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# TWO-DIMENSIONAL ULTRA-TODA INTEGRABLE MAPPINGS AND CHAINS (ABELIAN CASE)

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**Abstract**

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A new class of integrable mappings and chains is introduced. Corresponding (1+2) integrable systems invariant with respect to such discrete transformations are represented in an explicit form. Soliton like solutions of them are represented in terms of matrix elements of fundamental representations of semisimple $A_n$ algebras for the given group element.

**Аннотация**

Лезнов А.Н. Двумерные интегрируемые ультра-Тода отображения и цепочки: Препринт ИФВЭ 97-16. – Протвино, 1997. – 16 с., библиогр.: 12.

Введен новый класс двумерных интегрируемых отображений и цепочек. Указан путь построения иерархий (1+2) интегрируемых систем инвариантных относительно такого рода дискретных преобразований. Получены явные выражения для решений солитонного типа в терминах матричных элементов фундаментальных представлений полупростых алгебр.

# Introduction: 1. The key role of Toda chains in the theory of integrable systems

There are many different forms for the representation of infinite Toda chain. The best known and useful are the following ones:

$$(\ln v)_{xy} = \frac{\overleftarrow{v}}{v} - \frac{v}{\overrightarrow{v}}, \quad (\ln \theta)_{xy} = \overleftarrow{\theta} - 2\theta + \overrightarrow{\theta}, \tag{1}$$

where $v \equiv v_s, \overleftarrow{v} \equiv v_{s+1}, \overrightarrow{v} \equiv v_{s-1}$, $s$ is a natural number and $x, y$ are independent coordinates of the problem.

Equations (1) may be considered as a definition of some mapping: the law with the help of which two initial functions $v, \overrightarrow{v}$ $(\theta, \overrightarrow{\theta})$ are associated with two final ones $\overleftarrow{v}, v$ $(\overleftarrow{\theta}, \theta)$.

We restrict ourselves by the first system (1) and rewrite it in the following equivalent form:

$$\overleftarrow{u} = \frac{1}{v}, \quad \overleftarrow{v} = v(vu - (\ln v)_{xy}). \tag{2}$$

The remarkable property of the mapping (substitution) (2) consists in its integrability [1],[2]. This means that the corresponding symmetry equation (arising as variation derivative of the substitution by itself)

$$\overleftarrow{U} = -\frac{1}{v^2}V, \quad \overleftarrow{V} = v^2 U + (2uv - (\ln v)_{xy})V - v(\frac{V}{v})_{xy} \tag{3}$$

possesses the sequence of nontrivial solutions [3]. In (3) it is suggested that independent arguments of functions $U, V$ should be $u, v$ (from (2)) and its derivatives on space coordinates up to the definite order, while $\overleftarrow{U}, \overleftarrow{V}$ are the same functions the arguments of which are shifted with the help of (2).

Each solution of (3) may be connected with the integrable in the Liouville sense (infinite number of conservation "laws" in involution) system of evolution type equations

$$u_t = U, \quad v_t = V. \tag{4}$$

Moreover, the last systems are invariant with respect to substitution (2).

Most of integrable systems and equations resolved up to now by different methods (particular by inverse scattering one) are directly connected with Toda symmetry (2) (or its numerous auto-Backlund transformations) in above scheme [4].

Under appropriate boundary conditions the infinite Toda chain is interrupted and overgo into integrable finite dimensional system. The different classes solution of which can be represented in an explicit form. Best known ones are Toda chain with fixed ends $v_0^{-1} = v_N = 0$ or periodical Toda $v_0 = V_N$. In the first case it is possible to find general solution [5] (depending on the necessary number of arbitrary functions). In the second case – the parametric soliton-like subclass of solutions [6]. The general solution in this case can be represented in the form of infinite absolutely convergent series [7].

General solution of Toda chain with fixed ends in the case $N = 2m$ is directly connected with m-soliton solution of evolution type equations (4). Namely, $u_m, v_m$ of Toda chain with fixed ends is exactly m-soliton solution (under some additional restrictions on arbitrary functions) of evolution type equations (4).

So, we see that equations of Toda chain play an ambivalent role: it defines the form of integrable systems (as solution of its symmetry equation) and its interrupting version gives a possibility to find different classes solutions of such systems in an explicit form.

In papers [1], [2] it has been assumed that the theory of integrable systems is equivalent to the representation theory of the group of integrable substitutions with respect to which the Toda chain system is the simplest partial case.

In the present paper we introduce a new class of integrable mappings and chains. These chains differ from the usual Toda one by a greater number of unknown functions in each point of the lattice. These mappings are integrable and it is possible to construct the hierarchy of integrable systems, each one of which is invariant with respect to such discrete substitution. General solution for the case of fixed ends may be represented in terms of matrix elements of fundamental representations of semisimple $A_n$ algebras (groups).

## 2. Integrable chains and the graded algebras

In paper [8] (these results are literally repeated in the corresponding chapters of monograph [9]) a general method for the construction of exactly integrable systems connected with arbitrary graded (super) algebras has been proposed. We will use it below for the case of $A_n$ semisimple series.

As usual by $X^{\pm}_\alpha, h_\alpha$ we denote the generators of simple roots together with the corresponding Cartan elements. The $\pm s$ graded subspaces consist of generators of $A_n$ algebra, which can be constructed from the commutators of $s$ simple roots. The general equation of [8]

$$[\partial_x - \sum_{s=1}^{m_1} A^{-s}, \partial_y - (\rho h) - \sum_{s=1}^{m_2} A^{+s}] = 0 \tag{5}$$

in the case under consideration can be concretized to represent integrable chains in a more observable form. Generators of $\pm s$ graded subspaces are the following ones

$$Y^{+s}_\alpha = [X^+_\alpha ...[X^+_{\alpha+s-1}, X^+_{\alpha+s}]..], \quad Y^{-s}_\alpha = [[..[X^-_{\alpha+s}, X^-_{\alpha+s-1}]....X^-_\alpha] \tag{6}$$

with obvious commutation relations as a corollary of their definition:

$$[Y^{-i}_\alpha, Y^{+j}_\beta] = \delta_{\alpha+i,\beta+j} Y^{-i+j}_\alpha - \delta_{\alpha,\beta} Y^{-i+j}_{\alpha+j}, \quad i \leq j$$

$$[Y^{+i}_\alpha, Y^{+j}_\beta] = -\delta_{\alpha,\beta+j} Y^{i+j}_\alpha + \delta_{\alpha+i,\beta} Y^{i+j}_\beta, \tag{7}$$

$$[Y^{-i}_\alpha, Y^{+i}_\beta] = \delta_{\alpha,\beta} \sum_{s=0}^{i-1} h_{\alpha+i}.$$

