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QUANTUM VERTEX OPERATORS FOR SINE–GORDON MODEL

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Abstract

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On the base of the second Drinfeld realization of $U_q'(\widehat{sl_2})$ we construct the quantum version of vertex operators for the sine-Gordon model which are responsible for the generation of soliton solutions in the classics. The quantum vertex operators satisfy the relations of Zamolodchikov algebra and possess the quantum interaction function. We also discuss their possible role in the quantum soliton theory.

Аннотация

Савельв М.В., Зуевский А.Б. Квантовые вертексные операторы модели синус-Гордон: Препринт ИФВЭ 2000–4. – Протвино, 2000. – 21 с., библиогр.: 37.

На основе второй реализации Дринфельда $U_q'(\widehat{sl_2})$ мы строим квантовый вариант вертексных операторов модели синус-Гордон, которые ответственны за генерацию солитонных решений в классике. Квантовые вертексные операторы удовлетворяют соотношениям алгебры Замолодчикова и обладают квантовой функцией взаимодействия. Мы обсуждаем также их возможную роль в теории квантовых солитонов.

This work has been inspired by M.V. Saveliev as part of a concept on quantum groups applications to two-dimensional integrable systems and continuation of group-theoretical methods developed in [2,19]. Most of the ideas reflected in this paper originated from discussions with him. His friends and colleagues suffered an unbearable loss when he died last year. People will remember M.V. Saveliev as a very kind person, an outstanding scientist, and an excellent teacher. The second author is deeply indebted to M.V. Saveliev and would like to dedicate this work to his memory.

1. Introduction

Two dimensional integrable field theories have attracted much attention in the last years. Both in the classical and quantum cases they deliver examples of nice theories which have rather rich internal algebraic structure. In particular, a huge amount of work has been done in cases of conformal and affine Toda models. Among affine Toda systems, the case of the sine-Gordon equation that corresponds to the Lie algebra $\widehat{sl_2}$ is the most elaborated one. This theory is interesting from many points of view.

In the pioneering paper [1] a general solution to affine Toda systems was constructed on the base of the group-theoretical method [2]. It was shown how to obtain soliton solutions to affine Toda equations on the classical level starting with the general solution. The construction is crucially based on the existence of the principal or homogeneous Heisenberg subalgebra of an affine Lie algebra $\widehat{\mathcal{G}}$.

The quantum Toda field theory (in the compact or non-compact space) may be introduced in various ways [3,4,5,6,7,8,9,10,11]. In the Zamolodchikov's approach [12] the quantum sine-Gordon model was considered in the framework of the S-matrix formalism. This model is known as an example of the relativistic quantum field theory leading to the factorized scattering. Solitons and antisolitons are generated by non-commuting operators $F(\theta)$ and $\bar{F}(\theta)$ (θ denotes the rapidity of a soliton (antisoliton)) which act on the vacuum of the theory and form an associative algebra describing the scattering of corresponding particles in the theory (see Appendix E). In this, the spectrum of the quantum sine-Gordon theory consists of solitons, antisolitons and bound soliton-antisoliton states (quantum doublets).

In [13,14,15,16,17] the quantum sine-Gordon model was considered in the frames of the angular quantization approach. An attempt to search for rudiments of the classical algebraic structure in the quantum case was made in the paper [18]. The so-called quantum interaction function that had to play the role of the classical interaction function for some soliton vertex operators was introduced.

The purpose of this paper is to clarify further connections between quantum groups and quantum two dimensional integrable systems. By the example of the quantum sine-Gordon model we show that the structure of soliton operators in Zamolodchikov's approach is guided by the properties of corresponding quantum group. Starting from the second Drinfeld formulation of the quantized universal enveloping algebra $U'_q(\widehat{sl_2})$, we construct a particular realization of quantum vertex operators which possess the quantum interaction function introduced in [18] and satisfy the Zamolodchikov algebra. In contrast to vertex operators calculated in the angular quantization approach, the algebraic origin of our vertex operators is much more transparent and useful for further applications. We also calculate vertex operators in the antisoliton sector of the Zamolodchikov algebra subject to some condition. That restriction clarifies the existence range of bound soliton-antisoliton states (breathers). Our quantum vertex operators degenerate to the usual soliton-generating vertex operators in the classical limit. We also discuss their possible role in the theory of quantum solitons.

The paper is organized as follows. In Section 2 we recall affine Toda systems, the general solution and its solitonic specialization in the group-theoretical approach. Section 3 is devoted to the S-matrix for the sine-Gordon model and its relation to the quantum dilogarithm. In Section 4 we propose the quantum vertex operators and discuss their properties. The quantum antisoliton vertex operators are calculated in Section 5. In order to remind some basic facts about quantum groups, quantum dilogarithms, the S-matrix approach to the quantum sine-Gordon equation, and algebraic theory of Toda systems, we have added Appendices A–F. Appendix D is dedicated to an explicit construction of a soliton solution to the classical sine-Gordon model in the homogeneous gradation case.

2. Affine Toda systems and soliton specialization

Let us first recall some known results concerning affine Toda systems in the classical region (see Appendix C as a reminder on conformal Toda systems). Let $\mathcal{M}$ be a two dimensional manifold, say $\mathbb{R}^2$ or $\mathbb{C}^1$, with standard coordinates $z^{\pm} = t \pm x$ and derivatives $\partial_{\pm}$. In the $\mathbb{C}^1$ case we suppose that $z^- = (z^+)^*$. Let $\widehat{G}$ be a complex Lie group with a simple Lie algebra $\widehat{\mathcal{G}}$ of rank $r+1$ endowed with a $\mathbb{Z}$-gradation. In the principal gradation the subspace $\widehat{\mathcal{G}}_0$ in the decomposition $\widehat{\mathcal{G}} = \oplus_{m\in\mathbb{Z}}\widehat{\mathcal{G}}_m$ is abelian, while for the homogeneous gradation this is not the case.

