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S.N.Storchak

BOGOLUBOV TRANSFORMATION IN PATH INTEGRALS ON MANIFOLDS WITH A GROUP ACTION

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Abstract

Storchak S.N. Bogolubov Transformation in Path Integrals on Manifolds with a Group Action: IHEP Preprint 98-1. – Protvino, 1998. – p. 25, refs.: 17.

A path integral with respect to the measure, generated by the stochastic process, is used to describe the particle motion on a compact Riemannian manifold, on which a free effective and isometric action of a compact semi-simple Lie group is given.

By choosing with the Bogolubov coordinate transformation method the coordinates, adapted to the principal fibre structure, the transformation of the path integral is performed. The separation of variables in the obtained path integral is realized with the help of the nonlinear filtering equation from the stochastic process theory.

After factorizing the path integral measure, we get the integral relation between the path integral given on the total space of the principal fiber bundle and the path integral on the base space of this bundle — the orbit space of the group action.

Аннотация

Сторчак С.Н. Преобразование Боголюбова в континуальных интегралах на многообразиях с групповым действием: Препринт ИФВЭ 98-1. – Протвино, 1998. – 25 с., библиогр.: 17.

В континуальном интеграле, чья мера порождена случайным процессом и который используется для описания движения скалярной частицы на компактном римановом многообразии, на котором задано свободное эффективное и изометрическое действие полупростой компактной группы Ли, делается переход к выбранным по методу преобразования координат Н.Н.Боголюбова переменным, адаптированным к структуре главного расслоения.

В полученном континуальном интеграле для разделения переменных применяется уравнение оптимальной нелинейной фильтрации из теории случайных процессов.

Факторизация меры в континуальном интеграле приводит к интегральному соотношению между континуальными интегралами: исходным континуальным интегралом, заданным на тотальном пространстве главного расслоения, и континуальным интегралом на пространстве базы этого расслоения — пространстве орбит группового действия.

1. Introduction

Late in the 70s, a new kind of path integral transformation was proposed by L.D.Faddeev and V.N.Popov [1] to study the gauge theory quantization. With this transformation the relation between the path integrals defined over the orbit space of the gauge group and the path integrals over the whole space of the gauge potentials was established.

Since then and up to the present investigations have been carried out to reach a rigorous foundation of such kind of path integral transformations. We mention only recent papers from this field [2].

The path integral transformation from [1] was proposed for the path integrals of the quantum field theories. But in order to clarify all the difficulties of this transformation, it is useful (or necessary) to consider a similar transformation of the quantum mechanical path integrals. By this reason, many investigations has been also carried out in this field of path integration [3].

Of finite-dimensional mechanical systems there is one which has many properties that can be found in gauge theories. This is a mechanical system describing the particle motion on a manifold with the action of a group given. In the present paper we will investigate the transformation of path integrals related to this dynamical system. We will consider the case of a smooth compact Riemannian manifold (without the boundary) with a free, effective, isometric action of a semi-simple compact Lie group.

The considered system has an important property: Due to the symmetry it can be reduced to some classical mechanical system defined on the orbit space of the group action. In a quantum case, there is the relationship between original and reduced system too. When the system is quantized with the path integral method this relationship is expressed by some connection between the path integrals. We will investigate this connection between the Wiener-like path integrals. By the Wiener path integrals we mean the path integrals in which the integrations are performed over the measures that are generated by stochastic processes. The processes will be defined by the solution of the stochastic differential equations that are also given on the manifold. To determine the stochastic processes and the stochastic differential equation, we will follow the papers by

Beloplskaya and Dalecky [4]. It allows us to use mainly a local approach (provided that an additional analytical restrictions have been imposed) in the investigation of path integral transformations.

An introduction of "separated" coordinates is an important point in path integral transformations. It is well-known [5] that in the system under consideration the original manifold can be regarded as the total space of the principal fibre bundle over the orbit space. There are several ways to introduce the local coordinates that are adapted to a fibre bundle structure. These coordinates separate into the invariant coordinates – the coordinates on the base of the fibre bundle (on the orbit space) and on variable coordinates – the fiber coordinates.

In this paper we will use the Bogolubov coordinate transformation method [7] by which the invariant coordinates are defined. In this method it is supposed that an arbitrary gauge surface is given.

In our previous paper [8], where we investigated the reduction problem in path integrals for a scalar particle on a manifold with a group action, we found a way of separating invariant and variable coordinates in path integrals. It was shown there, that the separation problem was reduced to the solution of the nonlinear filtering equation, provided that the path integral is defined as an integral over the measure generated by a stochastic diffusion process. Due to the symmetry of the problem, this complicated nonlinear equation becomes a linear matrix equation.

The present paper continues the investigation begun in [8]. The contents of the paper is as follows:

In Section 1 the main definitions are given. Section 2 presents a way of introducing the local coordinates by means of the Bogolubov coordinate transformation method. In this section the transformation of the initial Riemannian metric due to the introduction of the coordinates adapted to the principal fibre bundle structure is under discussion.

Section 3 considers the path integral transformation originating from the replacement of the initial coordinates by fiber coordinates. This transformation is derived by using the transformation of the local stochastic differential equations representing the measure generated stochastic process on a manifold.

In Section 4 further transformation of a path integral is made. It relates an initial path integral to a path integral over the invariant variables.

In Appendix we present arguments that are used in [4] to define a stochastic process on a manifold.

2. Definition

We consider a particle movement on a smooth compact Riemannian manifold $\mathcal{P}$ (without boundary) on which a smooth isometric action of a semi-simple compact Lie group $\mathcal{G}$ is given.

We will assume that the action of a group $\mathcal{G}$ is effective, i.e. the homomorphism from $\mathcal{G}$ to the group of the transformation of a manifold $\mathcal{P}$ is an isomorphism, and free, i.e. for every $g \in \mathcal{G}$ there is some point $p_0 \in \mathcal{P}$ and element $a \in \mathcal{G}$ such that $p = p_0 a$.

With the particle movement we relate the backward Kolmogorov equation

$$\left\{\begin{array}{ll}\left(\dfrac{\partial}{\partial t_a}+\dfrac{1}{2}\mu^2\kappa\triangle_{\mathcal{P}}(p_a)+\dfrac{1}{\mu^2\kappa m}V(p_a)\right)\psi_{t_b}(p_a,t_a)=0, & \\ \psi_{t_b}(p_b,t_b)=\phi_0(p_b), & (t_b>t_a),\end{array}\right. \tag{1}$$

where $\mu^2=\frac{\hbar}{m}$, κ is a real positive parameter, the potential $V(p)$ is invariant under the action of the group $\mathcal{G}$: $V(pg)=V(p)$, $\triangle_{\mathcal{P}}(p_a)$ is the Laplace–Beltrami operator on a manifold $\mathcal{P}$. In local coordinates $Q=\varphi(p)$ given by the chart (U,φ) it has the following form

$$\triangle_{\mathcal{P}}(Q)=G^{-1/2}(Q)\frac{\partial}{\partial Q^A}G^{AB}(Q)G^{1/2}(Q)\frac{\partial}{\partial Q^B}, \tag{2}$$

where $G^{AB}(Q)$ are the components of the matrix which is inverse to the matrix G_{AB} of the components of the initial Riemannian metric given in the coordinate basis $\{\frac{\partial}{\partial Q^A}\}$, $G=det(G_{AB})$. In formula (2), as in the sequel, we assume that there is sum over the repeated indices. The indices denoted by the capital letters ranged from 1 to $n_{\mathcal{P}}$ ($n_{\mathcal{P}}=dim\mathcal{P}$).

Provided that the coefficients and the initial function of eq.(1) are properly bounded and satisfy the necessary smooth requirement, the solution of eq.(1), as it follows from [4], can be presented in the following form

$$\begin{aligned}\psi_{t_b}(p_a,t_a) &= \mathrm{E}\Big[\phi_0(\eta(t_b))\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}V(\eta(u))du\}\Big]\\ &= \int_{\Omega_-}d\mu^\eta(\omega)\phi_0(\eta(t_b))\exp\{\ldots\},\end{aligned} \tag{3}$$

where $\eta(t)$ is a stochastic process on a manifold $\mathcal{P}$, μ^η is the generated by this process measure in the path space $\Omega_-=\{\omega(t):\omega(t_a)=0,\eta(t)=p_a+\omega(t)\}$.

In Appendix we will present the main ideas of [4] concerning the definition of the stochastic process on a manifold. The basic elements of the approach are local stochastic processes given by the solution of the Stratonovich-like equations on charts of the atlas of a manifold. “Gluing” these local processes into a global stochastic process is made by a special method that are valid in a case of the manifold having a uniform atlas (see Appendix).

Neglecting the global effects (they could originate because of the nontrivial topology of the manifold), it is possible in many cases to predict the behavior of the global stochastic process by studying its local representatives. It takes place due to the fact that it is the local representatives of a global stochastic process that determine a character of such a behavior.

The local components $\eta^A(t)$ of the stochastic process $\eta(t)$ of eqs.(3) are defined by the map φ of the chart (U,φ) in accordance with the equality: $\varphi(\eta)=\eta_\varphi(t)\equiv\{\eta^A(t)\}$. They are satisfied by the following stochastic differential equation:

$$d\eta^A(t)=\mu^2\kappa G^{-1/2}\frac{\partial}{\partial Q^B}(G^{1/2}G^{AB})dt+\mu\sqrt{\kappa}\mathcal{X}^A_{\bar{M}}(\eta(t))dw^{\bar{M}}(t), \tag{4}$$

where $\mathcal{X}^A_{\bar{M}}$ is defined locally by $\sum_{\bar{K}=1}^{n_P}\mathcal{X}^A_{\bar{K}}\mathcal{X}^B_{\bar{K}}=G^{AB}$. Here and what follows by the barred indices we denote the Euclidean indices.

Notice that eq.(4) coincides with equation (42) of the Application, i.e. it is a Stratonovich-like equation.

From the general theory developed in [4] it follows that eq.(3) determines a semigroup in the space of a smooth and bounded function on a manifold $\mathcal{P}$. We assume that in our case there exists the fundamental solution $G_{\mathcal{P}}(p_b,t_b;p_a,t_a)$ of eq.(1). Then, we can present the semigroup as follows:

$$\psi_{t_b}(p_a,t_a)=\int G_{\mathcal{P}}(p_b,t_b;p_a,t_a)\phi_0(p_b)dv_{\mathcal{P}}(p_b), \tag{5}$$

where $dv_{\mathcal{P}}(p)$ is the volume element on the manifold $\mathcal{P}$.