Further

$$A^{-s} = \sum_\alpha Y^{-s}_\alpha f^s_\alpha, \quad A^{+s} = \sum_\beta Y^{+s}_\beta \bar{f}^s_\beta.$$

System of equations arising as a consequence of (5) takes the form

$$(f^s_\alpha)_y + (\sum_{k=0}^{s-1} \bar{\rho}_{\alpha+k}) f^s_\alpha - \sum_{k=1}^{m_1-s} (f^{s+k}_\alpha \bar{f}^k_{\alpha+s} - f^{s+k}_{\alpha-k} \bar{f}^k_{\alpha-k}) = 0,$$

$$\bar{\rho}_\alpha = -\rho_{\alpha+1} + 2\rho_\alpha - \rho_{\alpha-1}, \quad (\rho_\alpha)_x + \sum_{s=1}^{min(m_1,m_2)} \sum_{k=0}^{s-1} f^s_{\alpha-k} \bar{f}^s_{\alpha-k} = 0, \tag{8}$$

$$(\bar{f}^s_\alpha)_x - \sum_{k=1}^{m_2-s} (\bar{f}^{s+k}_\alpha f^k_{\alpha+s} - \bar{f}^{s+k}_{\alpha-k} f^k_{\alpha-k}) = 0.$$

We call (8) as UToda$(m_1, m_2)$ chain keeping in mind that the usual two-dimensional Toda system for $A_n$ series corresponds to the choice $m_1 = m_2 = 1$. The results of [8] assure that system (8) is exactly integrable and give a prescription how to integrate it.

## 3. UToda (2,2) system and its general solution

In this case in each point of the chain there are 5 unknown functions $\rho_\alpha, f^1_\alpha, \bar{f}^1_\alpha, f^2_\alpha, \bar{f}^2_\alpha$. But due to the gauge invariance not all of them are independent and after introducting the gauge invariant values $q_\alpha = f^1_\alpha \bar{f}^1_\alpha, p_\alpha = \frac{f^2_\alpha}{f^1_\alpha f^1_{\alpha+1}}, \bar{p}_\alpha = \frac{\bar{f}^2_\alpha}{\bar{f}^1_\alpha \bar{f}^1_{\alpha+1}}$, we rewrite (8) in the chain form with three independent functions in each point

$$(\ln p)_y + (\overset{\leftarrow 2}{q} \overset{\leftarrow}{p} - qp) + (\overset{\leftarrow}{q} p - \overset{\rightarrow}{q} \overset{\rightarrow}{p}) = 0,$$

$$-(\ln q)_{xy} + (\overset{\leftarrow}{q} \bar{p} - \overset{\rightarrow}{q} \overset{\rightarrow}{\bar{p}})_y + (\overset{\leftarrow}{q} p - \overset{\rightarrow}{q} \overset{\rightarrow}{p})_x + \hat{K}(q + \overset{\rightarrow}{q} q \overset{\rightarrow}{p} \overset{\rightarrow}{\bar{p}} + q \overset{\leftarrow}{q} p\bar{p}) = 0 \tag{9}$$

$$(\ln \bar{p})_x + (\overset{\leftarrow 2}{q} \overset{\leftarrow}{\bar{p}} - q\bar{p}) + (\overset{\leftarrow}{q} \bar{p} - \overset{\rightarrow}{q} \overset{\rightarrow}{\bar{p}}) = 0,$$

where $\hat{K}\theta = \overset{\leftarrow}{\theta} - 2\theta + \overset{\rightarrow}{\theta}$.

In the case $p = \bar{p} = 0$, we come back to the usual Toda chain system (UToda (1,1)). The case $\bar{p} = 0$ ( or equivalent to it $p = 0$) corresponds to UToda (1,2) chain with equations

$$-(\ln q)_{xy} + (\overset{\leftarrow}{q} p - \overset{\rightarrow}{q} \overset{\rightarrow}{p})_x + \overset{\leftarrow}{q} - 2q + \overset{\rightarrow}{q} = 0,$$

$$(\ln p)_y + (\overleftarrow{q}^2\overleftarrow{p} - qp) + (\overleftarrow{q}p - \overrightarrow{q}\overrightarrow{p}) = 0. \quad (10)$$

The last system is interesting by itself, but can always be considered as the reduction of UToda (2,2) chain under a definite choice of arbitrary functions defined by its solution.

Below we represent the general solution of system (9) after rewriting it in a more suitable and observable form.

Let us use the following substitutions

$$p = \exp(\overleftarrow{s} + s), \quad \bar{p} = \exp(\overleftarrow{t} + t), \quad \theta = q\exp(s+t).$$

In new variables system (9) takes the form

$$(\exp -s)_y = \theta \overleftarrow{\exp} -t - \theta \overrightarrow{\exp} -t, \quad (\exp -t)_x = \theta \overleftarrow{\exp} -s - \theta \overrightarrow{\exp} -s$$

$$-(\ln\theta)_{xy} + \hat{K}(\theta\exp -(s+t) + \theta\overleftarrow{\theta} + \theta\overrightarrow{\theta}) = 0$$

or finally after identification $p^{(1} = \exp -s$,$\bar{p}^{(1} = \exp -t$

$$(p^{(1)})_y = \overleftarrow{\bar{p}^{(1}}\theta - \overrightarrow{\bar{p}^{(1}}\theta,$$

$$(\bar{p}^{(1)})_x = \overleftarrow{p^{(1}}\theta - \overrightarrow{p^{(1}}\theta - (\ln\theta)_{xy} + \hat{K}(p^{(1}\bar{p}^{(1}\theta + \theta\overleftarrow{\theta} + \theta\overrightarrow{\theta}) = 0. \quad (11)$$

Anyone can agree that the last form is more attractive compared to (9), although both are equivalent to each other (at least in the case of interrupted chain).

To represent the general solution of (9) or (11) for us it will be necessary to know some facts from [8]. Here we reproduce them in a concise form. Two equations of S-matrix type are in foundation of the whole construction

$$(M_+)_y = M_+L_+ \equiv M_-(\sum_1^r Y_\alpha^{+1}\bar{\phi}_\alpha^1 + \sum_1^{r-1} Y_\beta^{+2}\bar{\phi}_\beta^2),$$

$$(12)$$

$$(M_-)_x = M_-L_- \equiv M_-(\sum_1^r X_\alpha^- \phi_\alpha^1 + \sum_1^{r-1} Y_\beta^{-2}\phi_\beta^2),$$

where $r$ is the rank of semisimple algebra, $Y^{\pm 2}$ are defined by (8).

