $G(E,k)$ in the E^2 plane, nearest to the physical one we will denote as (0)-upper and (2)-lower. The way to (0) from (1)-physical sheet is up from the lower rim of the cut, while the way to (2)-sheet is down from the upper rim of the cut. Correspondingly we can define the (0) and (2) sheets of $G(E,k)$ on the E-plane. In an unstable case the Green function has the pole singularity on (0) and (2) sheets which lie above the point $E^2 = k^2 + \mu_c^{\star 2}$ and $E^2 = k^2 + \mu_c^2$ correspondingly, where

$$\mu_c^2 = \mu^2 - i\mu\Gamma \tag{53}$$

and μ and Γ are real positive numbers defined by the formfactor $f(q)$.

The solutions of eigenvalue problem (40), corresponding to in-going boundary conditions are given in terms of the Green function:

$$b^+_{in}(E,k) = B^+(E,k) + 2\pi\alpha(\kappa(E,k))G(E+i\epsilon,k)\left[\int dE'\frac{\lambda\alpha}{\kappa}(E',k)\times\right.$$

$$\left.\times\left(\frac{B^+(E,k)}{E'-E-i\epsilon} - \frac{B(E',-k)}{E'+E}\right) - \frac{(E+\omega)a^+(k)+(E-\omega)a(-k)}{2\omega}\right]$$

for positive energy and

$$b_{in}(E,k) = B(E,k) + 2\pi\alpha(\kappa(E,k))G(E-i\epsilon,k)\left[\int dE'\frac{\lambda\alpha}{\kappa}(E',k)\times\right.$$

$$\left.\times\left(\frac{B(E',k)}{E'-E+i\epsilon} - \frac{B^+(E',-k)}{E'+E}\right) - \frac{(E+\omega)a(k)+(E-\omega)a^+(-k)}{2\omega}\right]$$

for negative energy

$$E \geq E_0(k) = (4m^2+k^2)^{1/2}. \tag{54}$$

The out-set of operators corresponds to change $\epsilon \to -\epsilon$ in (53).

Note that in the stable case, when the threshold E_0 lies above mass of the $\varphi(x)$-field the Green function $G(E,k)$ has a simple pole on real axis E which lies below threshold $E = E_0$ in the point $E = \omega(k)$. In this case there appears an additional solution of (40) which corresponds to perturbed operators $\tilde{a}(k)$ and $\tilde{a}^+(k)$. Together with operators $b(k,E)$ and $b^+(k,E)$ these operators form a complete set which is connected with initial one — $a(k), a^+(k); B(k,E), B^+(k,E)$ by the Bogolubov transformation. In the instable case the operators $\tilde{a}(k)$ and $\tilde{a}^+(k)$ are absent but nevertheless the Bogolubov transformation exists.

The commutation relations of new operators $b_{\substack{in\\out}}(k,E)$ and $b^+_{\substack{in\\out}}(k,E)$ are:

$$[b_{in}(k,E), b^+_{in}(k',E')] = [b_{out}(k,E), b^+_{out}(k',E')] = (2\pi)^4\kappa(k,E)\delta(E-E')\delta^3(\vec{k}-\vec{k}'),$$

$$[b_{out}(k,E), b^+_{in}(k',E')] = (2\pi)^4\kappa(k,E)\delta(E-E')\delta^3(\vec{k}-\vec{k}')\frac{G(E+i\epsilon,k)}{G(E-i\epsilon,k)},$$

$$[b_{in}(k,E), b^+_{out}(k',E')] = (2\pi)^4\kappa(k,E)\delta(E-E')\delta^3(\vec{k}-\vec{k}')\frac{G(E-i\epsilon,k)}{G(E+i\epsilon,k)}, \tag{55}$$

all other commutators vanish.