The affine Toda fields $\phi = \sum_{i=1}^{r} h_i\phi_i$ (here h_i, $i = 0, ..., r$ denote Cartan elements of $\widehat{\mathcal{G}}$, see Appendix A) satisfy the equations

$$\partial_+\partial_-\phi + \frac{4\eta^2}{\beta}\left(\sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta\alpha_i\cdot\phi} - \frac{\psi}{\psi^2} e^{-\beta\psi\cdot\phi}\right) = 0, \tag{2.1}$$

where α_i, $i = 0, ..., r$ are simple roots of $\widehat{\mathcal{G}}$, $\psi = -\alpha_0$ is the highest root, and $\frac{\psi}{\psi^2} = \sum_{i=1}^{r} m_i\frac{\alpha_i}{\alpha_i^2}$ defines m_i. Equations (2.1) are associated with an affine Lie algebra $\widehat{\mathcal{G}}$ in the principal gradation. In (2.1) η conventionally denotes a real inverse length scale and β is a coupling constant. The coefficients in (2.1) are arranged in such way that $\phi = 0$ is a constant solution. The very fact of the integrability of system (2.1) has been established on the base of a relevant flat connection form using more or less the same arguments as in the case of corresponding conformal (finite dimensional) Toda systems, see Appendix C, [2,19], and references therein.

The general solution to (2.1) is represented in a formal sense [1,20] as

$$e^{-\beta\lambda_j\cdot\phi}=\frac{<\Lambda_j|\gamma_+^{-1}\mu_+^{-1}\mu_-\gamma_-|\Lambda_j>}{<\Lambda_0|\gamma_+^{-1}\mu_+^{-1}\mu_-\gamma_-|\Lambda_0>^{m_j}}, \tag{2.2}$$

where $|\Lambda_j>$, $j=1,...,r$ is the highest vector of the j–th fundamental representation of $\widehat{\mathcal{G}}$ labelled by the fundamental weight λ_j, (m_j are marks on the Dynkin diagram of $\widehat{\mathcal{G}}$). The mappings $\mu_\pm(z^\pm):\mathcal{M}\to\widehat{G}_\pm$ satisfy

$$\partial_\pm\mu_\pm=\kappa_\pm\mu_\pm, \tag{2.3}$$

where

$$\kappa_\pm(z^\pm)=\sum_{i=0}^{r}\Psi_{\pm i}^{(0)}\cdot x_{\pm i}, \tag{2.4}$$

($x_{\pm i}$, $i=0,...,r$ are Chevalley generators of $\widehat{\mathcal{G}}$),

$$\Psi_{\pm j}^{(0)}=m_j e^{\mp\beta\sum\limits_{i=0}^{r}k_{ji}\phi_{\pm i}^{(0)}}, \tag{2.5}$$

where $\phi_{\pm i}^{(0)}$ are free fields and (2.4) can be also represented as

$$\kappa_\pm(z^\pm)=\gamma_\pm^{-1}\widehat{E}_{\pm1}\gamma_\pm, \tag{2.6}$$

with

$$\widehat{E}_{\pm1}=\sum_{i=0}^{r}\sqrt{m_i}x_{\pm i}. \tag{2.7}$$

In (2.2) the mappings $\gamma_\pm(z_\pm):\mathcal{M}\to\widehat{G}_0$ are

$$\gamma_\pm=e^{\sum\limits_{i=0}^{r}\phi_{\pm i}^{(0)}h_i}. \tag{2.8}$$

Affine Toda theories posses soliton solutions when β is imaginary. The sine-Gordon equation

$$\partial_+\partial_-\phi+\frac{4\eta^2}{\beta}\left(\frac{\alpha_1}{\alpha_1^2}e^{\beta\alpha_1\cdot\phi}-\frac{\psi}{\psi^2}e^{-\beta\psi\cdot\phi}\right)=0 \tag{2.9}$$

is a particular case associated with the affine Lie algebra $\widehat{\mathcal{G}}=\widehat{sl_2}$ both in the principal and homogeneous [21] gradations. In the homogeneous gradation the general solution to (2.9) has the same form (2.2). The soliton solution to equation (2.9) is

$$\phi_{sol}=arctg\left(e^{\frac{x-vt}{\sqrt{1-v^2}}}\right). \tag{2.10}$$

In [1] there was suggested a remarkable specialization of the general solution (2.2) in the principal gradation which leads to soliton solutions. The authors choose the mappings $\gamma_\pm$ in (2.2) to be unit elements of $\widehat{G}_0$. Therefore, because of (2.6), $\kappa_\pm=\widehat{E}_{\pm1}$. Then it is easy to integrate the equations (2.3). One finds

$$\mu_\pm=\mu_\pm(0)e^{\eta z^\pm\widehat{E}_{\pm1}}. \tag{2.11}$$

The general solution (2.2) reads

$$e^{-\beta\lambda_j\cdot\phi} = \frac{< \Lambda_j|e^{-\eta\widehat{E}_1 z^+}\mu(0)e^{-\eta\widehat{E}_{-1}z^-}|\Lambda_j >}{< \Lambda_0|e^{-\eta\widehat{E}_1 z^+}\mu(0)e^{-\eta\widehat{E}_{-1}z^-}|\Lambda_0 >^{m_j}}, \tag{2.12}$$

where $\mu(0) \equiv (\mu(0)_+)^{-1}\mu(0)_-$ is a constant mapping $\mathcal{M} \to \widehat{G}$ independent of the coordinates $z^\pm$. N-soliton solutions can be obtained by choosing the group element $\mu(0)$ as

$$\mu(0) = \prod_{n=1}^{N} e^{Q_n \widehat{F}^n(\zeta_n)}, \tag{2.13}$$

where $\xi_n = \frac{4\pi\theta_n}{\gamma}$, $\gamma = \frac{|\beta|^2}{1-\frac{|\beta|^2}{8\pi}}$ (θ_n are rapidities of solitons), and Q_n is the logarithm of the n-th soliton initial coordinate. In (2.13) $\widehat{F}^n(\zeta_n)$ are vertex operators in the principal gradation [22], i.e., elements of the principal vertex operator construction. These operators are eigenvectors with respect to elements of the principal Heisenberg subalgebra $\widehat{E}_{\pm k}$ (see (2.7) for $k = 1$). This fact helps us to eliminate $\widehat{E}_{\pm 1}$ from solution (2.12) commuting them with $\mu(0)$. Also note that exponentiation series of $\widehat{F}^n(\zeta_n)$ operators terminate after the order which coincides with the level of the highest weight representation. These two properties make solution (2.12) equal to classical soliton solution [23]. One sees that the vertex operators $Q_n\widehat{F}^n(\zeta_n)$ generate solitons in the classical theory. It should be mentioned that it is possible to choose the functions $\phi^{(0)}_{\pm i}$ in (2.8) to be $\phi^{(0)}_{\pm i} = m_i\phi^{(0)}$ where $\phi^{(0)}$ is an arbitrary function. Then $\gamma_\pm$ commutes with all the elements of the algebra $\widehat{\mathcal{G}}$. That leads to solution (2.11). However, under such a specialization, the final solution following from general solution (2.2) differs from the classical soliton solution (2.12) by the exponentiation of the function $\phi^{(0)}$.