The "Lagrangian" functions $L^{\pm}$ of equations (12) are correspondingly the upper and lower triangular matrixes and for these reasons the solutions of (12) can be represented in quadratures.

The solution of the problem may be expressed via matrix elements of the following $A_n$ group element

$$K = \exp(-h\Phi)M_-^{-1}M_+\exp -(h\bar{\Phi}) \equiv m_-^{-1}m_+. \quad (13)$$

As it follows from its definition groups elements $m_\pm$ satisfy the equations:

$$(m_+)_y = m_+(-(h\bar{\Phi})_y + \sum Y_j^{+1}(\bar{\nu}_{j+1} - \bar{\nu}_{j-1}) + \sum Y_j^{+2}) \equiv$$
$$m_+(\exp(h\bar{\Phi})L_+ \exp -(h\bar{\Phi}) - (h\bar{\Phi})_y),$$
$$(m_-)_x = m_-((h\Phi)_x + \sum Y_j^{-1}(\nu_{j+1} - \nu_{j-1}) + \sum Y_j^{-2}) \equiv$$
$$m_-(\exp(h\Phi)L_- \exp -(h\Phi) - (h\Phi)_x). \tag{14}$$

The last equalities determine all the introduced above values and relations between them.

By $\| i\rangle, (\langle i \|)$ we will denote the minimal vector of i-th fundamental representation of $A_n$ algebra with the properties

$$X_\alpha^- \| i\rangle = 0, \quad h_s \| i\rangle = -\delta_{s,i}, \quad \langle i \| X_\alpha^+ = 0, \quad \langle i \| h_s = -\delta s, i,$$
$$X_\alpha^+ \| i\rangle = \delta_{\alpha,i} X_i^+ \| i\rangle, \quad \langle i \| X_\alpha^- = \delta_{\alpha,i}\langle i \| X_i^- . \tag{15}$$

The following abbreviations will be used throughout the paper:

$$[i] = \langle i \| K \| i\rangle, \quad \theta_i = \frac{[i+1][i-1]}{[i]^2},$$

$$\alpha_{ij..l} \equiv \frac{\langle i \| X_i^- X_j^- ... X_l^- K \| i\rangle}{\langle i \| K \| i\rangle}, \quad \bar{\alpha}_{ij..l} \equiv \frac{\langle i \| K X_l^+ .... X_j^+ X_i^+ \| i\rangle}{\langle i \| K \| i\rangle}$$

In these notations the general solution of system (11) can be represented in the form

$$\bar{p}_i^{(1} = (\bar{\nu}_{i+1} - \bar{\alpha}_{i+1} - \bar{\nu}_{i-1} + \bar{\alpha}_{i-1} \equiv (\theta_i)^{-1}(\alpha_i)_y$$
$$\theta_i = \frac{[i+1][i-1]}{[i]^2}, \tag{16}$$
$$\bar{p}_i^{(1} = (\nu_{i+1} - \alpha_{i+1} - \nu_{i-1} + \alpha_{i-1} \equiv (\theta_i)^{-1}(\bar{\alpha}_i)_x.$$

In"old" variables solution of (9) may be expressed via (16) as

$$q_i = p_i^{(1}\bar{p}_i^{(1}\theta_i, \quad \bar{p}_i = \frac{1}{\bar{p}_i^{(1}\bar{p}_{i+1}^{(1}}, \quad p_i = \frac{1}{p_i^{(1}p_{i+1}^{(1}}.$$

For the checking of the validity of the represented above solution only one relation between matrix elements of different fundamental representations is necessary. Namely,

$$Det_2\begin{pmatrix} \langle i \| K \| i\rangle & \langle i \| X_i^- K \| i\rangle \\ \langle i \| K X_i^+ \| i\rangle & \langle i \| X_i^- K X_i^+ \| i\rangle \end{pmatrix} = [i+1][i-1]. \tag{17}$$

In fact, (17) is nothing but the famous Yakoby equality connecting the determinants of $i, i+1, i-1$ orders rewritten in a more concise form [5].

The proof of this relation the reader can find in [9] (see also Appendix). All other necessary relations are direct corollary of the last one.

With the help of these relations it is not difficult to prove that

$$(\ln[i])_{xy} = \theta_i\theta_{i+1} + \theta_i\theta_{i-1} + \bar{p}_i^{(1}p_i^{(1}\theta_i$$

and other equalities of the same kind (partially contained in equations of equivalence (16)). The details of corresponding calculations can be found in Section 5.

Solution of UToda(1,2) chain is contained in the constructed above. By the same kind of transformation as the overgo from (9) to (11) we obtain instead of (10)

$$\phi_y = \overleftarrow{\theta} - \overrightarrow{\theta}, \quad (\ln\theta)_{xy} = \overleftarrow{\phi}\theta - 2\phi\theta + \overrightarrow{\phi}\theta. \tag{18}$$

For the general solution of the last chain we have: $\phi$ coincides with $p_i^{(1}$ of (16) and in the expression for $\theta$ (16) it is necessary to make a little modification:

$$\theta_i = \nu_i\bar{\phi}_i^1 \frac{\langle i-1 \parallel K \parallel i-1\rangle\langle i+1 \parallel K \parallel i+1\rangle}{(\langle i \parallel K \parallel i\rangle)^2}.$$

Of course, in equation determining $M_\pm$ (12) it is necessary to put $\bar{\phi}_i^2 = 0$.

From an explicit form of solution (16) we see that it is defined by only one group element $K$ and so it is possible to expect that all the problems connected with UToda chains systems may be resolved at the level of their properties.

## 4. Parameters of evolution - Hamiltonians flows

In this Section we introduce the parameters of evolution and represent the way of constructing the systems of equations invariant with respect to UToda substitutions. We begin the discussion with the simplest case of the usual Toda chain for which the solution of the problem is known [3].

### 4.1. Two-dimensional Davey-Stewartson hierarchy

In this case the group element $m_+$ is defined by equation

$$m'_+ \equiv (m_+)_y = m_+(-(h\bar{\Phi})' + \sum Y_j^{+1} \tag{19}$$

and depends on the set of arbitrary functions $\bar{\Phi}_i$. Let us try to find the last functions in such a way that equation

$$\dot{m}_+ \equiv (m_+)_{t_2} = m_+(-(h\dot{\bar{\Phi}}) + \sum Y_j^{+1}\mu_j - \sum Y_j^{+2}) \tag{20}$$

would be self-consistent with (19). The Maurer-Cartan identity after its trivial resolution takes the following form:

$$\mu_i = -\bar{\Phi}'_{i+1} + \bar{\Phi}'_{i-1}, \quad \mu'_i + (k\dot{\bar{\Phi}})_i - \mu_i(k\bar{\Phi})' = 0, \tag{21}$$

where, as usual, $(kf)_i = -f_{i+1} + 2f_i - f_{i-1}$. Finally, (21) is equivalent to

$$\dot{\bar{\Phi}}_i - \dot{\bar{\Phi}}_{i-1} + \bar{\Phi}''_i + \bar{\Phi}''_{i-1} + (\bar{\Phi}'_i - \bar{\Phi}'_{i-1})^2 = 0 \tag{22}$$

(in all the cases from the condition $b_i = b_{i+1}$ we have made conclusion $b_i = 0$).