In the path integral for $G_{\mathcal{P}}$, the integration is carried out over the space of a such paths on a manifold that have the fixed values at times $t=t_a$ and $t=t_b$. The representation for $G_{\mathcal{P}}$ can be derived from (3) by substituting the delta-function for ϕ_0.

Symbolical formula (5) has a definite meaning if there is a some partition of unity $\mu_i(p)$, $\sum_i\mu_(p)=1$, subordinate to a locally finite covering U_i of $\mathcal{P}$: $\mathcal{P}=\bigcup_i U_i$. In that case formula (5) can be rewritten to give a representation of the semigroup, which acts on the functions defined in the corresponding domains of $R^{n_{\mathcal{P}}}$:

$$\psi_{i_a}(Q_a,t_a)=\sum_{i_b}\int_{\varphi_{i_b}(U_{i_b})}\tilde{\mu}_{i_b}(Q_b)G_{\mathcal{P}}(i_b,Q_b,t_b;i_a,Q_a,t_a)\phi_{0_{i_b}}(Q_b)dv_{\mathcal{P}}(Q_b) \tag{6}$$

($\psi_{i_a}=\psi\circ\varphi^{-1}_{i_a}$, $\phi_{0_{i_b}}=\phi_0\circ\varphi^{-1}_{i_b}$, $Q_{\substack{b\\a}}=\varphi_{i_{\substack{b\\a}}}(p_{\substack{b\\a}})$ and $dv_{\mathcal{P}}(Q)=\sqrt{G(Q)}dQ^1...dQ^{n_{\mathcal{P}}}$.

3. The principal bundle coordinates

In this section we consider the geometrical aspect of our problem. In our problem we have a smooth isometric effective and free action of a compact semi-simple Lie group $\mathcal{G}$ on a smooth compact manifold $\mathcal{P}$. This action maps the point p into the point $\tilde{p}=pg$. (We consider the right action of a group $\mathcal{G}$.) If in some chart the coordinates of the point p are $\{Q^A\}$, then the action of the group $\mathcal{G}$ in this chart is given by the functions F^A: $\tilde{Q}^A=F^A(Q^B,g^\alpha)$, $\alpha=1,\ldots,n_{\mathcal{G}}$, These functions have the well known properties, and we only recall here the one concerning the right multiplication of the group. Namely, for $(pg_1)g_2=p(g_1g_2)$:

$$F^A(F(Q,g_1),g_2)=F^A(Q,\Phi(g_1,g_2)),$$

where the function Φ is the group function which determines the multiplication law in the parameter space of the group.

An isometric action of the group $\mathcal{G}$ on $\mathcal{P}$ generates the vector fields K_α – the Killing vector fields. In coordinates $\{Q^A\}$, they can be written as follows: $K^A_\alpha=K^A_\alpha(Q)\frac{\partial}{\partial Q^A}$, $K^A_\alpha(Q)\equiv\frac{\partial F^A(Q,a)}{\partial a^\alpha}|_{a=e}$.

In our case these vector fields form a Lie algebra, which is isomorphic to the Lie algebra of a group $\mathcal{G}$:

$$[K^A_\alpha, K^A_\beta] = c^\gamma_{\alpha\beta} K^A_\gamma,$$

where $c^\gamma_{\alpha\beta}$ are the structure constants of a Lie algebra.

It is well known (see e.g. [5]) that if the action of a compact Lie group is free and effective, then the orbit space, denoted usually as $\mathcal{P}/\mathcal{G} = \mathcal{M}$, is also a smooth manifold. The natural projection $\pi : \mathcal{P} \to \mathcal{P}/\mathcal{G}$, which put in correspondence the orbit $p\mathcal{G}$ of a group $\mathcal{G}$ to every point p, determines the principal fibre bundle $P(\mathcal{M}, \mathcal{G})$.

Our main interest is the relationship between the movement on $\mathcal{P}$ and on $\mathcal{M}$. To consider this relation in detail, we, first of all, should introduce the corresponding coordinates. Since we deal with the principal fibre bundle, this implies that our manifold $\mathcal{P}$ can be locally presented as $\pi^{-1}(U_x) \sim U_x \times \mathcal{G}$, where U_x is a neighborhood of a point $x = \pi(p)$ belonging to a chart (U_x, φ_x) of the fibre bundle. It means that we should find local coordinates (x^i, a^α), $i = 1, \ldots, n_\mathcal{M}$, $n_\mathcal{M} = dim\mathcal{M}$, for each point p of the manifold $\mathcal{P}$. In finding such coordinates we should provide a one-to-one correspondence between the coordinates Q^A of a point p regarded as a point of the manifold $\mathcal{P}$ and the fiber coordinate (x^i, a^α) of this point. In addition to this requirement, we should have the compatibility condition of local coordinates on the overlapping of charts. In order words, we must construct the coordinate homeomorphisms of the principal bundle $P(\mathcal{M}, \mathcal{G})$.

It is clear, that there are several ways of introducing the bundle coordinates. As the local invariant bundle coordinates (the coordinates x^i) it is possible, for example, to take a complete set of invariants, obtained as a result of the action of a group $\mathcal{G}$ on $\mathcal{P}$. The existence of these coordinates follows from the fact that the principal fibre bundle $P(\mathcal{M}, \mathcal{G})$ can be locally regarded as a foliation of $\mathcal{P}$ formed by the vector fields on $\mathcal{P}$. These vector fields are the images of elements of a Lie algebra of a group $\mathcal{G}$. After completing these invariant coordinates to the full set, we get the necessary coordinates on a principal fibre bundle. We followed this way in our previous paper [8].

Here, we take another way, which has its origin in the Bogolubov coordinate transformation method. The method was applied by N.N.Bogolubov in the polaron problem to separate the translation invariant motion of a system considered as a whole and an internal motion of a system. Later, this method was generalized to include the systems that are invariant under an arbitrary Lie group [6]. Final generalization was given in [7], where, in addition, the geometrical content of the Bogolubov transformation method was revealed.

Here, we recall the main points of this method referring to [7] for details. In this method it is supposed that in each sufficiently small neighborhood of an arbitrary point p, there is the set of functions $\{\chi^\alpha(Q),\ \alpha = 1, \ldots, n_\mathcal{G}\}$ determining the submanifold. Given by the equation $\chi^\alpha(Q) = 0$ this submanifold, which we will also call as a surface, has a unique and transversal intersection with each orbit, i.e. $T\mathcal{P} = T\mathcal{M} + T\{\chi^\alpha = 0\}$.

Along with ordinary smoothness requirements posed on functions $\chi^\alpha(Q)$, they must satisfy an additional requirement that follows from the transversality condition: The matrix $(\Phi_\Pi)^\alpha_\beta$ (Faddeev–Popov matrix)

$$(\Phi_{\Pi})^{\alpha}_{\beta}(Q) = K^{A}_{\beta}\frac{\partial \chi^{\alpha}}{\partial Q^{A}}$$

should be invertible.

The coordinate homeomorphisms are defined by the following steps: Firstly, provided that the coordinates Q^A of a point p are given, from the equation

$$\chi^{\alpha}(F^{A}(Q, g^{-1})) = 0$$

one searches for the group element $g^{\alpha}(Q)$. This group element "connects" the point p having the coordinates Q^A, to that point, which belongs to the intersection of the submanifold $\{\chi^{\alpha}(Q) = 0\}$ with the orbit $p\mathcal{G}$.

In order to define the invariant coordinates $x^i(Q)$, the parametric form representation of the submanifold $\{\chi^{\alpha}(Q) = 0\}$ is used. The points of this submanifold are given by the equation $Q^A = Q^{*A}(x^i)$, where x^i are the surface coordinates and $\{\chi^{\alpha}(Q^*(x^i)) = 0\}$.

Then, by the invariant coordinates $x^i(Q)$ one should take the coordinates x^i of that point of the surface $Q^A = Q^{*A}(x^i)$ which results from the action of the element g^{-1} on the initial point p. These invariant coordinates are defined by the following equation:

$$Q^{*A}(x^i) = F^A(Q, g^{-1}).$$

Thus, for an arbitrary point p with the coordinates Q^A, we have introduced new coordinates $(x^i(Q), g^{\alpha}(Q))$. The coordinates $x^i(Q)$ define the orbit $p\mathcal{G}$ passing through the point p and the coordinates $g^{\alpha}(Q)$ tell us where on this orbit the point p is placed, provided that the surface $\{\chi^{\alpha} = 0\}$ is used as referrence surface.

In order words, if $\tilde{p} = pa$ and $\tilde{p} = \{\tilde{Q}^A\}$, then $x^i(Q) = x^i(\tilde{Q})$ and $g^{\alpha}(\tilde{Q}) = g^{\alpha}(Q)a$. These conditions provide the compatibility of the local coordinates in the intersections of charts of the principal fibre bundle.

This and the bijection between Q^A and (x^i, g^{α}) allows us to say that we have constructed the coordinate homeomorhisms of the principal fibre bundle $P(\mathcal{M}, \mathcal{G})$. By our initial assumption we only deal with the smooth functions. So, we suppose that the constructed coordinate homeomorhisms are diffeomorphisms.

Changing the coordinates Q^A for (x^i, a^{α}), $Q^A = F^A(Q^*(x^i), a^{\alpha})$, leads to the following components of the Riemannian metric G_{AB} in the basis $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial a^{\alpha}}\}$:

$$\begin{aligned}
G_{ij}(x,a) &= F^A_B(Q^*(x),a)Q^{*B}_i(x)G_{AC}(F(Q^*(x),a))F^C_D(Q^*(x),a)Q^{*D}_j(x),\\
G_{i\beta}(x,a) &= F^A_B(Q^*(x),a)Q^{*B}_i(x)G_{AC}(F(Q^*(x),a))F^C_{\beta}(Q^*(x),a),\\
G_{\alpha\beta}(x,a) &= F^A_{\alpha}(Q^*(x),a)G_{AB}(F(Q^*(x),a))F^B_{\beta}(Q^*(x),a).
\end{aligned}$$

Here we have introduced the notations $F^A_B(Q,a) \equiv \frac{\partial F^A}{\partial Q^B}(Q,a)$, $F^A_{\alpha}(Q,a) \equiv \frac{\partial F^A}{\partial a^{\alpha}}(Q,a)$, $Q^{*B}_i(x) \equiv \frac{\partial Q^{*B}}{\partial x^i}$.

