STATE RESEARCH CENTER OF RUSSIA

INSTITUTE FOR HIGH ENERGY PHYSICS

IHEP 98-81

S. M. Klishevich[1]

# MASSIVE FIELDS OF ARBITRARY INTEGER SPIN IN SYMMETRICAL EINSTEIN SPACE

Submitted to *Class. Quant. Grav.*

[1] E-mail address: klishevich@mx.ihep.su

Protvino 1998

UDK 539.1.01 M–24


## Abstract

Klishevich S.M. Massive Fields of Arbitrary Integer Spin in Symmetrical Einstein Space: IHEP Preprint 98-81. – Protvino, 1998. – p. 15, refs.: 23.

We study the propagation of gauge fields with arbitrary integer spins in the symmetrical Einstein space of any dimensionality. We reduce the problem of obtaining a gauge-invariant Lagrangian of integer spin fields in such background to algebraic problem of finding a set of operators with certain features using the representation of higher-spin fields in the form of vectors of pseudo-Hilbert space. We consider such construction at linear order in the Riemann tensor and scalar curvature and also present an explicit form of interaction Lagrangians and gauge transformations for massive particles of spins 1 and 2 in terms of symmetrical tensor fields.



## Аннотация

Клишевич С.М. Массивные поля произвольного целого спина в симметрическом пространстве Эйнштейна: Препринт ИФВЭ 98-81. – Протвино, 1998. – 15 с., библиогр.: 23.

Мы изучаем распространение массивных калибровочных полей произвольных целых спинов в симметрическом пространстве Эйнштейна произвольной размерности. Основываясь на представлении полей высоких спинов в виде векторов некоторого псевдогильбертового пространства, мы сводим проблему получения калибровочно-инвариантного лагранжиана полей целых спинов в таком фоновом пространстве к чисто алгебраической задаче отыскания некоторого набора операторов с определенными свойствами. Мы рассматриваем такое построение в линейном порядке по тензору Римана и скалярной кривизне и также приводим явный вид лагранжианов взаимодействия и калибровочные преобразования для массивных частиц со спинами 1 и 2 в терминах симметричных тензорных полей.

## Introduction

Problems of obtaining a consistent description of the gravitational interactions of higher-spin fields have a particular significance since it allows one to connect the fields of higher spins with the observable world.

It is well known that the gravitational interaction of massless fields with spins $s \geq 2$ does not exist in an asymptotically flat space-time [1]. For the covariant description of physical fields one must replace the ordinary derivatives with the covariant ones in the Lagrangian and gauge transformations. Since the covariant derivatives do not commute the gauge invariance fails and a residual appears. For the fields with spins $s \geq 2$ the residual is proportional to the Riemann tensor. In general, one cannot cancel such a residual in an asymptotically flat space-time by any changes of the Lagrangian and transformations in the linear approximation. Therefore, in such case this approximation does not exist. Since linear approximation does not depend on the presence of any other fields in the system, this means that the whole theory of interaction does not exist either.

This problem can be overcome in several ways. For instance, one can consider the massless fields in a constant curvature space. Then, the Lagrangian for gravitational would have the additional term $\delta\mathcal{L} \sim \sqrt{-g}\lambda$, where $\lambda$ is the cosmological constant. A modification of the Lagrangian and the transformations leads to a mixing of terms with different numbers of the derivatives. This allows one to compensate the residual with terms proportional to $R_{\mu\nu\alpha\beta}$. The complete theory will be represented as series in inverse value of the cosmological constant [2,3]. This means non-analyticity of the theory in $\lambda$ at zero, i.e. the impossibility of a smooth transition to the flat space. Such a theory was considered in Refs. [2,3,4].

Besides, massive higher-spin fields can have the gravitational interaction. For example, the string theory presents a consistent gravitational interaction of massive higher-spin fields. In Ref. [5] the interaction of fields at the lowest order was derived while investigating three point functions ot the type II superstring which has one graviton and two massive states.

In the literature the gravitational interactions of arbitrary spin fields were considered at the lowest order in the Riemann tensor [6,7,5]. When considering the interactions, the authors started from the free theory of massive fields in the conventional form [8]. The "minimal" introduction of interaction leads to contradictions, therefore, it is necessary to consider non-minimal terms in the interaction Lagrangian. Since, the massive Lagrangian for the spin-$s$ fields [8] is not gauge invariant, in this approach there are no restrictions on the form of non-minimal interaction. But, in a general case, such a theory is pathological, therefore, to build a consistent theory of interaction it is necessary to introduce an additional restrictions on non-minimal terms. So, for instance, when investigating the gravitational interactions [6,7], the authors required for the theory to have the tree-level unitarity up to the Planck scale.

In our opinion it is more convenient to use the gauge-invariant approach when one analyzes an interaction of the massive fields [9,10,11], [12] or [13]. Under such an approach the interaction is considered as a deformation of initial gauge algebra and Lagrangian[1][14, 11]. Although, generally speaking, the gauge invariance does not ensure the consistency of massive theories, but, anyway, it allows one to narrow the searches and conserves the appropriate number of physical degree of freedom. Besides, such an approach is quite convenient and practical.

Here we go along the line of Refs. [15,16] where the electromagnetic interaction of massive fields of integer and half-integer spins was investigated. We represent a free state with the arbitrary integer spin $s$ as state $|\Phi^s\rangle$ of Pseudo-Hilbert space[2][15]. The tensor fields corresponding to the particle with spin $s$ are coefficient functions of the state $|\Phi^s\rangle$. In the considered Fock space we introduce a set of operators by means of which we define the gauge transformations and necessary constraints for the state $|\Phi^s\rangle$. The gauge-invariant Lagrangian has the form of the expectation value of the Hermitian operator, which consists of the operators, in the state $|\Phi^s\rangle$.

In the considered approach the gauge invariance is a consequence of commutation relations of the introduced operators. For the covariant description of fields in the Riemann background, one must replace the ordinary derivatives with the covariant ones. This leads to a change of algebraic features of the operators and, as a consequence, to the loss of gauge invariance for higher-spin fields. We reduce the problem of recovering the invariance to algebraic problem of finding such modified operators which depend on the Riemann tensor and scalar curvature and satisfy the same commutation relations as initial operators in the flat space. In this, we should note that in the massless case one cannot realize such a construction in an asymptotically flat space. In section 3, for the massive theory we construct the set of operators having the algebraic features of free ones at linear order in the Riemann tensor and scalar curvature. Besides, in the next section we give an explicit form of interaction Lagrangian and transformations for the massive vector and spin-2 fields.