It is possible to express the solution of the chain system (22) via the $N$ ($N$ is the number of the points of interrupted chain) linear independent solutions of the single one-dimensional Schrodinger equation

$$\dot{\Psi} = \Psi'' + V(t_2, y)\Psi,$$

where $V$ is an arbitrary function of space $y$ and time $t_2$ coordinates [3].

Chain (22) induced the Davey-Stewartson system [11]. To explain this fact let us consider matrix element $[i]$ and calculate its derivatives with respect to arguments $y, t_2$ (we use notations introduced in (15) and below). As a consequence of (19), (20) and (22) we have

$$\begin{gathered}\dot{\ln[i]} = \dot{\bar{\Phi}}_i + \mu_i \bar{\alpha}_i - \bar{\alpha}_{i,i+1} + \bar{\alpha}_{i,i-1}, \\ (\ln[i])' = (\bar{\Phi}_i)' + \bar{\alpha}_i, \\ \frac{[i]''}{[i]} = (\bar{\Phi}_i)'' + (\bar{\Phi}'_i)^2 + (\bar{\Phi}_{i+1} + \bar{\Phi}_{i-1})'\bar{\alpha}_i + \bar{\alpha}_{i,i+1} + \bar{\alpha}_{i,i-1}.\end{gathered} \tag{23}$$

Excluding $\dot{\bar{\Phi}}_i, \bar{\Phi}'_i, \bar{\Phi}''_i$ from (22) with the help of (23) we come to the key equality

$$\dot{\ln\frac{[i]}{[i-1]}} + (\ln([i][i-1])'' + ((\ln\frac{[i]}{[i-1]})')^2 = 2(\bar{\alpha}_{i-1,i} + \bar{\alpha}_{i,i-1} - \bar{\alpha}_{i-1}\bar{\alpha}_i) \tag{24}$$

remarkable by the fact that its right-hand side is identically equal to zero due to recurrent relations between the matrix elements of different representations of $A_n$ groups (see Appendix).

Introducing the functions $v = \frac{[i]}{[i-1]}, u = \frac{[i-2]}{[i-1]}$, bearing in mind the main equation of Toda chain by itself

$$(\ln[i])_{xy} = \frac{[i+1][i-1]}{[i]^2}$$

and the fact that equality (24) is correct for arbitrary $i$, we conclude that functions $u, v$ satisfy the following system of equations

$$-\dot{u} + u_{yy} + 2u\int dx(uv)_y = 0 \quad \dot{v} + v_{yy} + 2v\int dx(uv)_y = 0. \tag{25}$$

This is exactly Davey-Stewartson system [11]. In one-dimensional limit – a usual nonlinear Schrodinger equation.

In a general case equation (20) defining algebra valued function $m_+^{-1}\dot{m}_+$, it is necessary to change the condition that this function is decomposed on generators of algebra the graded index of which is less than some given natural number, say $r$. In this case we will obtain a system of equations, which determine the dependence $\bar{\Phi}_i$ on the parameter $\bar{t}_r$

and obtain the corresponding system of equations of two-dimensional D-S hierarchy. By the different method this problem in an explicit form was solved in [3], [10].

The construction described above in one-dimensional case equivalent to multitime formalism and the corresponding technique of Hamiltonian flows [4].

Of course, it is possible to repeat all done above with respect to space coordinate $x$ in pair with group element $m_-$.

As a result we will obtain the sequence of right $\bar{t}_s$ and left $t_l$ evolution parameters, the corresponding system of equation invariant with respect to Toda discrete substitution and its particular explicit multisoliton type solutions.

### 4.2. UToda(2.2) case

Now we want to apply the technique of the last subsection for the construction of unknown up to the present example of integrable system invariant with respect to UToda(2,2) substitution (11). We omitted, as a rule, the calculations themselves, representing only the final results. All the necessary formulae for its independent verification the reader will find in Section 5 and Appendix. These calculations are not difficult, but very long in consequent rewriting (may be, because of bad notations or very straightforward attempts to realize them by the known for us methods).

In this case element $m_+$ satisfies the equation (see Section 3):

$$(m_+)' = m_+(-(h\bar{\Phi})' + \sum Y_j^{+1}\bar{\phi}_j + \sum Y_j^{+2}) \equiv$$

$$m_+(\exp(h\bar{\Phi})L_+ \exp -(h\bar{\Phi}) - (h\bar{\Phi})') \quad f' \equiv f_y, \quad \bar{\phi}_j = (\bar{\nu}_{j+1} - \bar{\nu}_{j-1}).$$

The corresponding operator of $t_2$ differentiation has the form

$$\dot{m}_+ = m_+((-(\dot{h\bar{\Phi}}) + \sum Y_j^{+1}\mu_j^{(1} + \sum Y_j^{+2}\mu_j^{(2} + \sum Y_j^{+3}\mu_j^{(3} - \sum Y_j^{+4}). \tag{26}$$

Condition of self-consistency (Maurer-Cartan identity) gives a possibility to express all functions $\mu_i^{(s}$ from (26) in terms of $\bar{\Phi}_i, \bar{\nu}_i$ and find the system of equations, which satisfy the last functions as functions of $y, t_2$ arguments.