Note that in the case of $\widehat{sl_2}$ algebra both the principal and homogeneous gradations lead to the same sine-Gordon equation (2.9). Therefore, one can construct soliton solutions using the general solution (2.2) in the case of the homogeneous gradation with the corresponding vertex operators based on the homogeneous Heisenberg subalgebra of $\widehat{sl_2}$ [22] (see Appendix D).

3. S-matrix of sine-Gordon model and quantum dilogarithm

In the paper [18] a remarkable dilogarithmic (see Appendix F) structure of the factorized S-matrix elements calculated in [12] (see Appendix E) was discovered. It turned out that the S-matrix elements $S(\theta)$, $S_T(\theta)$ and $S_R(\theta)$ could be expressed through a new function $X_q(x)$ which is composed of the ratio of two regularized quantum dilogarithms [24,25]. In addition, the function $X_q(x)$ (called the quantum interaction function) can be considered as a quantum analogue of so-called interaction function which plays a very important role on the classical level. The usual interaction function $X(\theta)$ [18,20] comes from a normal ordering of two vertex operators generating soliton solutions in the classical region,

$$F(\zeta_1)F(\zeta_2) = X(\theta_{12}) : F(\zeta_1)F(\zeta_2) :, \tag{3.1}$$

where $\theta_{12} = \theta_2 - \theta_1$, ($\theta_i$, $i = 1, 2$ are rapidities of two solitons), $\xi = e^{\frac{4\pi\theta_n}{\gamma}}$, $\gamma = \frac{|\beta|^2}{1-\frac{|\beta|^2}{8\pi}}$. It has been discovered earlier that the S-matrix for the sine-Gordon model coincides, up to a scalar function factor, with the R-matrix of $U_q(\widehat{sl_2})$, [26] (here we have used the notations of [18])

$$
\begin{aligned}
S^{11}(\theta) = v(x) & \begin{pmatrix} xq - x^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & x - x^{-1} & 0 \\ 0 & x - x^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & xq - x^{-1}q^{-1} \end{pmatrix} \\
& = \begin{pmatrix} S(\theta) & 0 & 0 & 0 \\ 0 & S_R(\theta) & S_T(\theta) & 0 \\ 0 & S_T(\theta) & S_R(\theta) & 0 \\ 0 & 0 & 0 & S(\theta) \end{pmatrix},
\end{aligned}
\tag{3.2}
$$

where $x = e^{\frac{8\pi}{\gamma}\theta}$, $q = e^{-i\frac{8\pi^2}{\gamma}}$. Here $S(\theta)$, $S_R(\theta)$ and $S_T(\theta)$ are given in Appendix E and differ from the R-matrix of $U_q(\widehat{sl_2})$ by a scalar function $v(x)$ [18]

$$
v(x) = \frac{q}{1 - x^2q^2} \frac{X_q(x)}{X_q(x^{-1})}, \tag{3.3}
$$

$$
S(\theta) = v(x)(xq - x^{-1}q^{-1}), \tag{3.4}
$$

$$
S_R(\theta) = v(x)(q - q^{-1}), \tag{3.5}
$$

$$
S_T(\theta) = v(x)(x - x^{-1}). \tag{3.6}
$$

We see that the deformation parameter q of the quantum group $U_q'(\widehat{sl_2})$ is related to the renormalized coupling constant γ which is pure by real. Here $X_q(x)$ is the above mentioned quantum interaction function. The most important thing is that the quantum interaction function $X_q(x)$ can be expressed as the ratio

$$
X_q(x) = \frac{S_{q^{-2}}^{reg}(e^{i\pi + 2mi\pi}x^2q^2)}{S_{q^{-2}}^{reg}(e^{i\pi+2mi\pi}x^2)}, \tag{3.7}
$$

where $m \in \mathbb{Z}$ is an arbitrary integer and $S_{q^{-2}}^{reg}(z)$ is the regularized quantum dilogarithm [24,25] (see Appendix F for the definition and properties of quantum dilogarithms).

4. Quantum vertex operators

The question one can pose now is what kind of object might correspond to classical vertex operators that are responsible for the creation of solitons in the classics. A natural idea is to find such a vertex operator which would have the interaction function as in (3.7) and satisfy the Zamolodchikov algebra (see Appendix E). The dilogarithmic structure of the quantum interaction function hints us an answer.

We propose the following form of the quantum vertex operator ${}_qF(\zeta)$

$$
{}_qF(\zeta) = {}_q\Phi(\zeta) e^{\frac{\alpha}{2}} \zeta^{\frac{1}{2}\partial_\alpha}. \tag{4.1}
$$

Here ${}_q\Phi(\zeta)$ is the product

$$ {}_q\Phi(\zeta) = \Phi_a(\zeta) \cdot \Phi_b(\zeta), \tag{4.2} $$

where

$$ \Phi_a(\zeta) = exp\left(\sum_{n=1}^{\infty} \frac{a_{-n}}{[2n]} q^{\frac{7}{2}n} \zeta^n\right) exp\left(-\sum_{n=1}^{\infty} \frac{a_n}{[2n]} q^{-\frac{5}{2}n} \zeta^{-n}\right), \tag{4.3} $$

$$ \begin{aligned} \Phi_b(\zeta) = exp & \left(\sum_{n=1}^{\infty} \frac{b_{-n}}{[2n]_{\widetilde{q}}} \widetilde{q}^{\frac{7}{2}n(2(1+2m)-\frac{\mu}{\pi})} \widehat{\zeta}^n\right) \\ & \cdot exp\left(-\sum_{n=1}^{\infty} \frac{b_n}{[2n]_{\widetilde{q}}} \widetilde{q}^{-\frac{5}{2}n(2(1+2m)-\frac{\mu}{\pi})} \widehat{\zeta}^{-n}\right). \end{aligned} \tag{4.4} $$

In (4.4) $\widehat{\zeta} = \zeta^{-\frac{\pi}{\mu}}$ and

$$ q^{-2} = e^{i\mu}, \qquad \widetilde{q} = e^{\frac{\pi^2}{\mu^2}}. \tag{4.5} $$