Because of the isometric action of the group $\mathcal{G}$ on the manifold $\mathcal{P}$, the previous expressions for metric components can be rewritten as follows:

$$\begin{aligned}
G_{ij}(x,a) &= Q^{*A}_i(x)G_{AB}(Q^*(x))Q^{*B}_j(x) = G_{ij}(x,e),\\
G_{i\beta}(x,a) &= Q^{*A}_i(x)G_{AB}(Q^*(x))K^B_\delta(Q^*(x))\bar{u}^\delta_\beta(a) = G_{i\delta}(x,e)\bar{u}^\delta_\beta(a),\\
G_{\alpha\beta}(x,a) &= \bar{u}^\gamma_\alpha(a)K^A_\gamma(Q^*(x))G_{AB}(Q^*(x))K^B_\delta(Q^*(x))\bar{u}^\delta_\beta(a) =\\
&= \bar{u}^\gamma_\alpha(a)G_{\gamma\delta}(x,e)\bar{u}^\delta_\beta(a). \qquad (7)
\end{aligned}$$

In (7) $\bar{u}^\alpha_\beta$ is an inverse matrix to the matrix $\bar{v}^\alpha_\beta(a) = \left.\frac{\partial\Phi^\alpha(b,a)}{\partial b^\beta}\right|_{b=e}$ (Φ is a group function which defines the group multiplication in the space of group parameters).

In deriving (7) the equality

$$F^A_\alpha(Q,g) = \bar{u}^\beta_\alpha(g)F^A_B(Q,g)K^B_\beta(Q)$$

was used.

In the considered problem there are enough data for determining the metric on the orbit space $\mathcal{M} = \mathcal{P}/\mathcal{G}$. In these, a significant role is played by the Killing vectors K_α which we have introduced earlier. By using the projectors $\Pi^A_B = \delta^A_B - K^A_\alpha d^{\alpha\beta}K_{\beta B}$ ($d^{\alpha\beta}$ is a metric which is inverse to the metric $d_{\alpha\beta} = K^A_\alpha G_{AB}K^B_\beta$ defined along the orbit), we split the initial metric G_{AB} into the horizontal metric $G^H_{AB} = \Pi^C_A G_{CD}\Pi^D_B$ and the vertical one: $G_{AB} = G^H_{AB} + G^V_{AB}$. The components G^H_{AB} of the horizontal metric satisfy the condition

$$K^A_\alpha(Q)G^H_{AB}(Q) = 0.$$

Projecting the horizontal metric G^H_{AB} onto the surface $Q^* = Q^*(x)$, we obtain the metric h_{ij} on the orbit space. In basis $\{\frac{\partial}{\partial x^i}\}$ its components are given as

$$h_{ij}(x) = Q^{*A}_{\ i}G^H_{AB}Q^{*B}_{\ j}, \qquad (8)$$

where $Q^{*A}_{\ i} \equiv \frac{\partial Q^{*A}}{\partial x^i}$.

Usually, to introduce the metric on the orbit space of the principal fibre bundle, one should make use of the metric of the total space and the principal fibre bundle connection. In that case the orbit space metric h_x is defined as follows

$$h_x(v,v') = G_p(v_H, v'_H), \quad p = \pi^{-1}(x), \qquad (9)$$

where v_H, v'_H are the horizontal lifts of the tangent vectors $v, v' \in T_x\mathcal{M}$.

In our problem, we also have a connection. In [9] it was called by mechanical connection. The connection is determined by the Lie algebra-valued connection one-form $\omega = \omega^\alpha \otimes e_\alpha$ given on a manifold $\mathcal{P}$ as

$$\omega^\alpha(Q) = d^{\alpha\beta}(Q)G_{AB}(Q)K^B_\beta(Q)dQ^A.$$

It has standard properties:

$$\omega^{\alpha}(Q)\left(K^A_{\beta}\frac{\partial}{\partial Q^A}\right) = \delta^{\alpha}_{\beta},$$
$$\omega^{\alpha}(F^A(Q,g)) = \rho^{\alpha}_{\mu}(g^{-1})\omega^{\mu}(Q).$$

In this formula $\rho^{\alpha}_{\mu}(a) = \bar{u}^{\alpha}_{\nu}(a)v^{\nu}_{\mu}(a)$ is a matrix of an adjoint representation of a Lie group $\mathcal{G}$.

In local coordinates (x^i, a^{α}), the one-form ω^{α} is written as

$$\omega^{\alpha}(F(Q^*(x),a)) = \rho^{\alpha}_{\nu}(a^{-1})A^{\nu}_i(x)dx^i + u^{\alpha}_{\beta}(a)da^{\beta},$$

where

$$A^{\nu}_i(x) = d^{\nu\sigma}(Q^*(x))G_{EB}(Q^*(x))K^B_{\sigma}(Q^*(x))\frac{\partial Q^{*E}(x)}{\partial x^i}$$

is the projection of the connection to the base of the fibre bundle.

Calculating the metric h_{ij} by eq.(9), we get the expression which coincides with that of eq.(8). This verifies the validity of definition of h_{ij} given by eq.(8).

Now, using the representation of h_{ij} and $A^{\nu}_i(x)$ in terms of the components G_{AB}, we rewrite G_{AB} in the following form:

$$\begin{pmatrix} h_{ij}(x) + A^{\mu}_i(x)A^{\nu}_j(x)\bar{\gamma}_{\mu\nu}(x) & A^{\mu}_i(x)\bar{u}^{\nu}_{\sigma}(a)\bar{\gamma}_{\mu\nu}(x) \\ A^{\mu}_i(x)\bar{u}^{\nu}_{\sigma}(a)\bar{\gamma}_{\mu\nu}(x) & \bar{u}^{\mu}_{\rho}(a)\bar{u}^{\nu}_{\sigma}(a)\bar{\gamma}_{\mu\nu}(x) \end{pmatrix}. \tag{10}$$

By $\gamma_{\mu\nu}(x)$ we denote $G_{\mu\nu}(x,e)$ of eq.(7).

From eq.(10) we see that in coordinates (x^i, a^{α}), the metric G_{AB} is a Kaluza–Klein metric [10]. And we find that the determinant of the metric G_{AB} is equal to

$$\det G_{AB} = (\det h_{ij}(x))\,(\det \bar{\gamma}_{\alpha\beta}(x))\,(\det \bar{u}^{\mu}_{\rho}(a))^2.$$

4. Point transformation of the path integral

In this section we will consider the transformation of the path integral (3) resulting from the replacement of the coordinates Q^A by (x^i, a^{α}). Since it mainly concerns with the transformation of the path integral measure, which is associated with the differential part of eq.(1), we discard the potential term in path integral of (3). The account of this potential term can be easily made in the final formulas by using the Girsanov theorem [4].

First we recall that in our paper the path integral of (3) is defined by the local approach. The semigroup, determined by this path integral, acts in the space of the smooth and bounded function on $\mathcal{P}$. The semigroup is obtained as a result of going to the limit in the superposition of the local semigroups (see eqs.(44) and (45) of Appendix):

$$\psi_{t_b}(p_a,t_a) = U(t_b,t_a)\phi_0(p_a) = \lim\nolimits_q \tilde{U}_{\eta}(t_a,t_1)\cdot\ldots\cdot\tilde{U}_{\eta}(t_{n-1},t_b)\phi_0(p_a). \tag{11}$$

And each local semigroup $\tilde{U}_{\eta}$ is build by using a stochastic family of local evolution mappings of the manifold $\mathcal{P}$.

The advantage of this approach lies in the fact that many properties of the initial global semigroup can be derived by analyzing the local semigroups $\tilde{U}_\eta$. But as for the local semigroups, they are completely determined by the stochastic differential equations whose solutions – the stochastic processes – generate the corresponding path integral measures.

Therefore, studying the transformation of the local stochastic differential equations enable us to conclude on the transformation of the path integrals and of the semigroups acting in the space of functions on a manifold.

Notice that in the case of the non-trivial topology of the manifolds such an approach for changing the coordinates Q^A for (x^i, a^α) in the path integral should be corrected. But in the paper we neglect the influence of the topology effects.

Let us consider the transformation of an arbitrary local semigroup $\tilde{U}_\eta$ of (11). This semigroup is defined by the equality:

$$\tilde{U}_\eta(s,t)\phi(p) = \mathrm{E}_{s,p}\phi(\eta(t)), \;\; s \leq t, \;\; \eta(s) = p, \tag{12}$$

where the stochastic process $\eta(t)$ is build by the Itô fields that are localizable at the neighborhood $\mathcal{V}_p$ of the point p. The mapping $\varphi^{\mathcal{P}}$ of the chart $(\mathcal{V}_p, \varphi^{\mathcal{P}})$ brings the process $\eta(t)$ to the corresponding neighborhood of $R^{n_{\mathcal{P}}}$:

$$\varphi^{\mathcal{P}}(\eta(t)) = \eta^{\varphi^{\mathcal{P}}}(t) \equiv \{\eta^A(t)\}.$$

By this mapping it is possible to change the process of the right-hand side of eq.(12) for the process $\eta^{\varphi^{\mathcal{P}}}(t)$. Then, we obtain

$$\tilde{U}_\eta(s,t)\phi(p) = \mathrm{E}_{s,\varphi^{\mathcal{P}}(p)}\phi((\varphi^{\mathcal{P}})^{-1}(\eta^{\varphi^{\mathcal{P}}}(t))), \;\; \eta^{\varphi^{\mathcal{P}}}(s) = \varphi^{\mathcal{P}}(p). \tag{13}$$

Now we introduce other coordinates in $R^{n_{\mathcal{P}}}$. We transform $Q = \varphi^{\mathcal{P}}(p)$ to (x, a): $Q^A = f^A(x^i, a^\alpha)$. In correspondence with this transformation, we have the tranformation of the components $\eta^A(t)$ of the stochastic process $\eta^{\varphi^{\mathcal{P}}}(t)$:

$$\eta^A(t) = F^A(Q^*(x^i(t)), a^\alpha(t)) \equiv f^A(x^i(t), a^\alpha(t)). \tag{14}$$

We can regard $(x^i(t), a^\alpha(t))$ as the components $\zeta^A(t)$ of the local stochastic process $\zeta^{\varphi^P}(t)$ which is also defined in $R^{n_{\mathcal{P}}}$. Then, eq.(14) represents the phase-space transformation of the stochastic process. From the stochastic process theory it is known that phase-space transformation of the stochastic processes does not change both the probabilities:

$$P(\eta^{\varphi^{\mathcal{P}}}(t) \in B) = P(\zeta^{\varphi^P}(t) \in f^{-1}(B))$$

(B is a Borel set of $\mathcal{B}(R^{n_{\mathcal{P}}})$) and the transition probabilities either. Taking this into account, we rewrite eq.(13) as

$$\tilde{U}_\eta(s,t)\phi(p) = \mathrm{E}_{s,\varphi^P(p)}\phi((\varphi^P)^{-1}(\zeta^{\varphi^P}(t))), \tag{15}$$

where $\varphi^P = f^{-1} \circ \varphi^{\mathcal{P}}$.