With the help of commutation relations (7) all the calculations are straightforward. Below the reader can find the result of them:

$$\mu_i^{(3} = \bar{\phi}_{i+2} + \bar{\phi}_i = \bar{\nu}_{i+3} - \bar{\nu}_{i-1} \quad \mu_i^{(2} = \bar{\Phi}'_{i+1} - \bar{\Phi}'_{i+2} - \bar{\Phi}'_i + \bar{\Phi}'_{i-1} + \bar{\phi}_i\bar{\phi}_{i+1},$$

$$\mu_i^{(1} = -(\bar{\nu}_{i+1} - \bar{\nu}_{i-1})(\bar{\Phi}'_{i+1} - \bar{\Phi}'_{i-1}) - (\bar{\nu}'_{i+1} + \bar{\nu}'_{i-1}).$$

The chain system of equations with respect to unknown functions $\bar{\Phi}, \bar{\nu}$ (compare with (22)) in this case has the form:

$$\dot{\bar{\Phi}}_{i+1} - \dot{\bar{\Phi}}_{i-1} + \bar{\Phi}''_{i+1} + \bar{\Phi}''_{i-1} + (\bar{\Phi}'_i - \bar{\Phi}'_{i+1})^2 + (\bar{\Phi}'_i - \bar{\Phi}'_{i-1})^2 = 2\bar{\nu}'_i(\bar{\nu}_{i+1} - \bar{\nu}_{i-1})$$

$$(\dot{\bar{\nu}_{i+1} - \bar{\nu}_{i-1}}) + (\bar{\nu}_{i+1} + \bar{\nu}_{i-1})'' - 2\bar{\nu}'_{i+1}(\bar{\Phi}'_i - \bar{\Phi}'_{i-1}) - 2\bar{\nu}'_{i-1}(\bar{\Phi}'_i - \bar{\Phi}'_{i+1}) +$$

$$(\bar{\nu}_{i+1}-\bar{\nu}_{i-1})\times \tag{27}$$
$$[(\dot{\bar{\Phi}}_{i+1}-2\dot{\bar{\Phi}}_i)+\dot{\bar{\Phi}}_{i-1}+\bar{\Phi}''_{i+1}-\bar{\Phi}''_{i-1}+(\bar{\Phi}'_{i+1})^2-(\bar{\Phi}'_{i-1})^2-2(\bar{\Phi}'_{i+1}-\bar{\Phi}'_{i-1})\bar{\Phi}'_i]=0.$$

In what follows in this section we deviate from the introduced in the previous section notations and consider UToda$(2,2)$ in variables

$$p_i=\alpha_i-\nu_i,\quad \bar{p}_i=\bar{\alpha}_i-\bar{\nu}_i.$$

In these variables as it follows from (16) UToda$(2,2)$ substitution takes the form

$$(p_i)_y=\theta_i(\bar{p}_{i-1}-\bar{p}_{i+1}),\quad (\bar{p}_i)_x=\theta_i(p_{i-1}-p_{i+1}),$$
$$(\ln\theta)_{xy}=\hat{K}(p_y\bar{p}_x\theta+\theta\overleftarrow{\theta}+\theta\overrightarrow{\theta}). \tag{28}$$

Let us define functions $v_i=\frac{[i+1]}{[i]}, u_i=\frac{[i-1]}{[i]}$. Obviously $\theta_i\equiv u_iv_i, u_{i+1}=v_i^{-1}$.

The following equalities are the direct corollary of all the introduced above definitions and may be verified directly (all the necessary formulae the reader can find in Section 5 and in Appendix)

$$\frac{\dot{v}_i}{v_i}+\frac{v''_i}{v_i}+2(\bar{\alpha}_{i,i-1}-\bar{\nu}_{i-1}\bar{\alpha}_i)'-2\bar{p}'_i\bar{p}_{i+1}=V_i(y,t_2)-$$
$$-\frac{\dot{u}_i}{u_i}+\frac{u''_i}{u_i}-2(\bar{\alpha}_{i,i+1}-\bar{\nu}_{i+1}\bar{\alpha}_i)'+2\bar{p}'_i\bar{p}_{i-1}=U_i(y,t_2),$$
$$\dot{p}_i=-p''_i-2(\ln v_{i-1})'+2\theta_i\bar{p}'_{i-1},$$
$$V_i=\dot{\bar{\Phi}}_{i+1}-\dot{\bar{\Phi}}_i+\bar{\Phi}''_{i+1}-\bar{\Phi}''_i+(\bar{\Phi}'_i-\bar{\Phi}'_{i+1})^2-2\bar{\nu}'_i\bar{\nu}_{i+1},$$
$$U_i=-\dot{\bar{\Phi}}_{i-1}+\dot{\bar{\Phi}}_i+\bar{\Phi}''_{i-1}-\bar{\Phi}''_i+(\bar{\Phi}'_i-\bar{\Phi}'_{i-1})^2+2\bar{\nu}'_i\bar{\nu}_{i-1}.$$

To have some closed system it is necessary to exclude from the last system of equalities the terms containing functions $\bar{\alpha}_{i\pm1},\bar{\alpha}_{i,i\pm1}$. The following additional equalities

$$(\bar{\alpha}_{i,i\pm1}-\bar{\nu}_{i\pm1}\bar{\alpha}_i)_x=\mp\theta_i\theta_{i\pm1}+\theta_i\bar{p}_{i\pm1}(\bar{p}_i)_x$$

solve this problem. After keeping in mind equations of substitution (28) it is possible to rewrite the previous system of equalities in the closed form for 8 "unknown" functions $v_{i+1},v_i,u_i,u_{i-1},p_i,p_{i+1},\bar{p}_i,\bar{p}_{i-1}$

$$\dot{v}_{i+1}+v''_{i+1}+2v_i\bar{p}'_{i+1}p'_{i+1}+2v_{i+1}\bar{p}_{i+1}\bar{p}'_i-2v_{i+1}\int dx[-u_iv_{i+1}+\bar{p}_{i+1}(\bar{p}_i)_x]'=0,$$
$$\dot{v}_i+v''_i-2v_i\bar{p}_{i+1}\bar{p}'_i+2v_i\int dx[v_iu_{i-1}+\bar{p}_{i-1}(\bar{p}_i)_x]'=0,$$
$$-\dot{u}_i+u''_i+2u_i\bar{p}_{i-1}\bar{p}'_i-2u_i\int dx[-u_iv_{i+1}+\bar{p}_{i+1}(\bar{p}_i)_x]'=0,$$
$$-\dot{u}_{i-1}+u''_{i-1}+2u_i\bar{p}'_{i-1}p'_{i-1}-2u_{i-1}\bar{p}_{i-1}\bar{p}'_i+2u_{i-1}\int dx[v_iu_{i-1}+\bar{p}_{i-1}(\bar{p}_i)_x]'=0,$$

$$\dot{p}_i + p_i'' - 2(\ln u_i)'p_i' - 2u_i v_i \bar{p}_{i-1}' = 0 \tag{29}$$
$$\dot{p}_{i-1} + p_{i-1}'' - 2(\ln u_i)'p_{i+1}' - 2\frac{u_{i-1}}{u_i}\bar{p}_i' = 0,$$
$$(\theta_i^{-1}\dot{(\bar{p}_i)_x}) = p_{i+1}'' + p_{i-1}'' + 2p_{i+1}'(\ln v_i)' + 2p_{i-1}'(\ln u_i)' - 2\bar{p}_i'(\theta_{i+1} - \theta_{i-1}),$$
$$(\theta_{i-1}^{-1}\dot{(\bar{p}_{i-1})_x}) = p_i'' + p_{i-2}'' + 2p_i'(\ln v_{i-1})' + 2p_{i-2}'(\ln u_{i-1})' - 2\bar{p}_{i-1}'(\theta_i - \theta_{i-2}).$$