Two other multipliers in (4.1) will be explained later. The operator $\Phi_a(\zeta)$ (4.3) is the vertex operator introduced in [27] (the values $\frac{7}{2}$ and $\frac{5}{2}$ in (4.3)-(4.4) are not important while their difference is). $\Phi_a(\zeta)$ contains elements of the homogeneous quantum Heisenberg subalgebra of $U_q(\widehat{sl_2})$. The operator $\Phi_b(\zeta)$ is constructed with the help of elements of the homogeneous Heisenberg subalgebra of the quantized universal enveloping algebra with the deformation parameter $\widetilde{q}$ associated with q. In (4.4) we have introduced the quantum homogeneous Heisenberg subalgebra of a quantzed universal enveloping algebra $U_{\widetilde{q}}(\widehat{sl_2})$. Its generators defined by $\{b_k, k \in \{\mathbb{Z} - 0\}, \widetilde{\gamma}^{\pm\frac{1}{2}}\}$ satisfy

$$ [a_l, b_k] = [x_n, b_k] = [K, b_k] = [\gamma, b_k] = 0, \tag{4.6} $$

$$ [b_k, b_m] = \delta_{k,-m} \frac{[2k]_{\widetilde{q}}}{k} \frac{\widetilde{\gamma}^{\frac{\mu k}{\pi}} - \widetilde{\gamma}^{-\frac{\mu k}{\pi}}}{\widetilde{q} - \widetilde{q}^{-1}}, \tag{4.7} $$

$k, m, l \in \{\mathbb{Z} - 0\}$, $n \in \mathbb{Z}$. Here

$$ [k]_{\widetilde{q}} = \frac{\widetilde{q}^k - \widetilde{q}^{-k}}{\widetilde{q} - \widetilde{q}^{-1}}. \tag{4.8} $$

The operators $\widetilde{\gamma}^{\pm\frac{1}{2}}$ belong to the center of $U_{\widetilde{q}}(\widehat{sl_2})$ and commute with gene
-rators of $U_q(\widehat{sl_2})$. The generators $\widetilde{\gamma}$ act as $\widetilde{\gamma}(f \otimes e^\beta) = \widetilde{q}(f \otimes e^\beta)$ on a representation vector of $U_{\widetilde{q}}(\widehat{sl_2})$ (see Appendix B).

Now we are going to explain the form of (4.2). The quantum vertex operator ${}_q\Phi(\zeta)$ consists of two commuting multipliers ${}_q\Phi(\zeta) = \Phi_a(\zeta) \cdot \Phi_b(\zeta)$. It is easy to verify that the product of two operators (4.3) with different parameters ζ_1 and ζ_2 satisfies

$$ \Phi_a(\zeta_1)\Phi_a(\zeta_2) = X_q^{unreg}(x) : \Phi_a(\zeta_1)\Phi_a(\zeta_2) :, \tag{4.9} $$

and

$$ \Phi_a(\zeta_1)\Phi_a(\zeta_2) = \frac{X_q^{unreg}(x)}{X_q^{unreg}(x^{-1})} \Phi_a(\zeta_2)\Phi_a(\zeta_1), \tag{4.10} $$

where the columns denote the normal ordering which means that all $\{a_{-n}, n \in \mathbb{N}\}$ generators are moved to the left with respect to $\{a_n, n \in \mathbb{N}\}$ generators. The function $X_q^{unreg}(x)$ (the superscript "unreg" meaning will be explained later) is given by

$$X_q^{unreg}(x) = exp\left(-\sum_{n=1}^{\infty} \frac{q^n x^{2n}[n]}{n[2n]}\right), \tag{4.11}$$

where

$$x^2 = \frac{\zeta_2}{\zeta_1}, \tag{4.12}$$

i.e., $\zeta_n = e^{\frac{4\pi\theta_n}{\gamma}}$.

Indeed, permuting the second exponential of the operator of $\Phi_a(\zeta_1)$ with the first exponential of $\Phi_a(\zeta_1)$ (see (4.3)), we obtain the interaction operator $exp\left(-\sum_{n=1}^{\infty} \frac{q^n x^{2n}}{n[2n]} \frac{\gamma^k - \gamma^{-k}}{q-q^{-1}}\right)$ that acts by the operator γ on a level one representation vector, giving (4.11). Similarly, the operator $\Phi_b(\zeta)$ permutes as

$$\Phi_b(\zeta_1)\Phi_b(\zeta_2) = X_{\widetilde{q}}^{unreg}(\widehat{x}) : \Phi_b(\zeta_1)\Phi_b(\zeta_2) :, \tag{4.13}$$

$$\Phi_b(\zeta_1)\Phi_b(\zeta_2) = \frac{X_{\widetilde{q}}^{unreg}(\widehat{x})}{X_{\widetilde{q}}^{unreg}(\hat{x}^{-1})}\Phi_b(\zeta_2)\Phi_b(\zeta_1), \tag{4.14}$$

where $\widehat{x} = x^{-\frac{\pi}{\mu}}$ and

$$X_{\widetilde{q}}^{unreg}(\widehat{x}) = exp\left(-\sum_{n=1}^{\infty} \frac{\widetilde{q}^{n(2(1+2m)-\frac{\mu}{\pi})} \quad \widehat{x}^{2n} \quad [\frac{\mu n}{\pi}]_{\widetilde{q}}}{n[2n]_{\widetilde{q}}}\right). \tag{4.15}$$

Since $\{a_k, k \in \{\mathbb{Z} - 0\}\}$ commute with $\{b_l, l \in \{\mathbb{Z} - 0\}\}$, the result of commutation of two operators ${}_q\Phi(\zeta_1)$ and ${}_q\Phi(\zeta_2)$ is

$${}_q\Phi(\zeta_1){}_q\Phi(\zeta_2) = \frac{X_q^{unreg}(x)}{X_q^{unreg}(x^{-1})} \frac{X_{\widetilde{q}}^{unreg}(\widehat{x})}{X_{\widetilde{q}}^{unreg}(\widehat{x}^{-1})} {}_q\Phi(\zeta_2){}_q\Phi(\zeta_1). \tag{4.16}$$

Let us denote by

$$X_q^{reg}(x) = X_q^{unreg}(x) \cdot X_{\widetilde{q}}^{unreg}(\widehat{x}) \tag{4.17}$$

the regularized quantum interaction function; the meaning of "regularized" will be clarified later. Then

$${}_q\Phi(\zeta_1){}_q\Phi(\zeta_2) = \frac{X_q^{reg}(x)}{X_q^{reg}(x^{-1})} {}_q\Phi(\zeta_2){}_q\Phi(\zeta_1). \tag{4.18}$$