In order words, φ^P are new coordinate homeomorphisms of the manifold $\mathcal{P}$. The index P suggests that they are attached to the principal bundle $P(\mathcal{M},\mathcal{G})$.

At last, we can introduce a new function $\tilde{\phi}=\phi\circ(\varphi^P)^{-1}$ and after this eq.(15) is as follows:

$$\tilde{U}_\eta(s,t)\phi(p)=\mathrm{E}_{s,\varphi^P(p)}\tilde{\phi}(\zeta^{\varphi^P}(t)). \tag{16}$$

Therefore, in eq.(11) each of the local semigroups is calculated by using the corresponding semigroup which acts on functions defined on $R^{n_{\mathcal{P}}}$.

Since we have introduced the fiber coordinates, then it is natural to expect that the stochastic process $\zeta^{\varphi^P}(t)$ is a local representative of the global stochastic process $\zeta(t)$ defined in the principal bundle $P(\mathcal{M},\mathcal{G})$. If it is the case, then local processes given in the overlapping charts must satisfy the compatibility condition: Under "the gluing" of the charts of the fibre bundle, these local stochastic processes transform into each other. In turn, this condition will be satisfied if the stochastic differential equations for the processes have the similar properties. Therefore, to elucidate this question, we must find the stochastic differential equation for the local components $\zeta^A(t)$ of the stochastic process $\zeta^{\varphi^P}(t)$. To find the coefficient of such a stochastic differential equation, we make use of the Itô differentiation formula (lemma).

Let the stochastic differential equation for $\zeta^A(t)=(x^i(t),a^\alpha(t))$ be given as follows:

$$\begin{cases} dx^i(t)=b^i(t)dt+X^i_{\bar{M}}dw^{\bar{M}} \\ da^\alpha(t)=b^\alpha(t)dt+X^\alpha_{\bar{M}}dw^{\bar{M}}. \end{cases} \tag{17}$$

An inversion of the homogeneous point canonical transformation $Q^A=f^A(x^i,a^\alpha)$ allows us to express the coordinates x^i and a^α in terms of the coordinates Q^A: $x^i=x^i(Q)$, $a^\alpha=a^\alpha(Q)$. In a similar manner, from formula (14) representing the phase-space transformation of the stochastic processes, we can derive the expressions of stochastic processes $x^i=x^i(\eta(t))$, $a^\alpha=a^\alpha(\eta(t))$.

Then, to obtain the equation for the definition of the coefficients b and X of eq.(17), we substitute the r.h.s. of the following equation

$$dx^i(t)=\frac{\partial x^i}{\partial Q^A}(\eta(t))d\eta^A(t)+\frac{1}{2}\frac{\partial^2x^i}{\partial Q^A\partial Q^B}(\eta(t))G^{AB}(\eta(t))dt \tag{18}$$

($d\eta^A(t)$ is given by eq.(4)) for dx^i in eq.(17). The same should be done to obtain the equation for $da^\alpha(t)$).

The partial derivatives $\frac{\partial x^i}{\partial Q^A}$ in eq.(18) (and $\frac{\partial a^\alpha}{\partial Q^A}$ in the corresponding equation for $da^\alpha(t)$) are defined by making use of the formula $Q^A=F^A(Q^*(x),a)$ by which we have introduced new coordinates on the manifold $\mathcal{P}$ [7]:

$$\frac{\partial x^i}{\partial Q^A}(F(Q^*(x),a))=F^B_A(F(Q^*(x),a),a^{-1})G^H_{BC}(Q^*(x))\frac{\partial Q^{*C}(x)}{\partial x^m}h^{mi}(x),$$

$$\frac{\partial a^\alpha}{\partial Q^A}(F(Q^*(x),a))=\bar{v}^\alpha_\beta(a)(\Phi^{-1}_{\Pi})^\alpha_\gamma(Q^*(x))\chi^\gamma_B(Q^*(x))F^B_A(F(Q^*(x),a),a^{-1}),$$

where $h^{mi}(x)$ is an inverse metric to the metric on the orbit space, Φ_{Π}^{-1} is an inverse matrix to the Faddeev-Popov matrix, $\chi^{\gamma}_{B} \equiv \frac{\partial \chi^{\gamma}}{\partial Q^B}$.

Notice that in the way we have used to define the coefficients b and X, the diffusion coefficients X are determined up to an arbitrary orthogonal transformation [11]. But, such a standard ambiguity does not interfere with the path integral transformation, since it is always possible to get rid of it in eq.(17) by changing the Wiener process with some orthogonal matrix: $(w^{\alpha})' = O^{\alpha}_{\beta} w^{\beta}$.

Remark, that the drift coefficients $b^A = (b^i, b^{\alpha})$ can be also derived by making use of the transformed metric $\tilde{G}_{AB}$. An initial drift coefficient of eq.(4), which is equal to $G^{-1/2}(Q)\frac{\partial}{\partial Q^B}(G^{1/2}(Q)G^{AB}(Q))$, is form-invariant under the "point transformation" performed with the Itô formula. In new coordinates (x^i, a^{α}), the drift coefficients are given by the same formula, but where the metric $\tilde{G}_{AB}$ is used instead of the metric G_{AB}.

In the sequel, it will be desirable for us to have equation (17) with $X^i_{\bar{\alpha}} = 0$. We achieve this form of the equation with the help of the orthogonal transformation of the Wiener process $w^{\bar{N}}$. As we have already remarked, it will not disturb the path integral measure. The replacement of the diffusion coefficients of eq.(17) is made as follows [12]:

$$
\begin{aligned}
&X^i_{\bar{k}} dw^{\bar{k}} + X^i_{\bar{\alpha}} dw^{\bar{\alpha}} = \tilde{X}^i_{\bar{k}} d\tilde{w}^{\bar{k}},\\
&X^{\alpha}_{\bar{k}} dw^{\bar{k}} + X^{\alpha}_{\bar{\beta}} dw^{\bar{\beta}} = \tilde{X}^{\alpha}_{\bar{k}} d\tilde{w}^{\bar{k}} + \tilde{X}^{\alpha}_{\bar{\beta}} d\tilde{w}^{\bar{\beta}},
\end{aligned}
$$

where

$$
\begin{aligned}
&\tilde{X}^i_{\bar{k}} = (X^i_{\bar{k}} X^j_{\bar{k}} + X^i_{\bar{\alpha}} X^j_{\bar{\alpha}})^{1/2} \equiv (X^i_{\bar{M}} X^j_{\bar{M}})^{1/2},\\
&\tilde{X}^{\alpha}_{\bar{k}} = (X^i_{\bar{k}} X^{\alpha}_{\bar{k}} + X^i_{\bar{\beta}} X^{\alpha}_{\bar{\beta}})(X^i_{\bar{M}} X^j_{\bar{M}})^{-1/2},\\
&\tilde{X}^{\alpha}_{\bar{\gamma}} = \left[(X^{\alpha}_{\bar{M}} X^{\beta}_{\bar{M}}) - X^{\alpha}_{\bar{M}} X^i_{\bar{M}})(X^i_{\bar{M}} X^j_{\bar{M}})^{-1/2}(X^{\beta}_{\bar{M}} X^j_{\bar{M}}) \right]^{1/2}.
\end{aligned}
$$

In these formulas the summations over the repeated indices are assumed.

Performing the transformation, we find that $X^i_{\bar{M}} X^j_{\bar{M}} = h^{ij}(x)$,

$$
\begin{aligned}
X^i_{\bar{k}} X^{\alpha}_{\bar{k}} + X^i_{\bar{\beta}} X^{\alpha}_{\bar{\beta}} &= \Pi^B_L {Q^*}^L_n h^{ni} \bar{v}^{\alpha}_{\beta} (\Phi^{-1}_{\Pi})^{\beta}_{\nu} \chi^{\nu}_B\\
&= -\bar{\gamma}^{\beta\nu} K^N_{\nu} G_{NM} {Q^*}^M_k h^{ki} \bar{v}\alpha_{\beta}\\
&= -A^{\mu}_k(x) h^{ki}(x) \bar{v}^{\alpha}_{\beta};
\end{aligned}
$$

and $\tilde{X}^{\alpha}_{\bar{\beta}} = [\bar{v}^{\alpha}_{\mu}(a) \bar{v}^{\beta}_{\nu}(a) \bar{\gamma}^{\mu\nu}(x)]^{1/2}$.

From these equalities it follows that in the transformed equation (17) we can choose the diffusion coefficients in the following form:

$$
\begin{aligned}
&\tilde{X}^i_{\bar{k}}(x) = (h^{ij}(x))^{1/2},\\
&\tilde{X}^{\alpha}_{\bar{k}}(x,a) = -A^{\mu}_n(x) \bar{v}^{\alpha}_{\mu}(a) \tilde{X}^n_{\bar{k}}(x),\\
&\tilde{X}^{\alpha}_{\bar{\beta}}(x,a) = \bar{v}^{\alpha}_{\mu}(a) \bar{Y}^{\mu}_{\bar{\beta}}(x)\\
&(\bar{Y}^{\mu}_{\bar{\beta}} \bar{Y}^{\nu}_{\bar{\beta}} = \bar{\gamma}^{\mu\nu}(x)).
\end{aligned}
$$

In what follows, for simplicity of the notation we will omit the tilde over X and w assuming that the additional orthogonal transformation of the Wiener process is already made. Taking this into account, we obtain that the equations for the local components of the stochastic process $\zeta^{\varphi^P}(t)$ will be the following ones:

$$
\begin{aligned}
&dx^i(t) = \frac{1}{2}\mu^2\kappa\Big[\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^n}(h^{ni}\sqrt{h\bar{\gamma}})\Big]dt + \mu\sqrt{\kappa}X^i_{\bar{n}}(x(t))dw^{\bar{n}}(t),\\
&da^\alpha(t) = \mu^2\kappa\Big[-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^\nu_m\right)\bar{v}^\alpha_\nu(a(t))\\
&+\frac{1}{2}(\bar{\gamma}^{\lambda\epsilon} + h^{ij}A^\lambda_i A^\epsilon_j)\bar{v}^\sigma_\lambda(a(t))\frac{\partial}{\partial a^\sigma}(\bar{v}^\alpha_\epsilon(a(t))\Big]dt\\
&+\mu\sqrt{\kappa}\bar{v}^\alpha_\lambda(a(t))\bar{Y}^\lambda_{\bar{\epsilon}}dw^{\bar{\epsilon}}(t) - \mu\sqrt{\kappa}X^i_{\bar{n}}A^\nu_i\bar{v}^\alpha_\nu(a(t))dw^{\bar{n}}(t).
\end{aligned}
\tag{19}
$$

Thus, due to the introduction of new coordinates (x^i, a^α) adapted to the fibre bundle structure, we have changed the local components $\eta^A(t)$ of the stochastic process $\eta(t)$ for the local components $\zeta^A(t) = (x^i(t), a^\alpha(t))$ of the stochastic process $\zeta^{\varphi^P}(t)$.