The last two equations are the direct consequence of the two previous ones. With the help of UToda $(2,2)$ transformation they may be represented in terms of only unknown functions. So, this system is closed, integrable and sequence of its particular solutions is given by formulae

$$p_i = \alpha_i - \nu_i, \quad \bar{p}_i = \bar{\alpha}_i - \bar{\nu}_i, \quad v_i = \frac{[i+1]}{[i]}, \quad u_i = \frac{[i-1]}{[i]}$$

as it was shown above.

Now we are able to clarify the situation with solution of chain system (27). It is obvious that system (31) possesses a particular solution of the form

$$u_0 = u_{-1} = p_0 = \bar{p}_0 = p_{-1} = \bar{p}_{-1} = 0$$

Indeed, in the case of final dimensional algebra $A_n$ all the matrix elements above are equal to 0. For the remaining unknown functions $v_0, v_1, p_1, \bar{p}_1$ as a corollary of (31), we obtain the following (closed) system of equations (after trivial manipulations)

$$\dot{v}_0 + v_0'' = V_0 v_0, \quad \dot{(p_1 v_0)} + (p_1 v_0)'' = V_0(p_1 v_0),$$
$$\dot{(\bar{p}_1 v_0)} + (\bar{p}_1 v_0)'' = U_0(\bar{p}_1 v_0), \quad \dot{v}_1 + v_1'' - U_0 v_1 = -2p_1'\bar{p}_1' v_0$$

(the arising of arbitrary functions $(U_0, V_0)$ is connected with ambiguity of $\int dx 0 = F(y, t_2)$).

So, we see that the functions are $(v_0, p_1 v_0)$ the solutions of the homogeneous Schrodinger equation with arbitrary potential function $V_0$. While $v_1$ and $\bar{p}_1 v_0$ are the solutions of inhomogeneous Schrodinger equation (with the known search function $-2p_1'\bar{p}_1' v_0$) and potential (also arbitrary function) $U_0$. In terms of different solutions of this pair of Schrodiger equations it is possible to represent a general solution of chain (27). We hope to come back to this problem elsewhere.

In conclusion of this section we represent the $(1+1)$ integrable system, which arises as a reduction of (31) on the one dimensional case ($\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$)

$$\dot{v}_{i+1} + v_{i+1}'' + 2v_i \bar{p}_{i+1}' p_{i+1}' + 2v_{i+1}^2 u_i = 0,$$
$$\dot{v}_i + v_i'' + 2\frac{1}{u_i}\bar{p}_i' p_i' + 2v_i^2 u_{i-1} = 0,$$
$$-\dot{u}_i + u_i'' + 2\frac{1}{v_i} p_i \bar{p}_i' + 2u_i^2 v_{i+1} = 0,$$

$$-\dot{u}_{i-1}+u''_{i-1}+2u_i\bar{p}'_{i-1}p'_{i-1}+2u^2_{i-1}v_i=0,$$
$$\dot{p}_i+p''_i-2(\ln u_i)'p'_i-2u_iv_i\bar{p}'_{i-1}=0,$$
$$\dot{p}_{i-1}-p''_{i-1}-2(\ln u_i)'p'_{i-1}-2\frac{u_{i-1}}{u_i}\bar{p}'_i=0, \tag{30}$$
$$\dot{\bar{p}}_i+\bar{p}''_i-2(\ln u_i)'\bar{p}'_i-2u_iv_ip'_{i-1}=0,$$
$$\dot{\bar{p}}_{i-1}-\bar{p}''_{i-1}-2(\ln u_i)'\bar{p}'_{i-1}-2\frac{u_{i-1}}{u_i}p'_i=0.$$

The equations of system (32) are invariant with respect to one-dimensional version of UToda$(2,2)$ substitution (28), possess the infinite number of conservation laws and can be considered by traditional methods [4].

## 5. UToda$(m_1,m_2)$ system and its general solution

In this section we represent some necessary auxiliary relations of representation theory of semisimple algebras. Construction of the equations of UToda $(m_1,m_2)$ chains together with its general solution after this takes the form of pure technical manipulations.

Let the pair of operators $m^{\pm}$ satisfy the equations, generalizing (14)

$$m^+_y=m^+(-(h\bar{\Phi})_y+\sum_{s=1}^{m_1}(Y^s\bar{\phi}^{(s)}))\equiv m^+L^+,$$
$$(Y^s\bar{\phi}^{(s)})\equiv\sum_i(Y^s_i\bar{\phi}^{(s)}_i),\quad (Y^s\phi^{(s)})^T\equiv\sum_i(Y^{-s}_i\phi^{(s)}_i), \tag{31}$$
$$m^-_x=m^-((h\Phi)_x-\sum_{s=1}^{m_2}(Y^s\bar{\phi}^{(s)})^T)\equiv m^-L^-,$$

where generators $Y^{\pm s}$ together with their commutation relations are defined by (8) and below.

Our nearest goal is to find recurrent relations (or equations) satisfying some matrix elements of the different fundamental representations of the single group element

$$k=m_-^{-1}m_+. \tag{32}$$

We have

$$(\ln[i])_{xy}=[i]^{-2}\begin{pmatrix}\langle i\parallel K\parallel i\rangle & \langle i\parallel L^-K\parallel i\rangle\\ \langle i\parallel KL^+\parallel i\rangle & \langle i\parallel L^-KL^+\parallel i\rangle\end{pmatrix}. \tag{33}$$