The crucial point is that the function $X_q^{reg}(x)$ in (4.17) coincides with so-called quantum interaction function $X_q(x)$ (3.7) introduced in [18]. In order to show that, let us use formula (F.4) (see Appendix F) which splits the regularized quantum dilogarithm into the product of two unregularized quantum dilogarithms

$$S_{q^{-2}}^{reg}(x) = S_{q^{-2}}^{unreg}(x) \cdot S_{\widetilde{q^{-2}}}^{unreg}(\widehat{x}). \tag{4.19}$$

Then we obtain

$$X_q(x) = \frac{S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2q^2)}{S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2)} \frac{S_{\widetilde{q^{-2}}}^{unreg}(e^{\widehat{i\pi+2mi\pi}}x^2q^2)}{S_{\widetilde{q^{-2}}}^{unreg}(e^{\widehat{i\pi+2mi\pi}}x^2)}. \tag{4.20}$$

This is the reason why we call the interaction function $X_q^{unreg}(x)$ (4.11) and $X_q^{reg}(x)$ (4.17) unregularized and regularized, respectively. Note that here $q^{-2} = e^{i\mu}$, $\widetilde{q} = q^{\frac{\pi^2}{\mu^2}}$ and $(\widehat{e^{i\pi+2mi\pi}} x^2 q^2) = (e^{i\pi+2mi\pi} x^2 q^2)^{-\frac{\pi}{\mu}}$ in (4.20).

Consider the first ratio in (4.20). Substituting the definition of the unregularized quantum dilogarithm (F.3), we obtain

$$
\begin{aligned}
\frac{S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2q^2)}{S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2)} &= exp\left(\sum_{k=1}^{\infty} \frac{q^k x^{2k}(q^k - q^{-k})}{k(q^{-2k} - q^{2k})}\right) \\
&= exp\left(-\sum_{k=1}^{\infty} \frac{q^k x^{2k}[k]}{k[2k]}\right) \\
&= X_q^{unreg}(x).
\end{aligned}
\tag{4.21}
$$

The second multiplier in (4.20) can be rewritten as

$$
\begin{aligned}
\frac{S_{\widetilde{q^{-2}}}^{unreg}(\widehat{e^{i\pi+2mi\pi}}x^2q^2)}{S_{\widetilde{q^{-2}}}^{unreg}(\widehat{e^{i\pi+2mi\pi}}x^2)} &= exp\left(-\sum_{k=1}^{\infty} \frac{\widetilde{q}^{k(2(1+2m)-\frac{\mu}{\pi})} \quad \widehat{x}^{2k} \quad [\frac{\mu k}{\pi}]_{\widetilde{q}}}{k[2k]_{\widetilde{q}}}\right) \\
&= X_{\widetilde{q}}^{unreg}(\widehat{x}).
\end{aligned}
\tag{4.22}
$$

We have used the fact that $\widehat{q^2} = -1$ and $(\widehat{e^{i\pi}}) = \widetilde{q}^2$.

In [18] there was discovered a remarkable connection between the elements of the exact quantum S-matrix for the sine-Gordon (see Appendix E) and the quantum interaction function $X_q(x)$. The S-matrix elements $S(\theta)$, $S_T(\theta)$, $S_R(\theta)$ associated with soliton-soliton (antisoliton-antisoliton) and soliton-antisoliton scattering transmission and reflection processes have a common multiplier $\frac{X_q(x)}{X_q(x^{-1})}$ (see (3.3),(3.4-3.6)) which is a ratio of two quantum interaction functions. We see that the commutation of two quantum vertex operators ${}_q\Phi(\zeta_n)$, $n = 1, 2$ (4.2) gives precisely that common factor in the right hand side of (4.16). That is, up to a scalar function, our operators (4.2) satisfy the algebra of quantum soliton operators introduced in [12] (see Appendix E).

Let us get convince now that $\left(e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}\right)$-part of the quantum vertex operator (4.1) corresponds to the above-mentioned scalar (R-matrix part) function. Consider the commutation of two operators of the form $\left(e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}\right)$ which act on the second part of the tensor product in the highest weight representation vectors $|1 \otimes e^{\frac{\alpha n}{2}} >$ of $U_q(\widehat{sl_2})$ (see Appendix B). We have

$$
\left(e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}\right)\left(e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}\right).e^{\beta} = \left(\frac{\zeta_2}{\zeta_1}\right)^{\frac{1}{2}(\alpha,\frac{\alpha}{2})}\left(e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}\right)\left(e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}\right).e^{\beta}. \tag{4.23}
$$

The normalization of the root lattice vector α is $(\alpha, \alpha) = 2$. Therefore, $\left(\frac{\zeta_2}{\zeta_1}\right)^{\frac{1}{2}(\alpha,\frac{\alpha}{2})} = \left(\frac{\zeta_2}{\zeta_1}\right)^{\frac{1}{2}} = \frac{1}{x}$. Then we obtain

$$
\left(e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}\right)\left(e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}\right) = \frac{1}{x}\left(e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}\right)\left(e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}\right). \tag{4.24}
$$

The S-matrix element corresponding to the scattering of two solitons (or two antisolitons) is

$$S(\theta) = -q\frac{xq - x^{-1}q^{-1}}{1 - q^2x^2}\frac{X_q(x)}{X_q(x^{-1})} = \frac{1}{x}\frac{X_q(x)}{X_q(x^{-1})}. \tag{4.25}$$

Therefore, the vertex operator (4.1) satisfies the algebra of quantum soliton operators in the sense of [12].

In the classical case the sine-Gordon vertex operators possess some important properties. In particular, they are eigenvectors with respect to the action of elements of the Heisenberg subalgebra of $U_q(\widehat{sl_2})$. That is the property which allows one to commute vertex operators in the exponential expressions in solution (D.5) to the sine–Gordon equation (see Appendix D). Moreover, the square of the vertex operator vanishes. This fact is responsible for the termination of the exponential series in (D.5). Then, this solution coincides with the well-known soliton solution to the sine-Gordon equation [23].