[1] Of course, one must consider only a non-trivial deformation of the free algebra and Lagrangian which cannot be completely gauged away or removed by a redefinition of the fields.

[2] The representation of the free fields of arbitrary integer spin in such a form was considered in Refs. [12,17]

## 1. Free Field with Spin $s$

**Massless fields.** Let us consider the Fock space generated by the creation and annihilation operators $\bar{a}_\mu$ and $a_\mu$ which are vectors on the $D$-dimensional Minkowski space $\mathcal{M}_D$ and which satisfy the following algebra

$$[a_\mu, \bar{a}_\nu] \;=\; g_{\mu\nu}, \quad a_\mu^\dagger = \bar{a}_\mu, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor with signature $\|g_{\mu\nu}\| = \mathrm{diag}(-1, 1, 1, ..., 1)$. Since the metric is indefinite, the Fock space that realizes the representation of the Heisenberg algebra (1) is Pseudo-Hilbert.

Let us consider the state in the introduced space:

$$|\Phi^s\rangle = \frac{1}{\sqrt{s}}\Phi_{\mu_1\ldots\mu_s}(x)\prod_{i=1}^{s}\bar{a}_{\mu_i}|0\rangle. \tag{2}$$

Coefficient function $\Phi_{\mu_1\ldots\mu_s}(x)$ is a symmetrical tensor of rank $s$ in space $\mathcal{M}_D$. For this tensor field to describe the state with spin[3]$s$ one has to imposes the condition:

$$\Phi_{\mu\mu\nu\nu\mu_4\ldots\mu_s} = 0. \tag{3}$$

In terms of such fields Lagrangian [18,19] has the form

$$\begin{aligned} \mathcal{L}_s \;=\; & \frac{1}{2}(\partial_\mu\Phi^s)\cdot(\partial_\mu\Phi^s) - \frac{s}{2}(\partial\cdot\Phi^s)\cdot(\partial\cdot\Phi^s) - \frac{s(s-1)}{4}(\partial_\mu\Phi'^s)\cdot(\partial_\mu\Phi'^s) \\ & - \frac{s(s-1)}{2}(\partial\cdot\partial\cdot\Phi^s)\cdot\Phi'^s - \frac{1}{8}s(s-1)(s-2)(\partial\cdot\Phi'^s)\cdot(\partial\cdot\Phi'^s) \end{aligned} \tag{4}$$

The following notation $\Phi' = \Phi_{\mu\mu\ldots}$ is used here while the point means the contraction of all indexes$\Phi^s\cdot\Phi^s \overset{def}{=} \Phi_{\mu_1\ldots\mu_s}\Phi^{\mu_1\ldots\mu_s}$.

This Lagrangian is invariant under the transformations

$$\delta\Phi_{\mu_1\ldots\mu_s} \;=\; \partial_{(\mu_1}\Lambda_{\mu_2\ldots\mu_{s-1})}, \tag{5}$$

$$\Lambda_{\mu\mu\mu_3\ldots\mu_{s-1}} \;=\; 0. \tag{6}$$

Let us introduce the following operators in our pseudo-Hilbert space

$$L_1 = p\cdot a, \quad L_{-1} = L_1^\dagger, \quad L_2 = \frac{1}{2}a\cdot a, \quad L_{-2} = L_2^\dagger, \quad L_0 = p^2. \tag{7}$$

Here $p_\mu = i\partial_\mu$ is the momentum operator that acts in the space of the coefficient functions.

Operators of such type appear as constraints of a two-particle system under quantization[4][20]. Operators (7) satisfy the commutation relations:

$$\begin{array}{rclrcl} [L_1, L_{-2}] & = & L_{-1}, & [L_1, L_2] & = & 0, \\ [L_2, L_{-2}] & = & N + \frac{D}{2}, & [L_0, L_n] & = & 0, \\ [L_1, L_{-1}] & = & L_0, & [N, L_n] & = & -\,nL_n, \quad n = 0, \pm1, \pm2. \end{array} \tag{8}$$

[3]We consider symmetric tensor fields only.

[4]It is also possible to regard operators (7) as a truncation of the Virasoro algebra.

Here $N = \bar{a} \cdot a$ is a level operator that defines the spin of states. So, for instance, for state (2)

$$N|\Phi^s\rangle = s|\Phi^s\rangle.$$

In terms of operators (8) condition (3) can be written as

$$(L_2)^2|\Phi^s\rangle = 0, \tag{9}$$

while gauge transformations (5) take the form

$$\delta|\Phi^s\rangle = L_{-1}|\Lambda^{s-1}\rangle. \tag{10}$$

Here, the gauge state

$$|\Lambda^{s-1}\rangle = \Lambda_{\mu_1 \dots \mu_{s-1}} \prod_{i=1}^{s-1} \bar{a}_{\mu_i}|0\rangle$$

satisfies the condition

$$L_2|\Lambda\rangle = 0. \tag{11}$$

This condition is equivalent to (6) for the coefficient functions.

Lagrangian (4) can be written as the expectation value of a Hermitian operator in state (2)

$$\mathcal{L}_s = \langle\Phi^s|\mathcal{L}(L)|\Phi^s\rangle, \quad \langle\Phi^s| = |\Phi^s\rangle^\dagger, \tag{12}$$

where

$$\begin{aligned} \mathcal{L}(L) &= L_0 - L_{-1}L_1 - 2L_{-2}L_0L_2 - L_{-2}L_{-1}L_1L_2 \\ &\quad + \{L_{-2}L_1L_1 + h.c.\}\,. \end{aligned} \tag{13}$$

Lagrangian (12) is invariant under transformations (10) as a consequence of the relation

$$\mathcal{L}(L)L_{-1} \sim (...)L_2.$$

**Massive fields** Let us consider the massive states of arbitrary spin $s$ in the similar manner. For that we have to extend our Fock space by introducing scalar creation and annihilation operators $\bar{b}$ and $b$, which satisfy the usual commutation relations

$$\left[b, \bar{b}\right] = 1, \quad b^\dagger = \bar{b}. \tag{14}$$

Operators (7) are modified as follows:

$$L_1 = p \cdot a + mb, \quad L_2 = \frac{1}{2}\left(a \cdot a + b^2\right), \quad L_0 = p^2 + m^2. \tag{15}$$

Here $m$ is an arbitrary parameter having the dimensionality of mass. In non-interacting case one can consider such transition as the dimensional reduction $\mathcal{M}_{D+1} \to \mathcal{M}_D \otimes S^1$ with the radius of sphere $R \sim 1/m$ (refer also to [12,17]).