In terms of new operators the energy momentum vector is diagonal, which follows from the fact that b and b^+ solve the eigenvalue problem. Also, direct calculation gives:

$$P_\mu = \int \frac{d^3k dE}{(2\pi)^4\kappa(k,E)} k_\mu b^+_{\substack{in\\out}}(k,E) b_{\substack{in\\out}}(k,E), \tag{56}$$

where $k_\mu = (E,\vec{k})$ and l.h.s. of (56) is given by eqs.(36) and (37) up to the irrelevant c-number constant in P_0, which arises due to operators ordering. Further important properties of new operators is its transformation. The part of it, corresponding to three dimensional rotations is trivial therefore we present only commutation of b-operators with Lorentz boosts:

$$[M_{oi}, b^{+}_{\substack{in\\out}}(E,k)] = i\left(E\frac{\partial}{\partial k_i} + k_i\frac{\partial}{\partial E}\right) b^{+}_{\substack{in\\out}}(E,k), \tag{57}$$

which manifest the correct law of Lorentz transformation of scalar function of 4-momenta $k_\mu = (E, \vec{k})$. Note that for calculation of (55)-(57) we used only an explicit form of $b(E,k)$ and fundamental commutation relation of initial operators — (28) and (33) together with the properties of Green function $G(E,k)$.

Commutation relations (55) promise a simple way for constructing the space of states of our system in the Fock space of unperturbed system formed by operators $B^+(E,k)$ and $a^+(k)$ above initial vacuum $|0>$, defined by

$$B(E,k)|0>= 0, a(k)|0>= 0. \tag{58}$$

The explicit form of $b(E,k)$ got us convinced that

$$b(E,k)|0>\neq 0, \tag{59}$$

i.e. there arises a new vacuum state which is a superposition of states with an arbitrary number of particles of B- and a-types. This new vacuum has the following form:

$$\Omega = e^V|0>, \tag{60}$$

where the exponential of dressing operator V is quadratic in creation operators $B^+(E,k)$ and $a^+(k)$. The new vacuum is defined by equation:

$$b_{\substack{in\\out}}\Omega = 0. \tag{61}$$

The solution of () is given with the help of factorization problem of Green function $G(E,k)$. The latter is formulated in the following way: let the Green function $G(E,k)$ have the properties stated above, find the function $\gamma(E,k)$, analytic in the right semiplane, such that

$$\gamma(E,k)\gamma(-E,k) = G(E,k) \tag{62}$$

on the whole complex E- plane with additional conditions which guaranties its uniqueness:

$$\frac{1}{\gamma(E,k)} + E + \omega + 2x + \int_{E_0}^{\infty} dE' \frac{\rho(E',k)\gamma(E',k)}{E'+E} = 0, \tag{63}$$

$$\int_{E_0}^{\infty} dE' \rho(E',k)\gamma(E',k) = 2x(\omega + x), \tag{64}$$

where $\rho(E,k)$ defined in (43), $\omega = (k^2 + M^2)^{1/2}$ and x is the function of k.

From (63) it is evident that $\gamma(E,k)$ has a cut in the left semiplane with discontinuity defined by discontinuity of $G(E,k)$, the asymptotic behaviour of $\gamma(E)$ is

$$\gamma(E,k) \to -\frac{1}{E} + \frac{\omega+2x}{E^2} + 0\left(\frac{1}{E^3}\right), \tag{65}$$

when $E \to \infty$. All the equations (62)-(64) hold true in the whole complex E-plane. For the real E we have to use an appropriate limiting procedure ($E \pm i\epsilon$).

Having the solution of the factorization problem of the Green function we can present the explicit form of the exponential of dressing operator

$$V(B^+, a^+) = \int \frac{d^3k}{(2\pi)^3 2(\omega+x)} \left\{ \frac{1}{2} \int_{E_0}^{\infty} dE dE' q(k,E) q(k,E') \left(1 + \frac{2(\omega+x)}{E+E'}\right), \right.$$

$$\left. B^+(k,E)B^+(-k,E) + \int_{E_0}^{\infty} dE q(E,k) B^+(k,E) a^+(-k) - \frac{x}{2\omega} a^+(k) a^+(-k) \right\}, \tag{66}$$

where

$$q(k,E) = \frac{\lambda\alpha(\kappa(k,E))}{\kappa(k,E)} \gamma(k,E). \tag{67}$$