The quantum vertex operator ${}_q\Phi(\zeta)$ has similar properties. First of all, it is easy to see that ${}_q\Phi(\zeta)$ is an eigenvector with respect to the action of the generators $a_{\pm k}$ of $U_q(\widehat{sl_2})$ (recall also that $a_{\pm k}$ commute with $b_{\pm n}$ for every $k, n \in \{\mathbb{Z} - 0\}$),

$$[a_{+k}, {}_q\Phi(\zeta)] = q^{\frac{7k}{2}}\frac{[k]}{k}\zeta^k {}_q\Phi(\zeta), \tag{4.26}$$

$$[a_{-k}, {}_q\Phi(\zeta)] = q^{-\frac{5k}{2}}\frac{[k]}{k}\zeta^{-k} {}_q\Phi(\zeta). \tag{4.27}$$

Secondly, the operator ${}_q\Phi(\zeta)$ satisfies

$${}_q\Phi(\zeta_1){}_q\Phi(\zeta_2) = X_q(x) : {}_q\Phi(\zeta_1){}_q\Phi(\zeta_2) : . \tag{4.28}$$

Moreover,

$$X_q(x) = X_q^{unreg}(x) \cdot X_{\widetilde{q}}^{unreg}(\widehat{x}), \tag{4.29}$$

and

$$X_q^{unreg}(x) = \frac{S_{q^{-2}}^{unreg}(e^{i\pi + 2mi\pi}x^2q^2)}{S_{q^{-2}}^{unreg}(e^{i\pi + 2mi\pi}x^2)}, \tag{4.30}$$

where

$$S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2q^2) = \prod_{k=0}^{\infty}\Big(1 + q^{-2(2k+1)}(e^{i\pi+2mi\pi}x^2q^2)\Big). \tag{4.31}$$

Therefore, when $\zeta_1 = \zeta_2$, i.e., $x = 1$, the first multiplier ($k = 0$) in the product (4.31) is zero and thus, $S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2q^2)|_{x=1} = 0$. At the same time $S_{q^{-2}}^{unreg}(e^{i\pi+2mi\pi}x^2)|_{x=1} \neq 0$. Similarly, $S_{\widetilde{q}^{-2}}^{unreg}(e^{i\pi+\widehat{2mi\pi}}x^2q^2)|_{x=1} = 0$ and $S_{q^{-2}}^{unreg}(e^{i\pi+\widehat{2mi}\pi}x^2)|_{x=1} \neq 0$. So, $X_q^{unreg}(1) = 0$ and $X_{\widetilde{q}^{-2}}^{unreg}(1) = 0$. Thus, we infer that $X_q(1) = 0$ which means that

$${}_q\Phi(\zeta) \cdot {}_q\Phi(\zeta) = 0. \tag{4.32}$$

These properties of the quantum vertex operators turn to be very useful in calculations of form-factors and correlation functions in the quantum sine-Gordon model. That will be a topic of a further paper. It is easy to see that in a classical limit the vertex operators (4.1) generate soliton solutions to the sine-Gordon equation. Indeed, take $q \to 1$. Then, due to (4.5) we have $\widetilde{q} = e^{\frac{1}{4l^2}}$, $l \in \mathbb{Z}$, i.e., when $l \to \pm\infty$, then $\widetilde{q} \to 1$ and one gets the soliton solution (D.15).

5. Antisoliton sector

The antisoliton sector in (E.5)-(E.6) can be obtained in a similar way. It is clear that a quantum antisoliton vertex operator should have almost the same form as ${}_qF(\zeta)$ in (4.1). The antisoliton solution in the classical case is given by

$$\phi_{antisol} = arctg\left(e^{-\frac{x-vt}{\sqrt{1-v^2}}}\right), \tag{5.1}$$

which can be obtained from the classical solution by changing the signs of x and t. On the base of the form of soliton solution (5.1), one can make a guess on a classical vertex operator that generates an antisoliton, and then construct a quantum vertex operator. We take

$${}_q\bar{F}(\zeta) = -{}_q\Phi(\zeta)e^{\frac{\alpha}{2}}\left(a(\zeta)\right)^{\frac{1}{2}\partial_\alpha}, \tag{5.2}$$

as such a quantum antisoliton vertex operator, where $a(\zeta)$ is some function. Then the permutation of two antisoliton operators gives

$${}_q\bar{F}(\zeta_1){}_q\bar{F}(\zeta_2) = \left(\frac{a(\zeta_1)}{a(\zeta_2)}\right)^{\frac{1}{2}} \frac{X_q^{reg}(x)}{X_q^{reg}(x^{-1})}{}_q\bar{F}(\zeta_2){}_q\bar{F}(\zeta_1). \tag{5.3}$$

Note that the operators $(a(\zeta_1))^{\frac{1}{2}\partial_\alpha}$ and $(a(\zeta_2))^{\frac{1}{2}\partial_\alpha}$ commute. Here we have made use of the formula

$$e^{\frac{\alpha}{2}}\left(a(\zeta)\right)^{\frac{1}{2}\partial_\alpha} = \left(a(\zeta)\right)^{-\frac{1}{2}(\alpha,\frac{\alpha}{2})}\left(a(\zeta)\right)^{\frac{1}{2}\partial_\alpha}e^{\frac{\alpha}{2}}. \tag{5.4}$$

But according to (E.5) antisolitons have the same scattering as solitons do. Therefore taking into account (4.25), we get a condition on the function $a(\zeta)$

$$\zeta_1 a(\zeta_2) = \zeta_2 a(\zeta_1). \tag{5.5}$$

The function $a(\zeta)$ can be determined with the help of rules (E.5)–(E.6). This function is given by

$$\left(a(\zeta)\right)^{\frac{1}{2}} = -\frac{1}{q}\zeta^{\frac{1}{2}}\left(\frac{1-q^2x}{1-x}\right), \tag{5.6}$$

where $x^2 = \frac{\zeta_2}{\zeta_1}$. In the limit $q \longrightarrow 1$ (5.2) tends to (D.16) (see Appendix D).