We shall describe the massive field with spin $s$ as the following vector in the extended Fock space:

$$|\Phi^s\rangle = \sum_{n=0}^{s} \Phi_{\mu_1\ldots\mu_n}(x)\bar{b}^{s-n}\prod_{i=1}^{n}\bar{a}_{\mu_i}|0\rangle. \tag{16}$$

Like the massless field case, this state satisfies the same condition (9), but in terms of operators (15). The algebra of operators (8) changes insignificantly, the only commutator modified is

$$[L_2, L_{-2}] = N + \frac{D+1}{2}. \tag{17}$$

Here, as in the massless case, the operator $N = \bar{a}\cdot a + \bar{b}b$ defines the spin of massive states. The Lagrangian describing the massive field of spin $s$ has the form of (13) as well, where the expectation value is taken in state (16). Such Lagrangian is invariant under transformations (10) with the gauge Fock vector

$$|\Lambda^{s-1}\rangle = \sum_{n=0}^{s-1} \Lambda_{\mu_1\ldots\mu_n}\bar{b}^{s-n-1}\prod_{i=1}^{n}\bar{a}_{\mu_i}|0\rangle,$$

which satisfies condition (6).

## 2. Propagation of Massive higher-spin Field in Symmertical Einstein Space

In this section we consider an arbitrary $D$-dimensional symmetrical Einstein space, i.e. the Riemann space defined by the following equations:

$$\mathcal{D}^{(\Gamma)}_{\mu} R_{\nu\alpha\beta\gamma} = 0, \tag{18}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = g_{\mu\nu}\lambda, \tag{19}$$

where $\mathcal{D}^{\Gamma}_{\mu}$ is the covariant derivative with the Cristofel connection ${\Gamma^{\alpha}}_{\nu\mu}$. We assume that the Greek indexes are global while the Latin ones are local. As usual, the derivative $\mathcal{D}^{\Gamma}_{\mu}$ acts on tensor fields with global indexes only.

To describe the massive higher-spin fields in the Riemann background, we must replace the ordinary derivatives with the covariant one, i.e. we make the substitution

$$p_\mu \to \mathcal{P}_\mu = i\left(\mathcal{D}^{\Gamma}_{\mu} + {\omega_\mu}^{ab}\bar{a}_a a_b\right), \tag{20}$$

where $\omega^{ab}_{\mu}$ is the Lorentz connection. We imply that the creation and annihilation operators primordially carry the local indexes. We also have to introduce the non-degenerate vielbein $e^{a}_{\mu}$ for the transition from the local indexes to the global ones and vice versa. As usual, we impose the conventional requirement on the vielbein

$$\mathcal{D}^{(\Gamma+\omega)}_{\mu} e^{a}_{\nu} = \partial_\mu e^{a}_{\nu} - {\Gamma^{\lambda}}_{\nu\mu} e^{a}_{\lambda} + {\omega_\mu}^{a}{}_{b} e^{b}_{\nu} = 0.$$

By means of this relation one can transfer from expressions with one connection to those with other. Besides, we should note that due to this relation the operator $\mathcal{P}_\mu$ commutes with the vector creation-annihilation operators with global indexes $\bar{a}_\nu = e_\nu^b\,\bar{a}_b$ and $a_\nu = e_\nu^b\,a_b$. This allows us not to care about the ordering of operators (15).

One can verify that the covariant momentum operator defined in this way properly acts on the states of type (16), indeed

$$\mathcal{P}_\mu|\Phi\rangle = i\mathcal{D}_\mu^{(\omega)}\Phi^{b_1\ldots b_n}\prod_{i=1}^{n}\bar{a}_{b_i}|0\rangle = i\mathcal{D}_\mu^{(\Gamma)}\Phi^{\nu_1\ldots\nu_n}\prod_{i=1}^{n}\bar{a}_{\nu_i}|0\rangle.$$

The commutator of covariant momenta defines the Riemann tensor:

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = R_{\mu\nu}{}^{ab}\,(\omega)\,\bar{a}_a a_b. \tag{21}$$

where $R_{\mu\nu}{}^{ab}\,(\omega) = \partial_\mu\omega_\nu{}^{ab} + \omega_\mu{}^a{}_c\,\omega_\nu{}^{cb} - (\mu \leftrightarrow \nu)$.

In the definition of operators (15), we replace the ordinary momenta with the covariant ones as well. As a result, the operators cease to obey algebra (8). Therefore, Lagrangian (13) loses the invariance under gauge transformations (10).

To recover the gauge invariance, we do not need to restore total algebra (8), it is enough to ensures the existence of the following commutation relations:

$$[L_1, L_{-1}] = L_0, \tag{22}$$

$$[L_2, L_{-1}] = L_1. \tag{23}$$

To restore these relations, let us represent operators (15) as normal ordered functions of the creation and annihilation operators as well as of $R_{\mu\nu\alpha\beta}$ and $R$, i.e.

$$L_i = L_i\left(\bar{a}_\mu, \bar{b}, a_\mu, b, R_{\mu\nu\alpha\beta}, R\right).$$

The particular form of operators $L_i$ will be defined from the condition recovering of commutation relations (22) and (23) by these operators. We should note that it is enough to define the form of operators $L_1$ and $L_2$, since the operators $L_0$ and $N$ can be expressed in terms of these operators.