Eq.(19) for $\zeta^A(t)$ are the coordinate representatives of the local stochastic differential process, given on a chart of the principal fibre bundle. If two local processes on overlapping charts are given and both of the processes are described (in the corresponding coordinates) by the stochastic differential equations (19), then it is not difficult to check, that gluing transformation of the charts transforms the local equations into each other. It gives the necessary conditions for the definition of the global stochastic process in the principal fibre bundle by the method of [4].

Solutions of the local stochastic differential equation (19) give rise to the local stochastic evolution family of the mappings of the total space $\mathcal{P}$ of the principal fibre bundle $P(\mathcal{M}, \mathcal{G})$. As in the case with the manifold these local families defined in charts of the principal fibre bundle, are the main elements that can be used in building the global stochastic evolution family of the mappings acting in $\mathcal{P}$. By definition, this global family is a global stochastic process in the principal fibre bundle.

Notice, however, since we haven't changed our manifold $\mathcal{P}$ but have introduced only a new coordinate system on it, the global process $\zeta(t)$ is the same global process $\eta(t)$, that is considered from the principal fibre bundle viewpoint.

From eq.(19) we see that the process $\zeta(t)$ has two components. The former describes the stochastic process on a base of the fibre bundle, while the latter — on a fiber of the principal fibre bundle (i.e., on a group $\mathcal{G}$). The stochastic evolution family $T(t,s)$ of mappings of the total space of the principal fibre bundle has the following properties:

$$
\pi \circ T(t,\tau) = S(t,\tau)\circ\pi,
\tag{20}
$$

where $S(t,\tau)$ is the stochastic evolution family of mappings of the fibre bundle base (i.e., the manifold $\mathcal{M}$). The first of eqs.(19) is the equation for the local components of the stochastic process $\xi(t)$ related to the family $S(t,s)$. Besides, according to eq.(20) we have $\pi(\zeta(t)) = \xi(t)$.

As in the case with the manifold, we can present the global stochastic evolution family of the mappings of the total space of the principal fibre bundle $\mathcal{P}$ in the form of the superposition of the local evolution families. This property can be extended to the global semigroup, defined by the process $\zeta(t)$, which acts in the corresponding function space.

In our case the global semigroup is the superposition of the local semigroups of such a form as in eq.(16). The local semigroups are obtained as a result of the transformation of the local semigroup associated with the process $\eta(t)$.

A global semigroup in the space of the smooth and bounded functions given on the total space of the principal fibre bundle is obtained as a result of going to the limit (under the refinement of the partition of the time interval) in the superposition of the local semigroups.

Concluding on the path integral transformation resulting from passing to the fibre bundle coordinates, we can say that this transformation enables us to present path integral (3) in the form of the limit of the local semigroups based on the process $\zeta(t)$:

$$\psi_{t_b}(p_a,t_a)=\lim{}_q\tilde{U}_{\zeta^{\varphi^P}}(t_a,t_1)\cdot\ldots\cdot\tilde{U}_{\zeta^{\varphi^P}}(t_{n-1},t_b)\tilde{\phi}_0(x_a,\theta_a), \tag{21}$$

where by $\tilde{U}_{\zeta^{\varphi^P}}$ we denote the r.h.s. of eq.(16), i.e.,

$$\tilde{U}_{\zeta^{\varphi^P}}(s,t)\tilde{\phi}(x_0,\theta_0)=\mathrm{E}_{s,(x_0,\theta_0)}\tilde{\phi}(x(t),a(t)),\ \ x(s)=x_0,\ \ a(s)=\theta_0. \tag{22}$$

Recovering the potential term, we present eq.(21) in the following symbolical form:

$$\psi_{t_b}(p_a,t_a)=\mathrm{E}\Big[\tilde{\phi}_0(\xi(t_b),a(t_b))\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(\xi(u))du\}\Big], \tag{23}$$

where $\xi(t_a)=x_a$, $a(t_a)=\theta_a$ and $\varphi^P(p_a)=(x_a,\theta_a)$.

Calculating the differential generator of the semigroup associated with the process $\zeta(t)$, we find that in (x^i,a^α)-coordinates, it can be written as follows:

$$\begin{aligned}&\frac{1}{2}\mu^2\kappa\{\triangle_M(x)+h^{ij}\frac{1}{\sqrt{\bar{\gamma}}}\Big(\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^i}\Big)\frac{\partial}{\partial x^j}+h^{ij}A_i^\alpha A_j^\beta\bar{L}_\alpha\bar{L}_\beta-2h^{in}A_n^\alpha\bar{L}_\alpha\frac{\partial}{\partial x^i}\\&-h^{in}\frac{\partial A_n^\alpha}{\partial x^i}\bar{L}_\alpha-h^{in}\frac{\partial\sqrt{h}}{\partial x^i}A_n^\alpha\bar{L}_\alpha-h^{in}\frac{1}{\sqrt{\bar{\gamma}}}\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^i}A_n^\alpha\bar{L}_\alpha-\frac{\partial h^{in}}{\partial x^i}A_n^\alpha\bar{L}_\alpha+\bar{\gamma}^{\alpha\beta}\bar{L}_\alpha\bar{L}_\beta\},\end{aligned} \tag{24}$$

where $\triangle_M$ is a Laplace–Beltrami operator on $\mathcal{M}$ and by $\bar{L}_\alpha$ we denote the right-invariant vector field $\bar{L}_\alpha=\bar{v}^\epsilon_\alpha(a)\frac{\partial}{\partial a^\epsilon}$.

We remark that the operator given by (24) can be rewritten in the following form:

$$\frac{1}{2}\mu^2\kappa\{\Box_H+h^{ij}\frac{1}{\sqrt{\bar{\gamma}}}\Big(\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^i}\Big)\frac{\partial}{\partial x^j}+\bar{\gamma}^{\alpha\beta}\bar{L}_\alpha\bar{L}_\beta-\frac{\partial h^{in}}{\partial x^i}A_n^\alpha\bar{L}_\alpha\},$$

where $\Box_H$ is the horizontal Laplace operator.

5. Separation of the variables in the path integral

In this section we continue the transformation of the path integral (21). As previously, the local semigroups that are used (by their superposition) in path integral definition will be in the center of our investigation.

The aim of the path integral transformation of this section is to separate the variables in the path integral. For this purpose we apply the method from our paper [8]. It was found there that local stochastic differential equations (in the present paper these are eqs.(19)) of the stochastic process $\zeta(t)$ given on the principal fibre bundle coincide with the stochastic differential equations that are used in the nonlinear filtering theory. The equation for $a^\alpha(t)$ describes the signal process (it cannot be directly observed in the experiment) and the equation for $x^i(t)$ — the observation process.

To estimate the existing difference between the observation process and the signal process in the experiment, the conditional expectation of the signal process given the sub-σ-field associated with the observation process is used. The evolution of this conditional expectation is given by the nonlinear filtering equation. It is this equation that enables us to perform the path integral transformation that separates the path integral variables.

To reveal the conditional expectation in the local semigroup (22) (in the local path integral), we make use of usual conditional expectation properties that are valid in our case due to the fact that the process $\zeta(t)$ is the Markov process. By these properties the path integral (22) can be transformed as follows:

$$\tilde{U}_{\zeta_{\varphi^P}}(s,t)\tilde{\phi}(x_0,\theta_0) = \mathrm{E}\Bigl[\mathrm{E}\bigl[\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s\bigr]\Bigr]. \tag{25}$$

Now, in eq.(25), under the "path integral sign", we have the conditional expectation

$$\hat{\tilde{\phi}}(x(t)) \equiv \mathrm{E}\Bigl[\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s\Bigr]$$

and we can write for it the nonlinear filtering equation. A comparison of our processes $x^i(t)$ and $a^\alpha(t)$ with the processes used in the nonlinear filtering equation of [13,12] leads to the following equation for the conditional expectation:

$$\begin{aligned}
d\hat{\tilde{\phi}}(x(t)) &= \Bigl[-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\Bigl(\sqrt{h\bar{\gamma}}h^{km}A^\mu_m\Bigr)\Bigr]\,\mathrm{E}[\bar{L}_\mu\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s]dt \\
&+\frac{1}{2}(\bar{\gamma}^{\mu\nu} + h^{ij}A^\mu_i A^\nu_j)E[\bar{L}_\mu\bar{L}_\nu\tilde{\phi}_0(x(t),a(t)) \mid (\mathcal{F}_x)^t_s]dt \\
&-A^\mu_k X^k_{\bar{m}}\,\mathrm{E}[\bar{L}_\mu\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F}_x)^t_s]dw^{\bar{m}}(t).
\end{aligned} \tag{26}$$

Due to the symmetry of our problem, equation (26) can be considerably simplified. To realize it, we expand the function $\tilde{\phi}(x,a)$ with the Peter–Weyl theorem in a series over the matrix irreducible representation ($\sum_q D^\lambda_{pq}(a)D^\lambda_{qn}(b) = D^\lambda_{pn}(ab)$) of a group $\mathcal{G}$:

$$\tilde{\phi}_0(x,a) = \sum_{\lambda,p,q} c^\lambda_{pq}(x) D^\lambda_{pq}(a) \tag{27}$$

and substitute this series in eq.(26).

Then, as it follows from the conditional expectation properties, we obtain that

$$\mathrm{E}\Big[\tilde{\phi}(x(t),a(t)) \mid (\mathcal{F})_x)^t_s\Big] = \sum_{\lambda,p,q} c^\lambda_{pq}(x(t))\, \mathrm{E}\Big[D^\lambda_{pq}(a(t)) \mid (\mathcal{F}_x)^t_s\Big].$$

The evolution of the conditional expectation $\hat{D}^\lambda_{pq}(x(t)) \equiv \mathrm{E}\Big[D^\lambda_{pq}(a(t)) \mid (\mathcal{F}_x)^t_s\Big]$[1]will be described by the linear matrix equation:

$$\begin{aligned} d\hat{D}^\lambda_{pq}(x(t)) &= \Gamma^\mu_1\ (J_\mu)^\lambda_{pq'} \hat{D}^\lambda_{q'q}(x(t))dt + \Gamma^{\mu\nu}_2\ (J_\mu)^\lambda_{pq'}(J_\nu)^\lambda_{q'q''}\hat{D}^\lambda_{q''q}(x(t))dt \\ &-(J_\mu)^\lambda_{pq'}\hat{D}^\lambda_{q'q}(x(t))A^\mu_k(x(t))X^k_{\bar{m}}(x(t))dw^{\bar{m}}(t), \end{aligned} \tag{28}$$

where the matrix elements $(J_\mu)^\lambda_{pn}$ are the generators of the representation $D^\lambda(a)$, $(J_\mu)^\lambda_{pq} \equiv (\frac{\partial D^\lambda_{pq}(a)}{\partial a^\mu})|_{a=e}$, with the properties

$$\bar{L}_\mu D^\lambda_{pq}(a) = \sum_{q'} (J_\mu)^\lambda_{pq'} D^\lambda_{q'q}(a).$$

In eq.(28), the explicit forms of the coefficients Γ^μ_1 and $\Gamma^{\mu\nu}_2$ can be easily obtained from the transformation of eq.(26).