Indeed, commutation (E.6) of soliton and antisoliton operators [12] (see Appendix E) in terms of the S–matrix elements (we keep notations of [18]) leads to

$$\begin{aligned}
&{}_q\Phi(\zeta_1)e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}\left(-{}_q\Phi(\zeta_2)\right)e^{\frac{\alpha}{2}}\left(a(\zeta_2)\right)^{\frac{1}{2}\partial_\alpha} \\
&= q\tfrac{x-x^{-1}}{1-x^2q^2}\tfrac{X_q(x)}{X_q(x^{-1})}\left(-{}_q\Phi(\zeta_2)\right)e^{\frac{\alpha}{2}}\left(a(\zeta_2)\right)^{\frac{1}{2}\partial_\alpha}{}_q\Phi(\zeta_1)e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha} \\
&+\tfrac{q^2-1}{1-x^2q^2}\tfrac{X_q(x)}{X_q(x^{-1})}{}_q\Phi(\zeta_2)e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}\left(-{}_q\Phi(\zeta_1)\right)e^{\frac{\alpha}{2}}\left(a(\zeta_1)\right)^{\frac{1}{2}\partial_\alpha}.
\end{aligned} \tag{5.7}$$

Thus because of (5.4) equation (5.7) is equivalent to

$$\begin{aligned}
\zeta_1^{\frac{1}{2}\partial_\alpha}\left(a(\zeta_2)\right)^{\frac{1}{2}\partial_\alpha}\zeta_1^{\frac{1}{2}} &= q\frac{x-x^{-1}}{1-x^2q^2}\left(a(\zeta_2)\right)^{\frac{1}{2}}\left(a(\zeta_2)\right)^{\frac{1}{2}\partial_\alpha}\zeta_1^{\frac{1}{2}\partial_\alpha} \\
&+\tfrac{q^2-1}{1-x^2q^2}\zeta_2^{\frac{1}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}\left(a(\zeta_1)\right)^{\frac{1}{2}\partial_\alpha}.
\end{aligned} \tag{5.8}$$

Therefore, using (5.5) we see that the function (5.6) is a solution to (5.8). Thus we have constructed the quantum vertex operator (5.2) that corresponds (in the sense of Zamolodchikov algebra) to an antisoliton. The spectrum of soliton-antisoliton bound states is given by $m_{br}(n) = 2m_{sol}\sin\left(\frac{n\gamma}{16}\right)$, $n = 1, 2, ... < \frac{8\pi}{\gamma}$, [12], where m_{sol} is the mass of a soliton. Note that according to (3.5) the reflection part $S_R(x)$ vanishes when $q = \pm 1$, i.e., $n = \frac{8\pi}{\gamma}$ and $(a(\zeta))^{\frac{1}{2}} = \pm\zeta^{\frac{1}{2}}$. Taking into account (5.6) and considering the classical soliton solution corresponding to the quantum antisoliton vertex operator (5.2), one sees that there is no bound states when $\gamma > 8\pi$.

Conclusions

We have constructed an explicit representation of the quantum sine-Gordon vertex operators that satisfy the Zamolodchikov algebra, generalize usual vertex operators, and generate soliton solutions in the classical limit. These quantum vertex operators possess the quantum interaction function $X_q(x)$ introduced in [18]. One has to mention that some other variants of quantum vertex operators for the sine-Gordon model was constructed in [17,28]. Nevertheless, they look rather ugly from a group-theoretical point of view, and their relation to soliton-generating vertex operators in the classics is not clear. In our quantum vertex operator construction the role of the quantum group $U_q(\widehat{sl_2})$ is much more transparent. The structure of these operators is based on two quantized universal enveloping algebras with the deformation parameters q and $\widetilde{q}$ related by (4.5). Though we restrict ourselves to the simplest case among affine Toda theories (the sine-Gordon model), it is quite obvious that such a construction can be extended to higher algebras (the quantum interaction function for $U_q(\widehat{sl_3})$ was calculated in [18]).

A natural question that one can ask is what a quantum soliton might be. In the Zamolodchikovs picture of the quantum sine-Gordon theory one has the algebra of soliton-generating operators and a vacuum state they act on. The quantum solitons are the states of the theory and they may be related (in an appropriate classical limit) to classical sine-Gordon soliton solutions.

Another approach to quantum conformal or affine Toda theories consists in choosing one of standard ways of quantization, say, the light-cone method [10]. In that case one has to define the quantum affine (conformal) Toda equations (e.g., by specifying a normal ordering of exponentials). A solution to the quantum affine Toda equations should be a Heisenberg field operator. The experience in that direction shows that a formal general solution to such equations may be constructed on the base of quantum groups [9]. Since vertex operators generate soliton solutions to affine Toda equations in the classics, one can think that the quantum situation is somewhat analogous and may try to construct some special operator solutions using quantum vertex operators. Unfortunately, except for an explicit representation of quantum vertex operators little is known about their role in such a specialization. In the classics, soliton solutions can be extracted from the general one with the help of elements of the principal or homogeneous (see Appendix D) Heisenberg subalgebra of the corresponding affine Lie algebra. Considering the second Drinfeld realization [29,30,31,27] of $U_q(\widehat{\mathcal{G}})$, we find that the homogeneous Heisenberg subalgebra comes out naturally in that formulation (see Appendix B). In the quantum case one may think that the situation is similar. However, unfortunately, it is not completely clear how to extract the principal Heisenberg subalgebra from a quantized universal enveloping algebra $U_q(\widehat{\mathcal{G}})$. In this paper we have made use of the homogenous quantum Heisenberg subalgebra in order to define our quantum vertex operators.

As an application of the quantum vertex operators constructed in this paper one may think of calculations of form-factors [13,15,16,32] and correlation functions [33,34] of the quantum sine-Gordon model. In [17] a way to do that using vertex operators was proposed. It involves calculations of traces of some vertex operator products. Having an explicit and algebraically transparent representation of quantum vertex operators, one is in a position to calculate the form-factors of the theory. This will be the topic of a further paper. The other algebraic construction in the frames of the angular quantization approach to the sine-Gordon model has been recently presented in [14].