In connection with (15) under the action on minimal state vector by the operators of the simple roots only $X^+_i\parallel i\rangle\neq 0,\langle i\parallel X^-_i\neq 0$. For this reason the action of operators $L^{\pm}$ on the minimal state vector may be represented in the form

$$L^+\parallel i\rangle=l^+_iX^+_i\parallel i\rangle,\quad \langle i\parallel L^-=\langle i\parallel X^-_il^-_i,$$

where $l_i^{\pm}$ are some operators polynomials in generators of positive (negative) simple roots. For instance,

$$(Y^2\bar{\phi}^{(2)}) \parallel i\rangle = (\phi^{(2}_{i-1}X^+_{i-1} - \phi^{(2}_i X^+_{i+1})X^+_i \parallel i\rangle$$

or in this case $l_i^+ = \phi^{(2}_{i-1}X^+_{i-1} - \phi^{(2}_{i+1}X^+_{i+1}$. Keeping this fact in mind and taking into account (8), we can rewrite (33) in the form

$$(\ln[i])_{xy} = [i]^{-2}(l_i^-)_l(l_i^+)_r[i-1][i+1], \tag{34}$$

where now $(l_i^-)_l, (l_i^+)_r$ are the same polynomials constructed from the generators of simple roots, correspondingly, of the left and right adjoint representations. Formulae below are an illustration of application of the last general equalities (34) to a concrete case under the choice $m_1 = m_2 = 3$ in (31)

$$(\alpha_i)_y = \theta_i(\bar{\phi}^{(1}_i + \bar{\phi}^{(2}_{i-1}\bar{\alpha}_{i-1} - \bar{\phi}^{(2}_i\bar{\alpha}_{i+1} + \bar{\alpha}_{i+1,i+2} - \bar{\alpha}_{i+1}\bar{\alpha}_{i-1} + \bar{\alpha}_{i-1,i-2}),$$

$$(\alpha_{i,i+1})_y = \alpha_{i+1}(\alpha_i)_y - \theta_i\theta_{i+1}(\bar{\phi}^{(2}_i + \bar{\alpha}_{i-1} - \bar{\alpha}_{i+2}), \tag{35}$$

$$(\ln[i])_{xy} = \theta_{i+2}\theta_{i+1}\theta_i + \theta_{i+1}\theta_i\theta_{i-1} + \theta_{i-2}\theta_{i-1}\theta_i +$$

$$\theta_{i+1}\theta_i(\phi^{(2}_i + \alpha_{i-1} - \alpha_{i+2})(\bar{....}) + \theta_{i-1}\theta_i(\phi^{(2}_{i-1} + \alpha_{i-2} - \alpha_{i+1})(\bar{....}),$$

$$\theta_i(\phi^{(1}_i + \phi^{(2}_{i-1}\alpha_{i-1} - \phi^{(2}_i\alpha_{i+1} + \alpha_{i+1,i+2} - \alpha_{i+1}\alpha_{i-1} + \alpha_{i-1,i-2})(\bar{....}),$$

where by $(\bar{....})$ we understand the same values as in the first multiplicator in which all functions are changed on the bar ones. The same relations as (35) obviously take place, when all the functions $\alpha$ are changed on the bar ones together with $y \to x$ and vice versa.

Now let us introduce the new functions $p^{(1,2}_i, \bar{p}^{(1,2}_i$ by the relations

$$p^{(1}_i = \theta^{1}_i(\bar{\alpha}_i)_y, \quad \bar{p}^{(1}_i = \theta^{1}_i(\alpha_i)_x$$

$$p^{(2}_i = \phi^{(2}_i + \alpha_{i-1} - \alpha_{i+2}, \quad \bar{p}^{(2}_i = \bar{\phi}^{(2}_i + \bar{\alpha}_{i-1} - \bar{\alpha}_{i+2}.$$

Using (35) once more, we obtain the chain of equations, whose functions $p^{(1,2}_i, \bar{p}^{(1,2}_i, \theta_i$ satisfy

$$(p^{(2)}_i)_y = \theta_{i-1}\bar{p}^{(1}_{i-1} - \theta_{i+2}\bar{p}^{(1}_{i+2}, \quad (\bar{p}^{(2)}_i)_x = \theta_{i-1}p^{(1}_{i-1} - \theta_{i+2}p^{(1}_{i+2},$$

$$(p^{(1)}_i)_y = \theta_{i-1}\bar{p}^{(1}_{i-1}p^{(2}_{i-1} - \theta_{i+1}\bar{p}^{(1}_{i+1}p^{(2}_i + \theta_{i-1}\theta_{i-2}\bar{p}^{(2}_{i-2} - \theta_{i+1}\theta_{i+2}\bar{p}^{(2}_{i+1},$$

$$(\bar{p}^{(1)}_i)_x = \theta_{i-1}p^{(1}_{i-1}\bar{p}^{(2}_{i-1} - \theta_{i+1}p^{(1}_{i+1}\bar{p}^{(2}_i + \theta_{i-1}\theta_{i-2}p^{(2}_{i-2} - \theta_{i+1}\theta_{i+2}p^{(2}_{i+1},$$

$$(\ln\theta_i)_{xy} = \hat{K}(\theta_i\theta_{i+1}\theta_{i+2} + \theta_i\theta_{i-1}\theta_{i-2} + \theta_i\theta_{i+1}p^{(2}_i\bar{p}^{(2}_i + \theta_{i-1}\theta_i p^{(2}_{i-1}\bar{p}^{(2}_{i-1} + \theta_i p^{(1}_i\bar{p}^{(1}_i.$$

This is exactly UToda $(3,3)$ chain system with the known general solution, which is determined by the set of arbitrary functions $(\Phi_i, \phi^{(1}_i, \phi^{(2)}_i)$ and $(\bar{\Phi}_i, \bar{\phi}^{(1}_i, \bar{\phi}^{(2)}_i)$ of single arguments (x,y), correspondingly.

In a general case repeating literally the calculations of this section or the corresponding places of Sections 2-3, we come finally to expressions for equations of integrable UToda $(2,2)$ substitution

$$
\begin{aligned}
(p^{(s)}_{\alpha})_y &= \sum_{k=1}^{m_1-s} (p^{(s+k}_{\alpha-k}\bar p^{(k}_{\alpha-k}\prod_{i=1}^{k}\theta_{\alpha-k+i-1} - p^{(s+k}_{\alpha}\bar p^{(k}_{\alpha+s}\prod_{i=1}^{k}\theta_{\alpha+s+i-1}),\\
(\ln\theta_\alpha)_{xy} &= \hat K \sum_{s=1}^{min(m_1,m_2)}\sum_{k=0}^{s-1} p^{(s}_{\alpha-k}\bar p^{(s}_{\alpha-k}\prod_{i=1}^{s}\theta_{\alpha-s+i-1}, \quad p^{(m_1}_{\alpha}=1, \quad \bar p^{(m_2}_{\alpha}=1,\\
(\bar p^{(s)}_{\alpha})_y &= \sum_{k=1}^{m_2-s} (\bar p^{(s+k}_{\alpha-k} p^{(k}_{\alpha-k}\prod_{i=1}^{k}\theta_{\alpha-k+i-1} - \bar p^{(s+k}_{\alpha} p^{(k}_{\alpha+s}\prod_{i=1}^{k}\theta_{\alpha+s+i-1}).
\end{aligned} \tag{36}
$$

## 6. Outlook

At first we summarize in a few words the construction of the present paper (excluding all details).