The solution of eq.(28) can be written in terms of the multiplicative stochastic integral [14,15]:

$$\hat{D}^\lambda_{pq}(x(t)) = (\overleftarrow{\exp})^\lambda_{pn}(x(t),t,t_a)\, \mathrm{E}\Big[D^\lambda_{nq}(a(s)) \mid (\mathcal{F}_x)^t_s\Big], \tag{29}$$

where

$$\begin{aligned} (\overleftarrow{\exp})^\lambda_{pn}(x(t),t,s) &= \overleftarrow{\exp} \int_s^t \Big\{ \Big[\frac{1}{2}\bar{\gamma}^{\mu\nu}(x(u))(J_\mu)^\lambda_{pr}(J_\nu)^\lambda_{rn} \\ &-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^\mu_m\right)(J_\mu)^\lambda_{pn}\Big] du - A^\mu_k(x(u))(J_\mu)^\lambda_{pn}X^k_{\bar{m}}(x(u))dw^{\bar{m}}(u) \Big\} \end{aligned} \tag{30}$$

($h, \bar{\gamma}$ depend on $x(u)$).

By the multiplicative stochastic integral of eq.(29) we mean the limit of the sequence of the time-ordered exponential multipliers. These multipliers are obtained as a result of partition of time interval (t,s). By the arrow we denote the order of the multipliers: The arrow is aimed at the multipliers under the greater times.

Using the series of eq.(27) and the representation of (29) in eq.(25), we obtain that

$$\tilde{U}_{\zeta^{\varphi^P}}(s,t)\tilde{\phi}(x_0,\theta_0) = \sum_{\lambda,p,q,q'} \mathrm{E}\Big[c^\lambda_{pq}(x(t))(\overleftarrow{\exp})^\lambda_{pq'}(x(t),t,s)\Big] D^\lambda_{q'q}(\theta_0). \tag{31}$$

[1] $\hat{D}^\lambda_{pq}(x(t))$ depends as well on $x^i_0 = x^i(s)$ and $\theta^\alpha_0 = a^\alpha(s)$. But, for brevity, this dependence was not explicitly shown in the notation of $\hat{D}^\lambda_{pq}(x(t))$.

In obtaining this equation, we have taken into account that at the initial moment of time it must be

$$\mathrm{E}\Big[D^{\lambda}_{nq}(a(s))\mid(\mathcal{F}_x)^t_s\Big]=D^{\lambda}_{nq}(a(s))=D^{\lambda}_{nq}(\theta_0).$$

Thus, we see that the expression under the expectation in eq.(31) depends on the stochastic process given on the base of the principal fibre bundle. Due to the symmetry of the problem the only trace of the fiber stochastic process $a(t)$ (the process on the group $\mathcal{G}$) is the dependence of the matrix element D^{λ} on the boundary value of this process at $t=s:\ a(s)=\theta_0$.

If after partition of the time interval $[t_a,t_b]$, we form the superposition of the local semigroups that are similar to the semigroups of eq.(31) and go to a limit by taking finer partition of the time interval, we get the global semigroup.

Similar to the symbolical formula (23), the resulting global semigroup can be also written in symbolical form as follows:

$$\psi_{t_b}(p_a,t_a)=\sum_{\lambda,p,q,q'}\mathrm{E}\Big[c^{\lambda}_{pq}(\xi(t_b))(\overleftarrow{\exp})^{\lambda}_{pq'}(\xi(t),t_b,t_a)\Big]D^{\lambda}_{q'q}(\theta_a) \tag{32}$$

$$(\xi(t_a)=\pi\circ p_a),$$

where the process $\xi(t)$ is a global process on a manifold $\mathcal{M}=\mathcal{P}/\mathcal{G}$. The stochastic equation of the local representatives of the process $\xi(t)$ is given by the first equation of (19).

Both the stochastic process $\xi(t)$ and multiplicative stochastic integral (30), which is an operator multiplicative functional of the process $\xi(t)$, determine the semigroup in the space of the sections $\Gamma(\mathcal{M},V^*)$ of the associated co-vector bundle (we consider the backward equation) $\mathcal{E}^*=P\times_G V^*_{\lambda}$. This semigroup is determined by the expectation value of eq.(32) standing under the sum. A scalar product in the space of the sections of the associated co-vector bundle is given by the following form:

$$(\psi_n,\psi_m)=\int_{\mathcal{M}}\langle\psi_n,\psi_m\rangle_{V^*_{\lambda}}\sqrt{\bar{\gamma}(x)}dv_{\mathcal{M}}(x), \tag{33}$$

($dv_{\mathcal{M}}(x)$ is an invariant volume measure on a manifold $\mathcal{M}$; in x^i-coordinates it is presented as $dv_{\mathcal{M}}(x)=\sqrt{h(x)}dx^1...dx^{n_{\mathcal{M}}}$).

It is not difficult to obtain the differential generator of the semigroup associated with the process $\xi(t)$. Its coordinate expression will be the following differential operator:

$$\frac{1}{2}\mu^2\kappa\Big\{\Big[\triangle_M+h^{ni}\frac{1}{\sqrt{\bar{\gamma}}}\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^n}\frac{\partial}{\partial x^i}\Big](I^{\lambda})_{pq}-2h^{ni}A^{\alpha}_n(J_{\alpha})^{\lambda}_{pq}\frac{\partial}{\partial x^i}$$
$$-\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^n}\left(\sqrt{h\bar{\gamma}}h^{nm}A^{\alpha}_m\right)(J_{\alpha})^{\lambda}_{pq}+(\bar{\gamma}^{\alpha\nu}+h^{ij}A^{\alpha}_iA^{\nu}_j)(J_{\alpha})^{\lambda}_{pq'}(J_{\nu})^{\lambda}_{q'q}\Big\}.$$

(Here $(I^{\lambda})_{pq}$ is a unity matrix.) This operator acts in the space of functions with the scalar product (33).

It is possible to inverse formula (32). In case of performing this, we will find how the kernel of the operator semigroup associated with the process $\xi(t)$ is related to the kernel of our initial semigroup (the Green function of eq.(1)) associated with the process $\eta(t)$ (or $\zeta(t)$). As a consequence of this relation, we get the connection between the corresponding path integrals.

We will carry out the inversion of formula (32) by using (6), where globally defined semigroup (5) was presented in terms of the kernels given on a locally finite covering of the manifold $\mathcal{P}$. We will make all of the variable replacements in the local integrals of (6) that lead us to formula (32).

After changing Q_b for (x^i_b, θ^α_b) performed both with the help of local maps of the charts and the local Bogolubov transformation

$$Q_b = \varphi^{\mathcal{P}}_{\alpha_b}(p_b), \;\; (x^i_b, \theta^\alpha_b) = f^{-1}(Q_b),$$

in which the local neighborhood $\varphi^{\mathcal{P}}_{\alpha_b}(\mathcal{U}^{\mathcal{P}}_{\alpha_b})$ transforms consequently into $(f_b^{-1}\circ\varphi^{\mathcal{P}}_{\alpha_b})(\mathcal{U}^{\mathcal{P}}_{\alpha_b}) \equiv \varphi^{P}_{\alpha_b}(\mathcal{U}^{P}_{\alpha_b})$ and $\varphi^{\mathcal{M}}_{\alpha_b}(\mathcal{U}^{\mathcal{M}}_{\alpha_b})\times\mathcal{G}$, for the l.h.s. of formula (32) restricted to a local neighborhood of the point p_a, we get the expression

$$\sum_{\alpha_b}\int\limits_{\varphi^{\mathcal{M}}_{\alpha_b}(\mathcal{U}^{\mathcal{M}}_{\alpha_b})\times\mathcal{G}} \tilde{\tilde{\mu}}_{\alpha_b}(x_b) G_{\mathcal{P}}(\alpha_b, f_b(x_b,\theta_b), t_b; \beta_a, f_a(x_a,\theta_a), t_a)\tilde{\phi}_0(x_b,\theta_b)dv(x_b)d\mu(\theta_b). \tag{34}$$

But, the r.h.s. of formula (32) can also be specified for the locally finite covering of the manifold $\mathcal{M}$:

$$\sum_{\alpha_b}\int\limits_{\varphi^{\mathcal{M}}_{\alpha_b}(\mathcal{U}^{\mathcal{M}}_{\alpha_b})} \tilde{\rho}_{\alpha_b}(x_b)\sum_{\lambda,p,q,q'} G^{\lambda}_{q'p}(\alpha_b, x_b, t_b; \beta_a, x_a, t_a) c^{\lambda}_{pq}(x_b) D^{\lambda}_{q'q}(\theta_a)dv(x_b). \tag{35}$$

Comparing (35) with (34), where the function $\tilde{\phi}_0$ has been expanded in series over the matrix elements of the irreducible representation of a group $\mathcal{G}$, we find that

$$\int_{\mathcal{G}} G_{\mathcal{P}}(\alpha_b, f_b(x_b,\theta_b), t_b; \beta_a, f_a(x_a,\theta_a), t_a) D^{\lambda}_{pq}(\theta_b)d\mu(\theta_b) =$$
$$\sum_{q'} G^{\lambda}_{q'p}(\alpha_b, x_b, t_b; \beta_a, x_a, t_a) D^{\lambda}_{q'q}(\theta_a).$$

After multiplying both sides of the above formula on $\bar{D}^{\lambda}_{mn}$ and integrating over the invariant Haar measure $d\mu(\theta_a)$ normalized to unity, we obtain the following relation:

$$G^{\lambda}_{mn}(\alpha_b, x_b, t_b; \beta_a, x_a, t_a) = \int_{\mathcal{G}} G_P(\alpha_b, x_b, \theta, t_b; \beta_a, x_a, e, t_a) D^{\lambda}_{nm}(\theta)d\mu(\theta) \tag{36}$$

with

$$G_P(\alpha_b, x_b, \theta, t_b; \beta_a, x_a, e, t_a) \equiv G_{\mathcal{P}}(\alpha_b, f_b(x_b,\theta_b), t_b; \beta_a, f_a(x_a,\theta_a), t_a).$$

By the letter e in eq.(36), we denote the unity element of the group $\mathcal{G}$.