Since we have turned to the extended universal enveloping algebra of the Heisenberg algebra, the arbitrariness in the definition of operators $a$ and $b$ appears. Besides, we should admit the presence of arbitrary operator functions depending on $a$, $b$, $R_{\mu\nu}{}^{ab}$, and $R$ in the right-hand side of (1) and (14). In this, such a modification of the operators must not lead to breaking the Jacobi identity and under the transition to flat space they must restore the initial algebra. However, one can make sure that using the arbitrariness in the definition of the creation and annihilation operators, we can restore algebra (1), (14) at linear order in the Riemann tensor and scalar curvature.

We shall search for operators $L_1$ and $L_2$ as series in the Riemann tensor and scalar curvature.

Let us consider linear approximation.

Operator $L_1$ should be no higher than linear in operator $\mathcal{P}_\mu$, since the presence of a greater number of these operator changes the type of gauge transformations and the number of physical degrees of freedom. Therefore, in this approximation we shall search for them in the form

$$
\begin{aligned}
L_1^{(1)} &= R\Big(h_0(\bar{b},b)\,b + h_1(\bar{b},b)\,b\,(\bar{a}\cdot a) + \bar{b}h_2(\bar{b},b)\,a^2 + h_3(\bar{b},b)\,b^3\bar{a}^2 \\
&\quad + h_4(\bar{b},b)\,(\mathcal{P}\cdot a) + h_5(\bar{b},b)\,b^2\,(\bar{a}\cdot\mathcal{P})\Big) + R^{\mu\nu ab}\Big(h_6(\bar{b},b)\,b\bar{a}_\mu\bar{a}_a a_\nu a_b \\
&\quad + h_7(\bar{b},b)\,\bar{a}_\mu a_\nu \mathcal{P}_a a_b + h_8(\bar{b},b)\,b_2\bar{a}_\mu\mathcal{P}_\nu\bar{a}_a a_b\Big). \qquad (24)
\end{aligned}
$$

At the same time the operator $L_2$ cannot depend on the momentum operators at all, since condition (9) defines the purely algebraic constraint on the coefficient functions. Therefore, at this order we choose the operator $L_2$ in the following form:

$$
\begin{aligned}
L_2^{(1)} &= R\left(h_9(\bar{b},b)\,b^2 + h_{10}(\bar{b},b)\,a^2 + h_{11}(\bar{b},b)b^2\,(\bar{a}\cdot a) + h_{12}(\bar{b},b)\,b^4\bar{a}^2\right) \\
&\quad + h_{13}(\bar{b},b)\,b^2\bar{a}^\mu\bar{a}^a a^\nu a^b R_{\mu\nu ab}. \qquad (25)
\end{aligned}
$$

Here $h_i(\bar{b},b)$ are normal ordered operator functions

$$
h_i(\bar{b},b) = \sum_{n=0}^{\infty} H_n^i \bar{b}^n b^n,
$$

where $H_n^i$ are arbitrary real coefficients.

Let us define a particular form of the functions $h_i$ from the condition of recovering commutation relations (22) and (23) by the operators $L_1$ and $L_2$.

We have to note that these operators can obey relations (22) and (23) up to the terms proportional to $L_2^{(0)} = \frac{1}{2}(a^2+b^2)$ at linear order, since this does not break the gauge invariance due to constraint (11).

Having calculated (22) and passing to normal symbols of the creation and annihilation operators, we obtain a system of differential equations for the normal symbols of operator functions $h_i$. For the normal symbols of operator functions we shall use the same notations. This does not lead to the mess since we consider the operator functions as the ones of two variables, while their normal symbols as the functions of one variable. Thereby, we have equations from (22)

$$
h_7''(x) + 2h_7'(x) + 4h_{13}(x) - 2h_8(x) = 0,
$$

$$
x\,(h_6''(x) + 2h_{13}'(x) + 2h_6'(x)) + 2\,(h_6'(x) + 2h_{13}(x)) = 0,
$$

$$
x^2\left(\frac{1}{2}h_8''(x) + h_8'(x)\right) + 2x\,(h_8'(x) + h_8(x)) + h_8(x) - 2h_7(x) = 0,
$$

$$
(h_2''(x) + 2h_{12}'(x)x + 2h_2'(x) + 8h_{12}(x) - 2h_3(x)) = 0,
$$

$$
\begin{aligned}
&h_4''(x) + 2h_4'(x) + 2h_{11}(x) - 2h_5(x) = 0,\\
&x\left(\frac{1}{2}h_1''(x) + h_{11}'(x) + h_1'(x)\right) + h_1'(x) + 2h_{11}(x) + 2h_2(x) = 0,\\
&x^2\left(\frac{1}{2}h_5''(x) + h_5'(x)\right) + 2x\left(h_5'(x) + h_5(x)\right) + 2h_{10}(x) + \frac{1}{D}h_7(x) + h_5(x) = 0,\\
&x^3\left(\frac{1}{2}h_3''(x) + h_3'(x)\right) + 3x^2\left(h_3'(x) + h_3(x)\right) + x\left(-\frac{1}{2}h_0''(x) + h_{10}'(x)\right.\\
&\left.- h_9'(x) - h_0'(x) + \frac{1}{D}h_6(x) + 3h_3(x) - h_2(x) + h_1(x)\right)\\
&- h_0'(x) - 2h_9(x) - Dh_2(x) = 0. \qquad (26)
\end{aligned}
$$

Here the prime denotes the derivative with respect to $x$, while $x = \bar{\beta}\beta$, where $\bar{\beta}$ and $\beta$ are the normal symbols of operators $\bar{b}$ and $b$, correspondingly.