The foundation of it are two groups elements $m_\pm$ belonging, correspondingly, to $\pm$ resolvable subgroups of some semisimple group. They are determined by the pair of $S$-matrix type equations

$$
l_+ \equiv m_+^{-1}(m_+)_y = \sum_{s=0}^{m_1} A^{+s}, \quad l_- \equiv m_-^{-1}(m_-)_x = \sum_{s=0}^{m_2} A^{-s}. \tag{37}
$$

The nature of these equations is purely algebraic - this is the condition for lagrangian operators to be decomposed on the operators of algebra with graded indexes less than $m_1,(m_2)$.

Matrix elements of different fundamental representations of the single group element $K=m_-^{-1}m_+$ satisfy the definite system of equalities, which can be interpreted as equations of exactly integrable UToda$(m_1,m_2)$ system.

At this step the construction of UToda $(m_1,m_2)$ integrable mapping (substitution) is closed.

The next step is connected with the introduction of evolution times parameters. Arbitrary up to now functions $\phi^{(s}_i(y),\phi^{(s}_i(x)$ are restricted by the conditions that elements $m_\pm$ satisfy an additional system of equations

$$
m_+^{-1}(m_+)_{\bar t_{d_1}} = \sum_{s=0}^{d_1} B^{+s}, \quad m_-^{-1}(m_-)_{t_{d_2}} = \sum_{s=0}^{d_2} B^{-s}. \tag{38}
$$

The condition of self-consistency of (37) with (38) determines an explicit dependence of functions $\bar\phi^{(s}_i \equiv \bar\phi^{(s}_i(y,\bar t_1,\bar t_2,...),\phi^{(s}_i \equiv \phi^{(s}_i(x,t_1,t_2,...)$ on evolution times parameters and space coordinates of the problem $x,y$.

As a result, we obtain the integrable hierarchies of evolution type equations (each system of which is determined by a different choice of $d_1,d_2$ under the fixed $m_1,m_2$) all invariant with respect to UToda $(m_1,m_2)$ substitution.

In one-dimensional case ($\frac{\partial}{\partial y}=\frac{\partial}{\partial x}$) this construction is equivalent to multitime formalism with the corresponding technique of Hamiltonians flows [4]

Now we want to enumerate the problems, which may be resolved in the context of the results of the present paper.

There are no doubts in a possibility of the direct (literally) generalization of this construction in the supersymmetrical case. As a result, it will be possible to obtain unknown up to now integrable hierarchies in (2|2) superspace.

Extremely interesting is the problem of generalization on the quantum domain. Heisenberg operators of the usual two-dimentional Toda chain (under canonical rules of quantization) [12] can be expressed as the matrix elements of single quantum group element $k=m_-^{-1}m_+$, where the elements of quantum groups $m_\pm$ are the solution of the following system of equations

$$(m_+)_y = m_+(\sum_{s=1}^r X_s^+ \exp(k\bar{\phi}(y))_s, \quad (m_-)_x = m_-(\sum_{s=1}^r X_s^- \exp(k\phi(x))_s. \tag{39}$$

$X_s^\pm$ are now the generators of the simple roots of quantum algebra, $\phi(x)+\bar{\phi}(y)$ is the quantum solution of two dimensional Laplace equation, $k$ — Cartan matrix of semisimple algebra.

Comparison (37) with (39) shows that the quantum version of UToda chains is not a fantastic assumption, but the problem of the nearest future.

In the present paper we have considered only one example of the system invariant with respect to UToda(2,2) substitution. We hope and are sure that the solution of symmetry equation in the case of UToda substitutions may be obtained by the similar methods as it was done in [3] in the case of the usual Toda chain. But now we are not ready to solve this problem.

And the last comment. The reader can mark a deep disconnection between the simplicity and pure algebraic nature of the construction foundation (only single group element $k$ and equations (37), (38)) on the one hand, and numerous nontrivial recurrent relations between the matrix elements of the different fundamental representation, which is necessary to prove by independent consideration (under approach of the present paper) on the other side. It is possible to hope that the last recurrent relations are the direct corollary of (37), (38), but how to extract from them this information is unknown to the author and, this may be the most interesting problem for the future investigation and the most important output of the present paper.

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### Appendix

Let us consider the function

$$R^i = \langle i \parallel KX^+_{i-1}X^+_i \parallel i\rangle[i-1] + \langle i-1 \parallel KX^+_i X^+_{i-1} \parallel i-1\rangle[i] -$$
$$\langle i \parallel KX^+_i \parallel i\rangle\langle i-1 \parallel KX^+_{i-1} \parallel i-1\rangle,$$

where $K$ is an arbitrary group element of $SL(n,R)$ group.

From its definition and properties of the minimal state vector $\| i\rangle$ (15), it follows that function $R$ is annihilated by generators of all right positive and left negative simple roots. This means that $R$ by itself is some linear combinations of the matrixes elements taken between the minimal state vectors. Calculations of left (right) Cartan elements shows that they take the definite values on $R$

$$h_r^s R^i = -(\delta_{s,i+1} + \delta_{s,i-2}) R^i, \quad h_l^s R^i = -(\delta_{s,i} + \delta_{s,i-1}) R^i.$$

From the last equalities it follows that from right and left $R^i$ belongs to different irreducible representation. This is impossible and so $R^i = 0$.

Now we enumerate the simplest state vectors of $i$ fundamental representation. They are different by the number of generators of the simple roots applied to the the minimal state vector. Zero order $\| i\rangle$ and first order $X_i^+ \| i\rangle$ states have the dimension one. Second order is two-dimensional $X_{i+1}^+ X_i^+ \| i\rangle, X_{i-1}^+ X_i^+ \| i\rangle$. There are three state vectors of the third order $X_{i+2}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_{i+1}^+ X_{i-1}^+ X_i^+ \| i\rangle, X_{i-2}^+ X_{i-1}^+ X_i^+ \| i\rangle$ and fife ones of the fourth order $X_{i+3}^+ X_{i+2}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_{i+2}^+ X_{i-1}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_i^+ X_{i+1}^+ X_{i-1}^+ X_i^+ \| i\rangle, X_{i-2}^+ X_{i-1}^+ X_{i+1}^+ X_i^+ \| i\rangle, X_{i-3}^+ X_{i-2}^+ X_{i-1}^+ X_i^+ \| i\rangle$. All other possibilities give the state vectors with zero norm.

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