Performing an inversion of formula (32), we have obtained the relation between the Green functions defined in charts of the locally finite covering of the manifold. Given in the intersection of the charts of this covering, the Green functions G^{λ}_{mn} satisfy the following relation:

$$G^{\lambda}_{qp}(\alpha_b,x_b,t_b;\beta_a,x_a,t_a)=\bar{D}^{\lambda}_{qn}(g_{\beta\mu}(x_a))G^{\lambda}_{ns}(\gamma_b,x_b,t_b;\mu_a,x_a,t_a)\bar{D}^{\lambda}_{sp}(g_{\gamma\alpha}(x_b)),$$

where the $\mathcal{G}$-valued functions $g_{\beta\mu}(x)$ on $\mathcal{M}$ are the transition functions of the principal fibre bundle.

On the overlapping of the charts labelled by the indices α and β, the corresponding functions ψ are connected by the equality:

$$(\psi_{\beta})_p(x_a)=\bar{D}^{\lambda}_{pn}(g_{\beta\alpha}(x_a))(\psi_{\alpha})_n(x_a). \tag{37}$$

From this it follows that ψ_n belongs to the space of the sections $\Gamma(\mathcal{M},V^*)$ of the associated bundle $\mathcal{E}^*$.

The consistency condition of the local Green functions given on the overlapping of the charts enables us to extend the local equality (36) to the global one. We write it as follows:

$$G^{\lambda}_{mn}(\pi(p_b),t_b;\pi(p_a),t_a)=\int_{\mathcal{G}}G_{\mathcal{P}}(p_b\theta,t_b;p_a,t_a)D^{\lambda}_{nm}(\theta)d\mu(\theta). \tag{38}$$

Notice that this formula determines the connection between the path integral for $G_{\mathcal{P}}$ and for G^{λ}_{mn}. The path integral for the Green function $G_{\mathcal{P}}$ is almost analogous in the form to the path integral of eq.(3) but with one exception concerning the domain of the integration. Now, it should be carried out over the paths with fixed values at the time $t=t_a$ and $t=t_b$. As for the path integral for the Green function G^{λ}_{mn}, it can be derived from the path integral determined by eq.(32) and we write it symbolically as

$$\begin{aligned}
&G^{\lambda}_{mn}(\pi(p_b),t_b;\pi(p_a),t_a)=\\
&\tilde{\mathrm{E}}_{\substack{\xi(t_a)=\pi(p_a)\\ \xi(t_b)=\pi(p_b)}}\Big[(\overleftarrow{\exp})^{\lambda}_{mn}(\xi(t),t_b,t_a)\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(\xi(u))du\}\Big]\\
&=\int\limits_{\substack{\xi(t_a)=\pi(p_a)\\ \xi(t_b)=\pi(p_b)}} d\mu^{\xi}\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}\tilde{V}(x(u))du\}\\
&\times\overleftarrow{\exp}\int_{t_a}^{t_b}\Big\{\Big[\frac{1}{2}\bar{\gamma}^{\mu\nu}(J_{\mu})^{\lambda}_{pr}(J_{\nu})^{\lambda}_{rs}-\frac{1}{2}\frac{1}{\sqrt{h\bar{\gamma}}}\frac{\partial}{\partial x^k}\left(\sqrt{h\bar{\gamma}}h^{km}A^{\mu}_m\right)(J_{\mu})^{\lambda}_{ps}\Big]du\\
&-A^{\mu}_k(J_{\mu})^{\lambda}_{ps}X^k_{\bar{m}}dw^{\bar{m}}\Big\}.
\end{aligned} \tag{39}$$

It is well known that there exists an isomorphism between the space of the sections of the associated vector bundle and the space of the covariant functions on the total space of the principal fibre bundle. From this it follows that the semigroup with the kernel given by eq.(39) acts in the space of the functions $\tilde{\psi}_n(p)$ for which the following relation holds:

$$\tilde{\psi}_n(pg)=D^{\lambda}_{mn}(g)\tilde{\psi}_m(p).$$

The connection between these functions and the functions of $\Gamma(\mathcal{M},V^*)$ for which equality (37) is valid, can be established with the Bogolubov transformation by the following local formula:

$$\tilde{\psi}_n(F(Q^*(x),e))=\psi_n(x).$$

Thus, as a result of the performed transformation, our initial problem has been reduced to the corresponding problem on the orbit space $\mathcal{M}=\mathcal{P}/\mathcal{G}$. To put it differently in terms of the constrained system point of view, we have made the quantum reduction of the dynamical system onto the momentum level λ, determined by the representation D^{λ}.

A considerable simplification of the obtained formulas can be gained in the case of the reduction onto the zero momentum level. Then, in this case $D^0_{pq}\equiv 1$, the semigroup with kernel given by eq.(39) acts in the space of the scalar equivariant functions on the total space of the principal fibre bundle. Now, the measure of the path integral of eq.(39) is determined by the stochastic process $\xi(t)$ and the multiplicative stochastic integral is equal to the unity. The differential generator of the process $\xi(t)$ is

$$\frac{1}{2}\mu^2\kappa\Bigl\{\triangle_{\mathcal{M}}+h^{ni}\frac{1}{\sqrt{\bar{\gamma}}}\frac{\partial\sqrt{\bar{\gamma}}}{\partial x^n}\frac{\partial}{\partial x^i}\Bigr\}.$$

By using the Girsanov transformation, we can change the measure in the path integral which is obtained from the path integral of eq.(39) in the case of $\lambda=0$. As a result of such a transformation, when the process ξ is replaced for $\tilde{\xi}$ which is locally described by the following stochastic differential equation:

$$d\tilde{x}^i(t)=\frac{1}{2}\mu^2\kappa\Bigl[\frac{1}{\sqrt{h}}\frac{\partial}{\partial x^n}(h^{ni}\sqrt{h})\Bigr]dt+\mu\sqrt{\kappa}X^i_{\bar{n}}(\tilde{x}(t))dw^{\bar{n}}(t),$$

we get the path integral for a semigroup that has the differential generator consisting of $\triangle_{\mathcal{M}}$ and an additional potential term – the Jacobian of the performed transformation.

Notice that the Jacobian depends on the orbit volume $\bar{\gamma}(x)$.

An obtained semigroup, determined by its kernel $G_{\mathcal{M}}$ from the equality

$$\bar{\gamma}(x_b)^{-1/4}\bar{\gamma}(x_a)^{-1/4}G_{\mathcal{M}}(x_b,t_b;x_a,t_a)=\int_{\mathcal{G}}G_{\mathcal{P}}(p_b\theta,t_b;p_a,t_a)d\mu(\theta),$$

$$G_{\mathcal{M}}(x_b,t_b;x_a,t_a)=\int d\mu^{\tilde{\xi}}\exp\{\frac{1}{\mu^2\kappa m}\int_{t_a}^{t_b}V(\tilde{\xi}(u))du+\int_{t_a}^{t_b}J(\tilde{\xi}(u))du\} \qquad (40)$$

$(x=\pi(p))$,
acts in the space of the scalar functions on $\mathcal{M}$ (or in the space of the scalar invariant functions on $\mathcal{P}$)) with the following scalar product: $(\psi_1,\psi_2)=\int\psi_1(x)\psi_2(x)dv_{\mathcal{M}}(x)$. In eq.(40), the Jacobian J is given by

$$J(x)=-\frac{\mu^2\kappa}{8}\Bigl[\triangle_M\ln\bar{\gamma}+\frac{1}{4}h^{ni}\frac{\partial\ln\bar{\gamma}}{\partial x^n}\frac{\partial\ln\bar{\gamma}}{\partial x^i}\Bigr].$$

As it has been shown in [16], this Jacobian can be presented by the expression depending on the mean curvature normal to the orbit at the point x.

Notice that the differential operator

$$\hat{H}_\kappa = \frac{\hbar\kappa}{2m}\triangle_M - \frac{\hbar\kappa}{8m}[\triangle_{\mathcal{M}}\ln\bar{\gamma} + \frac{1}{4}(\nabla_{\mathcal{M}}\ln\bar{\gamma})^2] + \frac{1}{\hbar\kappa}V$$

is the operator of the forward Kolmogorov equation for the Green function $G_{\mathcal{M}}$. When changing the forward Kolmogorov equation for the Schrödinger equation, the operator $\hat{H}_\kappa$ becomes the Hamilton operator $\hat{H} = -\frac{\hbar}{\kappa}\hat{H}_\kappa\Big|_{\kappa=i}$ of the corresponding Schrödinger equation.

Conclusion

The main result of the paper is formula (38) and formula (39) that represents the reduced Green function G^{λ}_{mn}. Our formulas generalize the well-known formula of [17] for the Green functions on the principal fibre bundle.

In obtaining our result, we made use of such an approach to the path integral, in which the path integral measure was determined by the stochastic process. This stochastic process was constructed by using the local stochastic differential equation defined on charts of the manifold. It has an advantage in performing the path integral transformation. However, in this approach the effects coming from the topology of the manifold are not explained and have not been considered in the paper. It is supposed to study this problem in further investigations.

Besides, in the paper a rather strong restrictions on a manifold, on a group, which acts on this manifold, and on differential equation are imposed. All these exclude an important case of the action of the noncompact group on a manifold. The investigation of the last case may be very important in attempts to generalize the obtained result to gauge theories. Perhaps, such a generalization to the noncompact case calls for some changes in the approach considered in the paper.

It is necessary to stress an important role of the nonlinear filtering equation in factorization of the path integral measure. At present, this field of the stochastic process theory is rather advanced. And after comprehension, one can derive from it the things that would be useful in quantization of the constrained systems.

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Appendix

Stochastic process on manifold

In case of regarding the manifold as the set of the charts together with their gluing mappings, it is possible to define the stochastic process on each chart by the solution of the corresponding stochastic differential equation in the Euclidean space. However, the problem arises when one is trying to construct the global process on a manifold. Its origin lies in the transformation law of the stochastic differentials: they transform in accordance with the Itô's formula but not as the tensor objects. If one makes use of the standard way of "gluing" the processes given on the overlapping of the charts, then one faces with violation of the compatibility condition for these stochastic processes.