Similarly, from (23) we derive the other system of equations:

$$
\begin{aligned}
&h_8'(x) = 0,\\
&h_8''(x)x + h_7''(x) + 3h_8'(x) + 2h_6'(x) = 0,\\
&h_8'(x)x + 3h_7'(x) + 2h_8(x) + 2h_6(x) = 0,\\
&x^2h_8''(x) + x\left(h_7''(x) + 4h_8'(x) + 6h_6''(x)\right) + h_7'(x) + 2h_8 + 6h_6 - 4 = 0,\\
&h_6''(x)x + 2h_6'(x) = 0,\\
&h_8'(x)x + \frac{1}{2}h_7'(x) + 2h_8(x) + h_6(x) = 0,\\
&x\left(h_5''(x) + 2h_3'(x)\right) + h_4''(x) + 3h_5'(x) + 4h_2'(x) + h_1'(x) + 6h_3(x) = 0,\\
&h_1''(x)x + 2h_1'(x) = 0,\\
&h_5'(x)x + 3h_4'(x) + 2h_5(x) + 2h_2(x) + h_1(x) = 0,\\
&h_5''(x)x^2 + \left(h_4''(x) + 4h_5'(x) + 2h_2'(x) + 3h_1'(x)\right)x + h_4'(x) + 2h_5(x)\\
&+ 2h_2(x) + 3h_1(x) = 0,\\
&h_3''(x)x^2 + \left(h_2''(x) + 6h_3'(x)\right)x + 2h_2'(x) + 6h_3(x) = 0,\\
&h_3''(x)x^3 + \left(h_2''(x) + 4h_3'(x)\right)x^2 - 2h_0''(x)x - 4h_0'(x) = 0. \qquad (27)
\end{aligned}
$$

Having solved the whole system[5] of equations (26) and (27), we obtain the particular

[5] We search for finite at $x \to 0$ solutions only.

form of the operators $L_1$ and $L_2$:

$$
\begin{aligned}
L_1^{(1)} &= \frac{1}{6} R^{\mu\nu\alpha\beta} \bar{\alpha}_\alpha \alpha_\mu \left\{ \mathcal{P}_\nu \alpha_\beta \left(1 + 2\bar{\beta}\beta\right) - 5\bar{\alpha}_\nu \alpha_\beta \beta + 2\bar{\alpha}_\nu \mathcal{P}_\beta \beta^2 \right\} \\
&\quad + R \left\{ c_2 \left(\mathcal{P} \cdot \alpha\right) + c_1 \beta - \frac{1}{2} \alpha^2 \bar{\beta}^2 h_5'(x) \beta - \alpha^2 \bar{\beta} h_5(x) \right. \\
&\quad \left. + \left(\bar{\alpha} \cdot \mathcal{P}\right) h_5(x) \beta^2 + \frac{1}{2} \bar{\alpha}^2 h_5'(x) \beta^3 \right\}, \\
L_2^{(1)} &= R \left\{ \alpha^2 \left( -\frac{1}{4} \bar{\beta}^2 h_5''(x) \beta^2 - \frac{1}{2} \bar{\beta}^2 h_5'(x) \beta^2 - \bar{\beta} h_5'(x) \beta - \bar{\beta} h_5(x) \beta \right. \right. \\
&\quad \left. - \frac{1}{2} h_5(x) + \frac{1}{6D} \bar{\beta}\beta + \frac{1}{12D} \right) + \frac{1}{4} \bar{\alpha}^2 \left( h_5''(x) + 2h_5'(x) \right) \beta^4 \\
&\quad \left. + \left(\bar{\alpha} \cdot \alpha\right) h_5 \beta^2 + \frac{1}{2} h_5(x) \beta^2 D + \frac{1}{3D} \bar{\beta} \beta^3 \right\},
\end{aligned}
\tag{28}
$$

where $c_1$ and $c_2$ are arbitrary real parameters and $h_5(x)$ is an arbitrary function regular at $x \to 0$, while $\bar{\alpha}_\mu$ and $\alpha_\mu$ are normal symbols of the operators $\bar{a}_\mu$ and $a_\mu$. One can verify that this function corresponds to the rest of arbitrariness in the redefinition of creation and annihilation operators when initial Heisenberg algebra (1), (14) is fixed. Therefore, we can set $h_5(x) = 0$.

The transition to the operator functions is realized in the conventional manner:

$$
:O(\bar{a}, \bar{b}, a, b) := \exp\left(\bar{a} \cdot \frac{\partial}{\partial \bar{\alpha}}\right) \exp\left(\bar{b} \frac{\partial}{\partial \bar{\beta}}\right) \exp\left(a \cdot \frac{\partial}{\partial \alpha}\right) \exp\left(b \frac{\partial}{\partial \beta}\right) O(\bar{\alpha}, \bar{\beta}, \alpha, \beta) \Big|_{\substack{\alpha^\# \to 0 \\ \beta^\# \to 0}} .
$$

Thus, we have obtained the general form of the operators $L_n$, which satisfy commutation relations (22) and (23) in the linear approximation. This means that Lagrangian (13) is invariant under gauge transformations (10) at this order. The form of operator $L_2$ has changed in this approximation, hence, the conditions

$$
L_2 L_2 |\Phi^s\rangle = 0, \qquad L_2 |\Lambda^{s-1}\rangle = 0
$$

undergo the nontrivial modifications.

## 3. Examples

In this section we will apply the proposed algebraic scheme to the description of propagation of the massive states with spin 1 and 2 in the Symmetrical Einstein space.

**Vector massive field.** This case is quite interesting since it is practically the only massive bosonic field among the other higher-spin states which was observed in the experiment. Let us consider the state of the Hilbert space that corresponds to the massive state with spin 1.

$$
|1\rangle = \left( (v \cdot \bar{a}) + \varphi \bar{b} \right) |0\rangle.
$$

It is not difficult to compute the expectation value of operator (13) in this state. Having made this, one derives the following Lagrangian[6] in the linear approximation

$$\begin{aligned} \mathcal{L}_{s=1} &= (1+2c_2R)\left(\bar{v}^\alpha\mathcal{P}^2v_\alpha - \bar{v}^\beta\mathcal{P}_\beta\mathcal{P}_\alpha v^\alpha + \bar{\varphi}\mathcal{P}^2\varphi\right) + (1+2c_1R)\,\bar{v}^\alpha v_\alpha \\ &\quad - (1+(c_1+c_2)R)\,(\bar{\varphi}\mathcal{P}_\alpha v^\alpha + h.c.) + \frac{1}{3}\bar{v}_\delta R^{\alpha\delta\gamma\beta}\mathcal{P}_\alpha\mathcal{P}_\gamma v_\beta. \end{aligned} \tag{29}$$

For the Lagrangian to describe the massive vector field properly, we have to impose the constraints

$$1+2c_1R \geq 0, \qquad 1+2c_2R > 0. \tag{30}$$

The former constraint ensures the given state not to be the tachyon, while the latter one provides the right sign of kinetic terms.