This operator V defines new, Lorentz invariant vacuum state. Note, that the Ω is a new vacuum for both in- and out- operators $b(k,E)$. Now we can construct new Fock space acting on Ω with creation operators $b^+(k,E)$. In particular, one-particle state is

$$b^+_{in}(k,E)\Omega = \left\{ B^+(k,E) + \frac{2\pi\lambda\alpha(\kappa(k,E))}{2(\omega+x)} \gamma(-E-i\epsilon) \times \right.$$

$$\left. \times \left[\int dE' q(k,E') \left(1 + \frac{2(\omega+x)}{E'-E-i\epsilon}\right) B^+(k,E) + a^+(k) \right] \Omega \right\}, \tag{68}$$

where we have used the explicit form of $B^+(k,E)$ and Ω to eliminate the annihilation operators $B(k,E)$ and $a(k)$. This equation has to be compared with equation (7). If we forget for a time being that the creation operators act in (68) on the new vacuum Ω, there is a correspondence between (7) and (68) and now we get knowledge what different objects in nonrelativistic case are remnants of the relativistic one. In particular, the one-particle resolvent $\eta_+^{-1}(E)$ is what remained of $\gamma(k,-E)$ — "square root" if the Green function or, more precisely — the solution of the factorization problem. Their properties, however, are similar. For example, let us consider the Green function $G(E,k)$ in the vicinity of its complex pole of the point μ_c^2 — eq.(53)

$$G(E,k) \simeq \frac{c^2}{-E^2 + (\mu_c^2 + k^2)}, \tag{69}$$

when $E^2 - k^2 \sim \mu_c^2$, with c being some constant. At this point the solution of factorization problem with the required asymptotic properties apparently has the following form:

$$\gamma(E) \simeq -\frac{c}{E + (\mu_c^2 + k^2)^{1/2}},$$

$$\gamma(-E) \simeq + \frac{c}{E-(\mu_c^2+k^2)^{1/2}}, \tag{70}$$

when $E \sim E_c$, which coincides with the behaviour of $\eta^{-1}(E)$.

Now we are ready to discuss the main point of the present paper — an instable relativistic particle. As was already stated the latter manifests itself in the structure of operators $b^{+}_{\substack{in\\out}}(k,E)$ and, consequently in the energy dependence of eigenstates of total hamiltonian P_0. Consider, for example, state (68). Due to the presence of $\gamma(-E-i\epsilon)$, the state $\Phi_{in}(k,E)$

$$\Phi_{in}(k,E) = b_{in}(k,E)\Omega \tag{71}$$

has a pole out the second sheet of complex E-plane, where we can start from real axis on the physical sheet, from the upper rim of the cut. This statement, as well as all other considerations of the present section acquire its rigorous mathematical foundation in the framework of rigged Hilbert space [2] and its generalization — rigged Fock space [11]. Referring the reader for details to these papers we shall very briefly point out the needed definition in the due course. In the vicinity of point $E = E_c(k)$

$$E_c(k) = (\vec{k}^2+\mu^2-i\mu\Gamma)^{1/2} =$$

$$= A(k) - iB(k), \tag{72}$$

where

$$A(k) = \left\{\frac{[(\vec{k}^2+\mu^2)^2+\mu^2\Gamma^2]^{1/2}+(\vec{k}^2+\mu^2)}{2}\right\}^{1/2},$$

$$B(k) = \left\{\frac{[(\vec{k}^2+\mu^2)^2+\mu^2\Gamma^2]^{1/2}-(\vec{k}^2+\mu^2)}{2}\right\}^{1/2}$$

the state $\Phi_{in}(E,k)$ could be written as

$$\Phi_{in}(E,k) \simeq \frac{1}{E-E_c(\vec{k})}\varphi_G^{in}(\vec{k}), \tag{73}$$

where index G stands for Gamov. According to the general consideration of Bohm and Gadella [2] the Gamov vector $\varphi_G(\vec{k})$ should be understood as follows. Consider the state $\varphi(E,\vec{k})$:

$$\varphi(E,\vec{k}) = \frac{1}{2(\omega+x)\gamma(E,k)}\left[\int_{E_0}^{\infty} dE' q(E',k)\left(1+\frac{2(\omega+x)}{E'-E}\right)B^{+}(k,E')+\right.$$

$$\left. a^{+}(k)\right]\Omega. \tag{74}$$

If we formally set $E = E_c$ in (74), we as well formally obtain the residue in the pole $\varphi_G^{in}(\vec{k})$, but this couldn't be done because the r.h.s. of (74) defined only for E on the physical sheet — the integration path is along the real axis, and to reach $E = E_c(k)$ we have to make analytic extension of (74). This analytic extension cannot be made for creation operator $B^+(k,E)$, but according to the RHS philosophy considering $\varphi(E,\vec{k})$ as an antilinear functional for the appropriate space of test function (states) we can define analytic continuation of scalar product $(f\varphi(E,\vec{k}))$ to the point $E = E_c(\vec{k})$. It is clear that this continuation is based on that of