In [4], to solve this problem the following construction has been proposed. First of all, it has been shown there, that for each manifold $\mathcal{M}$ it is possible to define a special fibre bundle $I(\mathcal{M})$. The fibers of this fibre bundle are formed from vector and operator functions (a, A), that are defined in the corresponding spaces. The "gluing" mapping of this fibre bundle which acts on overlapping of the charts (U_1, φ_1) and (U_2, φ_2) is given as follows

$$I(f_{\varphi_1\varphi_2})(a^{\varphi_1}, A^{\varphi_1}) \to \Big(f^{'}_{\varphi_1\varphi_2}a^{\varphi_1} + \frac{1}{2}Trf^{''}_{\varphi_1\varphi_2}(A^{\varphi_1}, A^{\varphi_1}), f^{'}_{\varphi_1\varphi_2}A^{\varphi_1}\Big),$$

$(f_{\varphi_1\varphi_2} = \varphi_2 \cdot {\varphi_1}^{-1})$.

Locally, that is on a chart (U, φ), each pair (a, A) determines the germ $\mathcal{R}_x(a, A)$ of the diffusion processes that don't leave the chart to which this pair belongs. The diffusion processes of the germ $\mathcal{R}_x(a, A)$ are stochastically equivalent to the processes obtained as a result of the solution of the stochastic differential equations

$$d\xi(s) = a(s, \xi(s))ds + A(s, \xi(s))dw(s),$$

which coefficients $a(s, x)$ and $A(s, x)$ have the supports in U and satisfy the equalities:

$$a(0, x) = a, \quad A(0, x) = a.$$

It is important, that under the local mappings of the manifold, the germ $\mathcal{R}_x(a, A)$ transforms into the germ $\mathcal{R}_{f(x)}(I(f)(a, A))$. It allows one to associate the fiber of the

Itô fibre bundle $I(\mathcal{M})$ with the space of the germs of the diffusion processes. The more important fact is that it is possible to extend the functorial character of the correspondence $U \to I_x(U)$ to $I_x(U) \to \mathcal{R}_x(I_x(U))$. Therefore, if we have the Itô fibre bundle $I(\mathcal{M})$, then we are able to construct the germs of the diffusion processes on a manifold $\mathcal{M}$.

Now we must introduce a new notion called the Itô field. It comes from the particular case of the Itô bundle — the vector Itô bundle. The vector Itô bundle is a sum of tangent bundle and an operator bundle. The fibers of this vector bundle are again given by the pairs (a_x, A_x), where $a_x \in T_x\mathcal{M}$, and A_x is a linear map from R^n into $T_x\mathcal{M}$. But since now we have a vector bundle, the law of the "gluing" transformation becomes as follows

$$(a_x, A_x) \to (f^{'}(x)a_x, f^{'}(x)A_x).$$

By definition, the Itô field is a section of the introduced vector Itô bundle.

However, by constructing with the Itô field $I(\mathcal{M})$ the germs of the stochastic processes on a manifold $\mathcal{M}$, we find that they don't have the necessary law of the transformation on overlappings of the charts. So, we must change our construction. And the solution of the problem concerning the transformation law is to take the Itô field $I_{(x,0)}(T_x\mathcal{M})$ (for the tangent bundle $(T_x\mathcal{M})$) rather than the Itô field $I(\mathcal{M})$.

By the general approach, the Itô field $I_{(x,0)}(T_x\mathcal{M})$ leads us to the germs $\mathcal{R}_{(x,0)}(a, A)$ of the stochastic processes in the neighborhood of the zero ($y = 0$ in (x, y)) of the tangent bundle $T_x\mathcal{M}$. Then, by supposing the existence of the linear connection on a manifold, the germs $\mathcal{R}_{(x,0)}(a, A)$ are carried onto the manifold with the help of the exponential mapping. These diffusion processes germs

$$\widetilde{exp}_x\mathcal{R}_{(x,0)}(a, A) \equiv \mathcal{R}_x\Big(I(exp_x)(a, A)\Big) \ \Big(\equiv exp_x(a_x dt + A_x dw)\Big),$$

which in coordinates is presented as follows

$$\varphi \cdot \widetilde{exp}_x\mathcal{R}_{(x,0)}(a, A) = \mathcal{R}^{\varphi}_x(a^{\varphi} - \frac{1}{2}Tr\Gamma^{\varphi}_{\varphi(x)}\Big(A^{\varphi}_x, A^{\varphi}_x), A^{\varphi}_x)\Big),$$

are called the stochastic differentials.

Because of the functorial character of the previous construction, these stochastic differentials transform as necessary under the "gluing" mapping of the charts. That is, the Itô field transformation

$$(a_x, A_x) \to (f'(x)a_x, f'(x)A_x),$$

leads to the following transformation of the germs of the diffusion processes on $\mathcal{M}$:

$$\tilde{f} \cdot \widetilde{exp}_x\mathcal{R}_{(x,0)}(a, A) \to \widetilde{exp_{f(x)}}\mathcal{R}_{(f(x),0)}\Big(f^{'}(x)a_x, f^{'}(x)A_x\Big).$$

Now, if we let the Itô field be dependent on time as well, we will be able to construct the field of the compatable germs. In this case we may put a question concerning the existence of such a process $\xi(t)$ on a manifold $\mathcal{M}$ for which at every moment of time and at every point x it would be possible to find a neighborhood of x where the process coincides (almost surely) with one of the processes belonging to the germ $\widetilde{exp}_x\mathcal{R}(a_x(t), A_x(t))$. By definition, the process in question is the integral process of the Itô field $(a_x(t), A_x(t))$.

The problem of finding the integral process $\xi(t)$ can be symbolically written as the following (symbolical) equation:

$$d\xi(t) = exp_{\xi(t)}(a_{\xi(t)}(t)dt + A_{\xi(t)}(t)dw(t)). \tag{41}$$

An integral process of the Itô field (the solution of eq.(41)) is determined in two steps. In the first step, one solves the equation defined by the local Itô field, by which is meant that the Itô fields is supported to some chart of the manifold.

Let eq.(41) be given by the local Itô field. Then, this equation determines the germ of the diffusion processes that don't leave the chart where the support of the local Itô field is. If the considered chart of the point x is $(\mathcal{V}, \varphi)$, then the stochastic differential equation for $\xi^\varphi = \varphi(\xi)$ in the neighborhood $V^\varphi = \varphi(\mathcal{V})$ of the Euclidean space will be the following equation:

$$d\xi^\varphi(t) = \{a^\varphi(\xi^\varphi(t)) - \frac{1}{2}\Gamma^\varphi_{\xi^\varphi(t)}(A^\varphi(\xi^\varphi(t))\,\cdot, A^\varphi(\xi^\varphi(t))\,\cdot)\}dt + A^\varphi(\xi^\varphi(t))dw(t), \tag{42}$$

with the initial data: $\xi^\varphi(s) = \varphi(x)$.

Being the Stratonovich-like equation, eq.(42) is invariant under the changing of the charts of the manifold. The solution of this equation defines the stochastic evolution family $T_x(t,s)$ $(x \in \mathcal{V})$ of mappings of the manifold. In the exterior of $\mathcal{V}$, this family consists of the identity mappings and in the interior of $\mathcal{V}$, it is given by the solution of eq.(42): $\varphi(T_x(t,s)\cdot x) \equiv \varphi(T(t,s;x))$.

In the second step, by making use of the local stochastic evolution families of mappings of the manifold $\mathcal{M}$, the global solution of the stochastic differential equation (41) is built. It is made as follows: Taking the partition $q = (s = t_0 \leq t_1 \leq \ldots \leq t_n = t)$ of the time interval $[s,t]$ and forming the consequent superposition of local stochastic mappings $T_{\xi(t_{i-1})}(t_{i-1}, t_i; \xi(t_{i-1}))$, the stochastic evolution family T^q of mappings of the manifold $\mathcal{M}$ is defined as

$$T^q(t,s;x) = T_{\xi_{n-1}(t_{n-1})}(t, t_{n-1}; \xi_{n-1}(t_{n-1})),$$
$$\xi_k(\tau) = T_{\xi_{k-1}(t_{k-1})}(\tau, t_{k-1}; \xi_{k-1}(t_{k-1})), \;\; \xi_0 = x, \;\; \tau \in [t_{k-1}, t_k].$$

In [4] it was shown that under the refinement of the partition q, the family T^q of the mappings has a limit. It is this limit mapping that is a solution of the stochastic differential equation (41). The existence of the limit was proven with rather general assumptions concerning the structure of the manifold. Namely, it was assumed that there was a uniform atlas[2]on a manifold $\mathcal{M}$.

Obtained as a result of the limiting procedure, the evolution family T has the following evolution property:

$$T(t,s;x) = T(t,\tau;T(\tau,s;x)).$$

[2]In each chart (V_x, φ_x) of the point x there must be two embeded neighborhoods $V_x^2 \subset V_x^1 \subset V_x$, such that if $y \in V_x^2$ then $V_x^2 \subset V_y^1$. In the neighborhood $\varphi_x(V_x^2)$ it must be a ball of the fixed radius. Also, the linear connection coefficients and the Itô field coefficients must be uniformly bounded. These conditions are satisfied in the case of the compact manifold.

It means that the stochastic process $\xi_{s,x}(t) = T(t,s;x)$ is a Markov stochastic process. Then, from the general theory it follows, that there is a family of the operators

$$U(\tau,t)f(y) = \mathrm{E}f(\xi_{\tau,y}(t)) = \mathrm{E}f(T(t,\tau;y)), \tag{43}$$

defined in the space $\mathcal{B}(\mathcal{M})$ of continuous and bounded functions on $\mathcal{M}$. This family has an evolution property

$$U(\tau,t) = U(\tau,s)U(s,t), \quad \tau \leq s \leq t.$$

The expectation value of formula (43) denotes the integration with respect to the measure defined in the space of paths on $\mathcal{M}$ constructed by the Kolmogorov theorem, that is

$$\mathrm{E}f(\xi_{\tau,y}(t)) = \int_{M^t_{\tau,y}} f(x(t))d\mu^{\xi_{\tau,y}}(x(\cdot)),$$

where $M^t_{\tau,y} = \{x(s), s \in [\tau,t], x(\tau) = y\}$.

By its construction, the family of the operators $U(t,s)$ has the following superposition properties:

$$U(\tau,t) = \mathrm{lim}_q \tilde{U}(\tau,t_1)\cdot\tilde{U}(t_1,t_2)\cdot\ldots\cdot\tilde{U}(t_{n-1},t_n). \tag{44}$$

This property enables one to calculate $U(t,s)$ by using such local operators as

$$\tilde{U}(\tau,t)f(x) = U_x(\tau,t) = \mathrm{E}f(T_x(t,\tau;x)). \tag{45}$$

Notice that in [4], the same approach was applied to the construction of the stochastic processes on vector bundles and on principal fibre bundles.

С.Н.Сторчак
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