The gauge vector for the massive spin-1 state is

$$|\Lambda, 1\rangle = \eta b|0\rangle$$

and the gauge transformations for the massive field are

$$\begin{aligned} \delta v_\alpha &= (1+c_2R)\,\mathcal{P}_\alpha\eta, \\ \delta\varphi &= (1+c_1R)\,\eta. \end{aligned}$$

For the vector massive field it is not difficult to generalize the linear approximation to the general case[7] of arbitrary symmetrical Einstein space. For that we make the following substitution:

$$c_1R \to f_1(R), \qquad c_2R \to f_2(R).$$

But the gauge invariance requires the functions be equal to each other. Thereby, the whole Lagrangian describing the propagation of vector field in the considered background is

$$\begin{aligned} \mathcal{L}_{s=1} &= (1+f(R))\left(\bar{v}^\alpha\mathcal{P}^2v_\alpha - \bar{v}^\beta\mathcal{P}_\beta\mathcal{P}_\alpha v^\alpha + \bar{\varphi}\mathcal{P}^2\varphi + \bar{v}^\alpha v_\alpha - (\bar{\varphi}\mathcal{P}_\alpha v^\alpha + h.c.)\right) \\ &\quad + \frac{1}{3}\bar{v}_\delta R^{\alpha\delta\gamma\beta}\mathcal{P}_\alpha\mathcal{P}_\gamma v_\beta. \end{aligned} \tag{31}$$

There is no reason to be surprised, since, due to the gauge invariance, we cannot obtain a different result by virtue of the fact that $R$ is constant.

Now we can consider two variants. The first is when $1 + f(R) > 0$[8]. Then, the Lagrangian is invariant under the usual gauge transformations for the massive vector fields

$$\begin{aligned} \delta v_\mu &= \mathcal{P}_\mu\eta, \\ \delta\varphi &= \eta. \end{aligned}$$

[6] We suppress the usual multiplier $\sqrt{-g}$.

[7] Of course, this is only one possibility among others.

[8] The massive vector field becomes a ghost when $1 + f(R) < 0$.

In principle, in this case we can include the multiplier $1 + f(R)$ into the normalization of fields. After that we obtain the usual Lagrangian "minimally" coupled to the Riemann background with the single "non-minimal" term.

A different quite unusual situation is realized when $1 + f(R) = 0$. Then, the whole Lagrangian consists of the single term

$$\mathcal{L} = \frac{1}{3} R^{\mu\nu\alpha\beta} \bar{V}_{\mu\nu} V_{\alpha\beta},$$

where $V_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$. Obviously, the Lagrangian is invariant under the transformation $\delta v_\mu = \mathcal{P}_\mu \eta$. One can notice that the transition to an arbitrary Riemann space does not break this invariance.

Let us discuss the causality for the massive vector field in the given background. Having fixed the gauge invariance by

$$\varphi = 0,$$

from (29) we derive the following equations

$$\frac{\delta \mathcal{L}}{\delta \bar{v}^\mu} = (1 + 2c_2 R) \left( \mathcal{P}^2 v_\mu - \mathcal{P}_\mu (\mathcal{P} \cdot v) \right) + (1 + 2c_1 R) v_\mu - \frac{1}{3} R_\mu{}^{\nu\alpha\beta} \mathcal{P}_\nu \mathcal{P}_\alpha v_\beta.$$

Having taken the divergence of these equations one obtains the constraint

$$\mathcal{P}^\mu \frac{\delta \mathcal{L}}{\delta \bar{v}^\mu} = \left( 1 + \left( 2 (c_1 - c_2) - \frac{1}{D} \right) R \right) (\mathcal{P} \cdot v) + \mathcal{O} \left( R^2 \right) = 0.$$

From here we see that if we impose the requirement

$$m^2 + \left( 2 (c_1 - c_2) - \frac{1}{D} \right) R \neq 0 \tag{32}$$

we get the necessary constraint on the vector field in this order. Here we have restored the dimensional parameter $m$. Calculating the characteristic determinant[9], we obtain as a result

$$D(n) = \left( 3Dm^2 + R \right) \left( n^2 \right)^D + \mathcal{O}(R^2),$$

where $n_\mu$ is the normal vector to the characteristic surface. The equations of motion will be causal (hyperbolic) if the solutions $n^0$ to $D(n) = 0$ are real for any $\vec{n}$. Thereby, in our case, from the condition $D(n) = 0$ we have the usual light cone as the solution for $n_\mu$ if we impose the following condition:

$$3Dm^2 + R \neq 0 \tag{33}$$

Of course, our consideration essentially depends on the higher orders in the Riemann tensor and scalar curvature.

Thus, we can see that in such theory there are restrictions on $m$ and $R$ similar to Ref. [22,23].

[9]The determinant is entirely determined by the coefficients of the highest derivatives in equations of motion after gauge fixing and resolving all the constaraints [21].

**Massive field with spin 2.** Now we obtain the Lagrangian describing the propagation of massive spin-2 field in the symmetrical Einstein space. The following state of the Fock space corresponds to such field

$$|2\rangle = \{(\bar{a}\cdot h\cdot\bar{a}) + (v\cdot\bar{a})\,\bar{b} + \eta b^2\}|0\rangle.$$

It is easy to see that this state trivially satisfies condition[10] (9).

Having calculated the expectation value of operator (13) in this state we derive the following Lagrangian