$$f(\vec{k},E) \equiv (f, B^+(k,E)\Omega) \tag{75}$$

to the lower half plane. In this case

$$(f, \varphi(E_c(k),\vec{k})) = \frac{1}{2(\omega+x)\gamma(E_c,k)} \left[\int_{E_0}^{\infty} dE' q(E',k) \left(1 + \frac{2(\omega+x)}{E'-E_c(k)}\right) f(k,E') - \right.$$

$$\left. -f(\vec{k}) \right] \Omega, \tag{76}$$

where we denote

$$f(\vec{k}) \equiv (f, a^+(k)\Omega),$$

has a meaning as the complex number.

The simplest choice for $f(\vec{k},E)$, which does not depend on the location of $E_c(\vec{k})$ is, as in nonrelativistic case [2], a square integrable Hardy function of energy from below. The Hardly class H^2_- from below is formed by function on real axis which are boundary values of functions analytic in the lower half plane and square integrable.

Alternatively one can define the state vector $\varphi^{out}_G(\vec{k})$, extending $\Phi_{out}(E,\vec{k})$ in the vicinity of point $E = E^\star_c(\vec{k})$:

$$\Phi_{out}(E,\vec{k}) = \frac{1}{E - E^\star_c(k)} \varphi^{out}_G(\vec{k}). \tag{77}$$

In this case the space of test functions states which make sense of analytic extension of (74) consists of such f, that (75) has analytic continuation to the upper half plane i.e. belongs to the Hardly class H^2_+.

As was pointed out in the discussion of nonrelativistic case [2]–[6] the states $\varphi^{in}_G(\vec{k})$ are generalized eigenstates of hamiltonian P_0 with complex eigenvalues $E_c(k)$ and $E^\star_c(k)$ correspondingly. Also these states exhibit a remarkable property to have one way evolution under action $u_t = \exp(-iP_0 t)$. The same properties hold true in the relativistic case as well.

So far we have dealt with generalized ket-vectors $\varphi^{in}_G(\vec{k})$ and $\varphi^{out}_G(\vec{k})$ which were antilinear functionals on the space of test functions f_- and f_+ with properties

$$(f_\pm, B^+(k,E)\Omega) = f_\pm(k,E) \in H^2_\pm. \tag{78}$$

Alternatively we can define the generalized bra-vectors $\varphi^{in}_G(\vec{k})$ and $\varphi^{out}_G(\vec{k})$ as linear functionals whose space of test functions are exchanged.

Going back to the definition of creation and annihilation operators $b^+_{in(out)}(k,E)$ and $b_{in(out)}(k,E)$ we can also define operators which "create" and "annihilate" Gamov states. Indeed, let us consider

$$\psi^+(k,E) = \int_{E_0}^{\infty} dE' \frac{\lambda\alpha}{\kappa}(E',k)\left(\frac{B^+(k,E')}{E'-E} - \frac{B(-k,E')}{E'+E}\right) -$$

$$- \frac{(E+\omega)a^+(k) + (E-\omega)a(-k)}{2\omega} \tag{79}$$

and

$$\psi(k,E) = \int_{E_0}^{\infty} dE' \frac{\lambda\alpha}{\kappa}(E',k)\left(\frac{B(k,E')}{E'-E} - \frac{B^+(-k,E')}{E'+E}\right) -$$

$$- \frac{(E+\omega)a(k) + (E-\omega)a^+(-k)}{2\omega}. \tag{80}$$

Apparently that analytic continuation described above for the state $\psi^+(k,E)\Omega$ produces states $\varphi^{in}_G(\vec{k})$ or $\varphi^{out}_G(\vec{k})$, and we can consider $\psi^+(\vec{k},E_c(k))$ and $\psi^+(\vec{k},E^\star_c(\vec{k}))$ as operators which create Gamov ket-states. In the same sense $\psi(\vec{k},E_c(\vec{k}))$ and $\psi(\vec{k},E^\star_c(k))$ create Gamov bra-states. The nonvanishing commutation relations for these operators are:

$$[\psi(\vec{k},E_c(k)),\psi^+(\vec{k}',E_c(k'))] = (2\pi)^3\delta(\vec{k}-\vec{k}')2E_c(k)Z,$$

$$[\psi(\vec{k},E^\star_c(k)),\psi^+(\vec{k}',E^\star_c(k'))] = (2\pi)^3\delta(\vec{k}-\vec{k}')2E^\star_c(k)Z^\star, \tag{81}$$

where

$$Z = \frac{d}{dE^2}G^{-1}(E,k)|_{E^2-k^2=\mu_c^2}.$$

These commutation relation may be a starting point for the introduction of the Gamov field, which will be considered elsewhere.