$$\begin{aligned}
\mathcal{L}_{s=2} \;=\; & (1+2c_2R)\Big\{\bar{h}^{\alpha\beta}\mathcal{P}^2h_{\alpha\beta} - 2\bar{h}\mathcal{P}^2h - 2\bar{h}^{\beta\gamma}\mathcal{P}_\beta\mathcal{P}_\alpha h^\alpha_\gamma + \Big\{2\bar{h}\mathcal{P}_\alpha\mathcal{P}_\beta h^{\alpha\beta} \\
& + \bar{\varphi}\mathcal{P}_\alpha\mathcal{P}_\beta h^{\alpha\beta} - \bar{h}\mathcal{P}^2\varphi + h.c.\Big\} + \frac{1}{2}\bar{v}^\alpha\mathcal{P}^2v_\alpha - \frac{1}{2}\bar{v}^\beta\mathcal{P}_\alpha\mathcal{P}_\beta v^\alpha\Big\} \\
& - (1+(c_1+c_2)\,R)\left\{\bar{v}_\beta\mathcal{P}_\alpha h^{\alpha\beta} - \bar{h}\mathcal{P}_\alpha v^\alpha + h.c.\right\} \\
& + (1+2c_1R)\left(\bar{h}^{\alpha\beta}h_{\alpha\beta} - \bar{h}h\right) - \frac{3}{8}R^{\alpha\gamma\beta\delta}\bar{h}_{\gamma\mu}\mathcal{P}_\alpha\mathcal{P}_\delta h^\mu_\beta \\
& - \frac{1}{2}R^{\alpha\gamma\beta\delta}\bar{v}_\gamma\mathcal{P}_\alpha\mathcal{P}_\delta v_\beta - \frac{1}{6}\Big\{2R^{\alpha\gamma\beta\delta}\bar{h}_{\beta\gamma}\mathcal{P}_\alpha\mathcal{P}_\mu h^\mu_\delta - R^{\alpha\gamma\beta\delta}\bar{h}_{\beta\gamma}\mathcal{P}_\alpha\mathcal{P}_\delta h \\
& - 5R^{\alpha\gamma\beta\delta}\bar{h}_{\beta\gamma}\mathcal{P}_\alpha\mathcal{P}_\delta\varphi + 5R^{\alpha\gamma\beta\delta}\bar{h}_{\beta\gamma}\mathcal{P}_\alpha v_\delta + h.c.\Big\} + \frac{4}{3}R^{\alpha\gamma\beta\delta}\bar{h}_{\alpha\beta}h_{\gamma\delta} \\
& + \frac{1}{3D}R\Big\{ - \bar{h}\mathcal{P}^2h + \Big\{\frac{1}{2}\bar{h}\mathcal{P}_\alpha\mathcal{P}_\beta h^{\alpha\beta} - \frac{1}{2}\bar{h}\mathcal{P}^2\varphi + h.c.\Big\} + 3\bar{\varphi}\mathcal{P}^2\varphi \\
& + \Big\{\frac{1}{2}\bar{v}^\alpha\mathcal{P}_\alpha h - 3\bar{\varphi}\mathcal{P}_\alpha v^\alpha + h.c.\Big\} - 4\bar{h}h + 3\bar{v}^\alpha v_\alpha\Big\}
\end{aligned} \tag{34}$$

where $h = g^{\mu\nu}h_{\mu\nu}$. It is not difficult to notice that we have to impose the same restrictions (30) for the proper description of the massive state.

The gauge vector for the massive spin-2 state is

$$|\Lambda, 2\rangle = \{(\xi\cdot\bar{a}) + \eta b\}|0\rangle.$$

Condition (11) is the non-trivial constraint for the gauge vectors of massive states with spin 3 and higher only.

From (10) we obtain the following gauge transformations

$$\begin{aligned}
\delta h_{\alpha\beta} &= (1+c_2R)\,\mathcal{P}_{(\alpha}\xi_{\beta)} - \frac{1}{6}R_{(\alpha}{}^{\gamma}{}_{\beta)}{}^{\delta}\mathcal{P}_\gamma\xi_\delta, \\
\delta v_\alpha &= (1+c_2R)\,\mathcal{P}_\alpha u + (1+c_1R)\,\xi_\alpha, \\
\delta\varphi &= (1+c_1R)\,\eta.
\end{aligned}$$

[10] One can verify that condition (9) imposes a not-trivial restriction only on the states with spin 4 and higher.

Let us fix the gauge invariance by means of the gauge condition

$$v_\mu = 0, \qquad \varphi = 0.$$

Now we have the following equations of motion

$$\begin{aligned}
\frac{\delta \mathcal{L}}{\delta h^{\mu\nu}} &= (1+2c_2R)\Big(\mathcal{P}^2 h_{\mu\nu} - 2\mathcal{P}_{(\mu}\left(\mathcal{P}\cdot h\right)_{\nu)} + \mathcal{P}_{(\nu}\mathcal{P}_{\mu)}h \\
&\quad - g_{\mu\nu}\left(\mathcal{P}^2 h - (\mathcal{P}\cdot\mathcal{P}\cdot h)\right)\Big) - (1+2c_1R)\left(h_{\mu\nu} - g_{\mu\nu}h\right) \\
&\quad + \frac{R}{6D}\left(\mathcal{P}_{(\nu}\mathcal{P}_{\mu)}h - g_{\mu\nu}\left\{2\mathcal{P}^2 h - (\mathcal{P}\cdot\mathcal{P}\cdot h) - 8h\right\}\right) \\
&\quad - \frac{1}{6}g_{\mu\nu}R^{\alpha\delta\beta\gamma}\mathcal{P}_\delta\mathcal{P}_\gamma h_{\alpha\beta} + \frac{1}{3}R_{\alpha(\mu|\beta}{}^{\gamma}\mathcal{P}_{|\nu)}\mathcal{P}_\gamma h^{\alpha\beta} - \frac{1}{6}R^{\beta}{}_{(\mu}{}^{\gamma}{}_{\nu)}\mathcal{P}_\gamma\mathcal{P}_\beta h \\
&\quad + \frac{1}{3}R_{\beta(\mu}{}^{\gamma}{}_{\nu)}\mathcal{P}_\gamma\left(\mathcal{P}\cdot h\right)^\beta - \frac{2}{3}R^{\alpha\gamma\beta}{}_{(\mu|}\mathcal{P}_\gamma\mathcal{P}_\beta h_{|\nu)\alpha} - \frac{4}{3}R_{\alpha\mu\beta\nu}h^{\alpha\beta}.
\end{aligned} \tag{35}$$

From these equations we can obtain the constraints:

$$\begin{aligned}
&\left(\mathcal{P}\cdot h\right)_\mu = \mathcal{P}_\mu h\left(1 + \frac{R}{6D}\right) + \frac{5}{6}R_\mu{}^{\nu\alpha\beta}\mathcal{P}_\alpha h_{\nu\beta} \\
&\left(3 + 2\left(3c_1 D + \frac{2}{D}\right)R\right)h = 0.
\end{aligned} \tag{36}$$

Now one can see that when

$$3 + 2\left(3c_1 D + \frac{2}{D}\right)R \neq 0$$

we have the appropriate number of constraints and, correspondingly, the appropriate number of degree of freedom at this order.