Now let us discuss the Lorentz transformation of Gamov vectors. Commutation relations (55) together with Lorentz invariance of vacuum state provide us with infinitesimal transformation of state (71):

$$M_{oi}\Phi_{in}(\vec{k},E) = i\left(k_i\frac{\partial}{\partial E} + E\frac{\partial}{\partial k_i}\right).\Phi_{in}(\vec{k},E). \tag{82}$$

For the finite boost transformation this gives

$$U(\alpha)\Phi_{in}(\vec{k},E) = \exp i\alpha_i M_{oi}\Phi_{in}(k,E) =$$

$$\Phi_{in}(\vec{k}'_\alpha,E'_\alpha), \tag{83}$$

where

$$E'_\alpha = Ech\alpha - (\vec{n}\vec{k})sh\alpha,$$
$$\vec{k}_\alpha = \vec{k} - \vec{n}(\vec{k}\vec{n}) + \vec{n}((\vec{k}\vec{n})ch\alpha - Esh\alpha) \tag{84}$$
$$\alpha_i = \alpha n_i,\ \vec{n}^2 = 1.$$

The r.h.s. of (83) has a pole at the point $E'_\alpha = E_c(k'_\alpha)$ which, of course, coincides with $E = E_c(k)$ on the E-plane because the complex mass does not change. Comparing the pole terms in both sides of (85) we come to conclusion that

$$U(\alpha)\varphi_G^{in}(\vec{k}) = \varphi_G^{in}(\vec{k}'_\alpha) \tag{85}$$

with

$$\vec{k}'_\alpha = \vec{k} - \vec{n}(\vec{k}\vec{n}) + \vec{n}((\vec{k}\vec{n})ch\alpha - E_c(k)sh\alpha).$$

This equation, being understood literary leads to nonsense because the transformed space components of the momentum become complex. The difficulty could be resolved making use again of the RHS. Indeed, the only successive definition of the Gamov state is the analytic continuation of antilinear functional $(f, \varphi(\vec{k}, E))$ to the point $E = E_c(k)$, so, instead of (83) we should define the Lorentz transformation of φ_G^{in} as the following:

$$(f, U(\alpha)\varphi_G^{in}(k)) \equiv (U^\star(\alpha)f, \varphi(\vec{k}, E))|_{E\to E_c(\vec{k})}. \tag{86}$$

The only thing we should take care of is that the transformed f stays in the admitted space of test functions.

The same procedure holds true for generalized ket-$\varphi_G^{out}(k)$ as well as for Gamov bra-vectors.

With this definition of Lorentz transformations of Gamov states (three-dimensional rotations brings no difficulty) we can consider incorporation of a semigroup of time translation into the Poincaré group. The action of the general element of translation subgroup $T(a)$ parameterized by 4-vector a_μ on Gamov ket $\varphi_G^{in}(\vec{k})$ is defined by equation, similar to (86):

$$(f, T(a)\varphi_G^{in}(\vec{k})) \equiv (T(a)^+ f, \varphi_G^{in}(\vec{k})) \tag{87}$$

with

$$T(a) = \exp(-iP_\mu a^\mu),$$

where P_μ is energy-momentum. The remarkable property of translations, as was pointed out in nonrelativistic case, is that it leaves the space of test functions for $\varphi_G^{in}(k)$ unchanged only if

$$a_0 \geq 0, \tag{88}$$

i.e. the state $\varphi_G^{in}(\vec{k})$ may evolve only in a positive time direction, while the state $\varphi_G^{out}(\vec{k})$ may evolve only backward in time, making the collapse of wave function impossible.

Now let us combine the translation $T(a)$ with the Lorentz transformation

$$\begin{aligned}(f, T(a)U(\alpha)\varphi_G^{in}(\vec{k})) &\equiv (U^+(\alpha)T^+(a)f, \varphi_G^{in}(\vec{k})) = \\ &= (U^+(\alpha)T^+(a)U^{+-1}(\alpha)U^+(\alpha)f, \varphi_G^{in}(\vec{k})) = \\ &= (T^+(a'(-\alpha))U^+(\alpha)f, \varphi_G^{in}(\vec{k})) = \\ &= e^{i(a'_0(-\alpha)E_c(k) - \vec{a}(-\alpha)\vec{k})}(U^+(\alpha)f, \varphi_G^{in}(\vec{k})),\end{aligned} \tag{89}$$

where in the last equation (89) we have used the fact that $\varphi_G^{in}(\vec{k})$ is the generalized eigenvector of P_μ with eigenvalue $(E_c(k), \vec{k})$. The $a'_\mu(\alpha)$ in (89) is given by:

$$a'_0(-\alpha) = a_0 ch\alpha + \vec{n}\vec{a} sh\alpha,$$

$$\vec{a}'(-\alpha) = \vec{a} - \vec{n}(\vec{a}\vec{n}) + \vec{n}(\vec{a}\vec{n} ch\alpha + a_0 sh\alpha). \qquad (90)$$

Apparently, the space of test functions will stay unchanged only if

$$a'_0(-\alpha) \geq 0. \qquad (91)$$

This inequality gives us the set of translations whose action on $\varphi_G^{in}(\vec{k})$ may be defined: indeed, if the vector a_μ is time-like with nonnegative zero component then it has this property in each frame of reference. So, in the relativistic case the permitted translations of $\varphi_G^{in}(\vec{k})$ belong to a future cone V_+. Alternatively permitted translations of $\varphi_G^{out}(\vec{k})$ belong to a past cone V_-. Note, that in the nonrelativistic case all possible translations were separated into future and past oriented ones (being nevertheless the remnants of $V_+ \cup V_-$).

In the relativistic case there exist the translations onto space-like vectors, which have no analogs in the nonrelativistic case and which cannot be defined either for $\varphi_G^{in}(\vec{k})$, or for $\varphi_G^{out}(\vec{k})$.

Taking the semidirect product of $\{T_\pm(a) : a_\mu \in V_\pm\}$ and the Lorentz group we obtain the Poincaré semigroup $P_\pm$ which is the motion group for the Gamov states. It will be very interesting to construct the whole set of representations for this semigroup.

The work was supported in part by Russian Foundation for Fundamental Research (Grant 95-01-00647a) and Solvay Institute.

References

[1] G.Gamov. Z.Phys., v.51, 1928, 510.

[2] A.Bohm, M.Gadella. "Dirac kets, Gamov vectors and Gel'fand triplets", Springer-Verlag, 1989.

[3] I.Prigogine. Phys. Rep., v.219 , 1992, 93;
I.Antoniou. Nature, v.338,1989, 210.

[4] A.Bohm. J. Math. Phys., v.22, 1981, 2813; Lett. Math. Phys., v.3, 1974, 455.

[5] E.C.G.Sudarshan, C.Chiu, V.Gorini. Phys. Rev. v.D18 1978, 2914;
T.Petrovsky, I.Prigogine, S.Tasaki. Physica, v.A173, 1991,175.

[6] A.Bohm."Quantum Mechanics", Springer 1993, 3-d edition.

[7] K.Friedrichs. Comm. Pure Appl. Math. v.1, 1948, 361.

[8] I.Antoniou, M.Gadella, G.P.Pron'ko (to be published).

[9] Ph.Droz-Vincent. In: Lecture Notes on Physics. V.162, Springer-Verlag (1982), 75.

[10] A.Komar. Phys. Rev., v.D18 (1981), 1887; 3017 (1978); v.D19 (1979) 2908;
I.Todorov. Phys.Rev., v.3 (1971), 2351.

[11] M.Gadella. J.Math.Phys., v.26, 1985, 725.

Received November 18, 1996

Г.П.Пронько
Релятивистские векторы Гамова и полугруппы Пуанкаре.

Оригинал-макет подготовлен с помощью системы LaTeX.
Редактор Е.Н.Горина. Технический редактор Н.В.Орлова.

Подписано к печати 21.11.96. Формат $60 \times 84/8$. Офсетная печать.
Печ.л. 2,12. Уч.-изд.л. 1,63. Тираж 250. Заказ 844. Индекс 3649.
ЛР №020498 17.04.97.

ГНЦ РФ Институт физики высоких энергий
142284, Протвино Московской обл.

Индекс 3649

П Р Е П Р И Н Т 96-95, И Ф В Э, 1996