Now we consider the question of causality of the massive spin-2 state in the linear approximation. Using relations (36) and equations of motion (35), one can obtain the characteristic determinant

$$D(n) = \left(n^2\right)^{\frac{D^2(D+1)^2}{4}}\left(1 + \frac{D+5}{3D}R\right) + \mathcal{O}\left(R^2\right).$$

Thereby, when $\left(1 + \frac{D+5}{3D}R\right) \neq 0$ from the condition $D(n) = 0$, we have the usual light cone in this approximation, i.e. the causal propagation of massive spin-2 state in the considered Riemann background.

For massive states with higher spins, one can derive a similar result. Quite obviously that the causal propagation of these states in considered background imposes some restrictions on the mass of states and the scalar curvature only.

## 4. Conclusion

We have applied the algebraic scheme proposed in Ref. [15,16] to the description of propagation of the gauge massive fields in the arbitrary symmetrical Einstein space of arbitrary dimensionality in the lowest approximation in the Riemann tensor and scalar curvature. This approach is quite convenient since it allows one to reduce the cumbersome problem of searching for the gauge invariant action of higher-spin fields to the pure algebraic problem of finding the appropriate modification of some operators. In principal, this approach can be applied to the description of fermionic massive fields in such background as well.

## Acknowledgment

The author is grateful to Prof. Yu. M. Zinoviev for the numerous useful discussions and the help in the work.

## References

[1] C. Aragone and S. Deser. Consistency problems of hypergravity. *Phys. Lett.*, **86B** (1979), 161.

[2] M. A. Vasilev. *Yad. Fiz.* **45** (1987) 1784; *ibid.* **47** (1988) 831; *ibid.* **48** (1988) 1478.

[3] M. A. Vasilev and E. S. Fradkin. *Jetp Lett.* **44** (1986) 484 (in russian); *Phys. Lett.* **B189** (1987) 89; *Nucl. Phys.* **B291** (1987) 141.

[4] M. A. Vasilev. Higher spin gauge theories in four-dimensions, three-dimensions, and two-dimensions. *Int. J. Mod. Phys.*, **D5** (1996), 763, hep-th/9611024.

[5] James T. Liu Ioannis Giannakis and Massimo Porrati. Massive higher spin states in string theory and the principle of equivalence. hep-th/9809142.

[6] Massimo Porrati. Massive spin 5/2 fields coupled to gravitational: Tree level unitarity as the equivalence principle. *Phys. Lett.*, **B304** (1993), 77–80, gr-qc/9301012.

[7] A. Cucchieri, M. Porrati, and S. Deser. Tree level unitarity constraints on the gravitational couplings of higher spin massive fields. *Phys. Rev.*, **D51** (1995), 4543–4549, hep-th/9408073.

[8] L. P. C. Singh and C.R. Hagen. *Phys. Rev*, **D9** (910), 898, 1974.

[9] Yu. M. Zinoviev. Gauge invariant description of massive high spin particles. IHEP preprint 83-91 (Protvino, 1983).

[10] Yu. M. Zinoviev, *The Role of Poincare Group in Elementary PArticle Physics*, XVII Seminar on High Energy Physics and Field Theory, June 1994, Protvino.

[11] S. M. Klishevich and Yu. M. Zinoviev. On electromagnetic interaction of massive spin-2 particle. *Phys. Atom. Nucl.*, **61** (1998), 1527–1537, hep-th/9708150.

[12] A.I. Pashnev. Composite system and free field theory of the Regge trajectory. *Teor. Mat. Fiz.*, **78**, 1989 (in Russian).

[13] P. C. Argyres and C. R. Nappi. Massive spin 2 bosonic string states in an electromagnetic background. *Phys. Lett.*, **B224** (1989), 89.

[14] J. Fang and C. Fronsdal. Deformation of gauge groups. Gravitation. *J. of Mat. Phys (N.Y.)*, **20** (1979), 2264.

[15] S. M. Klishevich. Massive fields of arbitrary integer spin in homogeneous electromagnetic field. 1998, IHEP 98-24, hep-th/9810228.

[16] S. M. Klishevich. Massive fields of arbitrary half-integer spin in constant electromagnetic field. 1998, IHEP 98-70, hep-th/9811030.

[17] A. Pashnev and M. Tsulaya. Dimensional reduction and BRST approach to the description of a Regge trajectory. *Mod. Phys. Lett.*, **A12** (1997), 861–870, hep-th/9703010.

[18] C. Fronsdal. Massless fields with integer spin. *Phys. Rev.*, **D18** (1978), 3624.

[19] T. Curtright. Massless field supermultiplets with arbitrary spin. *Phys. Lett*, **85B** (1979), 219.

[20] A. Barducci and D. Dominici. A model for the supersymmetric relativistic oscillator. *Nuovo Cim.*, **37A** (1977), 385.

[21] G. Velo and D. Zwanziger. Propagation and quantization of Rarita-Schwinger waves in an external electromagnetic potential. *Phys. Rev.*, **186** (1969) 1337.

[22] Atsushi Higuchi. Forbidden mass range for spin 2 field theory in de sitter space-time. *Nucl. Phys.*, **B282** (1987), 397.

[23] I. Bengtsson. Note on massive spin 2 in curved space. *J. Math. Phys.*, **36** (1995), 5805, gr-qc/9411057.

*Received November 25, 1998*

С.М. Клишевич
Массивные поля произвольного целого спина в симметрическом пространстве Эйнштейна.

Оригинал-макет подготовлен с помощью системы LaTeX.
Редактор Е.Н.Горина. Технический редактор Н.В.Орлова.

Подписано к печати 26.11.98. Формат $60 \times 84/8$. Офсетная печать.
Печ.л. 1,87. Уч.-изд.л. 1,44. Тираж 180. Заказ 38. Индекс 3649.
ЛР №020498 17.04.97.

ГНЦ РФ Институт физики высоких энергий
142284, Протвино Московской обл.

Индекс 3649

П Р Е П Р И Н Т 98-81, И Ф В Э, 1998