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Appendix

A. Affine Lie algebras

Let $\widehat{G}_0$ and $\widehat{G}_\pm$ be subspaces of the group $\widehat{G}$ corresponding to the subspaces to $\widehat{\mathcal{G}}_0$ and $\oplus_{m>1}\widehat{\mathcal{G}}_{\pm m}$ of the simple Lie algebra $\widehat{\mathcal{G}}$. Denote by h_i and $x_{\pm i}$ the Cartan and Chevalley generators, i.e., the elements of the subspaces $\widehat{\mathcal{G}}_0$ and $\widehat{\mathcal{G}}_{\pm 1}$ of $\widehat{\mathcal{G}}$ endowed with the principal gradation. They satisfy the defining relations

$$[h_i, h_j] = 0, \tag{A.1}$$

$$[h_i, x_{\pm j}] = \pm k_{ij} x_{\pm j}, \tag{A.2}$$

$$[x_{+i}, x_{-j}] = \delta_{ij} h_i, \tag{A.3}$$

$1 \le i, j \le r$; where k_{ij} are the elements of the Cartan matrix of $\widehat{\mathcal{G}}$. The generators in (A.1-A.3) also satisfy Serre relations. If $\widehat{\mathcal{G}}$ is an affine Kac–Moody Lie algebra, say of rank $r+1$, then the matrix k in (A.1- A.3) is affine, i.e., degenerated with a single zero eigenvalue. For more details on affine Lie algebras see [22]. It is convenient and traditional to enlarge the Cartan subalgebra of $\widehat{\mathcal{G}}$ by a derivative element d such that

$$[d, h_i] = 0, \qquad [d, x_{\pm i}] = \pm x_{\pm i},$$

and then the completed Cartan subalgebra has dimension $r+2$. Positive integers m_i in (2.2) are defined as the lowest for which

$$\sum_i k_{ji} m_i = 0. \tag{A.4}$$

Then

$$c = \sum_{i=0}^{r} m_i h_i, \tag{A.5}$$

belongs to the center of $\widehat{\mathcal{G}}$.

B. Second Drinfeld realization of quantized universal enveloping algebra $U'_q(\widehat{sl_2})$

A quantized universal enveloping algebra $U'_q(\mathcal{G})$ has, instead of (A.1 - A.3), the commutation relations, e.g., in Jimbo–Drinfeld form [29,30,31,36,27,37])

$$[h_i^q, h_j^q] = 0, \tag{B.1}$$

$$[h_i^q, x_{\pm j}^q] = \pm k_{ij} x_{\pm j}^q, \tag{B.2}$$

$$[x_i^q, x_j^q] = \delta_{ij} \frac{q_i^{h_i^q} - q_i^{-h_i^q}}{q_i - q_i^{-1}}, \tag{B.3}$$

where q_i is defined as $q_i = e^{d_i \hbar}$ in terms of the Planck constant $\hbar$ and coprime integers d_i such that $d\, k$ is a symmetric matrix. There are also analogues of Serre relations.

Let us recall the second Drinfeld realization of the quantized universal enveloping algebra $U'_q(\widehat{sl_2})$, (i.e., $U_q(\widehat{sl_2})$ without a grading operator) [29,30,27], which is a natural quantum analogue of the algebra $\widehat{sl_2}$ in the loop realizations. $U'_q(\widehat{sl_2})$ is an associative algebra generated by $\{x_k^\pm, k \in$

$\mathbb{Z}; a_n, n \in \{\mathbb{Z} - 0\}; \gamma^{\pm\frac{1}{2}}, K\}$, where $\gamma^{\pm\frac{1}{2}}$ belong to the center of the algebra, satisfying the commutation relations

$$[K, a_k] = 0, \tag{B.4}$$

$$[a_k, a_l] = \delta_{k,-l}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \tag{B.5}$$

$$K x_k^{\pm} K^{-1} = q^{\pm 2} x_k^{\pm}, \tag{B.6}$$

$$[a_n, x_k^{\pm}] = \pm\frac{[2n]}{n}\gamma^{\mp\frac{|n|}{2}} x_{n+k}^{\pm}, \tag{B.7}$$

$$[x_k^+, x_n^-] = \frac{\gamma^{(k-n)/2}\psi_{k+n} - \gamma^{(n-k)/2}\phi_{k+n}}{q - q^{-1}}, \tag{B.8}$$

$$x_{k+l}^{\pm} x_l^{\pm} - q^{\pm 2} x_l^{\pm} x_k^{\pm} = q^{\pm 2} x_k^{\pm} x_{l+1}^{\pm} - x_{l+1}^{\pm} x_k^{\pm}. \tag{B.9}$$

The generators ϕ_k and ψ_{-k}, $k \in \mathbb{Z}_+$ are related to a_k and a_{-k} by means of the expressions

$$\sum_{k=0}^{\infty} \psi_m z^{-m} = K exp\left((q - q^{-1})\sum_{n=1}^{\infty} a_k z^{-k}\right), \tag{B.10}$$

$$\sum_{k=0}^{\infty} \phi_{-m} z^{m} = K^{-1} exp\left(-(q - q^{-1})\sum_{n=1}^{\infty} a_{-k} z^{k}\right), \tag{B.11}$$

i.e.,

$$\psi_m = 0, \qquad m < 0, \tag{B.12}$$

$$\phi_m = 0, \qquad m > 0. \tag{B.13}$$

Here $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$.

It is easy to define the grading operators corresponding to the principal and homogeneous gradation of $U_q'(\widehat{sl_2})$ by analogy with the grading of $U_q'(\mathcal{G})$ where $\mathcal{G}$ is a simple Lie algebra [37]. The principal gradation can be realized with the help of the operator

$$D_p x = \frac{1}{2} q K^{-1}\left(\frac{d}{dq}(K x K^{-1})\right) K + 2\lambda\frac{d}{d\lambda}x, \tag{B.14}$$

where $x \in U_q(\widehat{sl_2})$ and λ is an affinization parameter. The power of λ is denoted by the subscript of $U_q'(\widehat{sl_2})$ generators. Then the grading subspaces are

$$\begin{aligned} {}_q\widehat{\mathcal{G}}_0 &= \{K, \gamma\}, \\ {}_q\widehat{\mathcal{G}}_{2n+1} &= \{x_n^+, x_{n+1}^-, n \in \mathbb{Z}\}, \\ {}_q\widehat{\mathcal{G}}_{2n} &= \{a_n, n \in \{\mathbb{Z} - 0\}\}. \end{aligned} \tag{B.15}$$

The grading operator for the homogeneous gradation is

$$D_h x = 2\lambda\frac{d}{d\lambda}x, \tag{B.16}$$

so that the grading subspaces are

$$
\begin{aligned}
&{}_q\widehat{\mathcal{G}}_0 = \{K, \gamma, x_0^+, x_0^-\}, \\
&{}_q\widehat{\mathcal{G}}_n = \{x_n^+, x_n^-, a_n, n \in \{\mathbb{Z} - 0\}\}.
\end{aligned}
\tag{B.17}
$$

The level one irreducible integrable highest weight representation of $U_q'(\widehat{sl_2})$ can be constructed in the following way [31,27]. Let $P = \mathbb{Z}\frac{\alpha}{2}$, $Q = \mathbb{Z}\alpha$ be the weight/root lattice of sl_2. Consider the group algebras $F[P]$, $F[Q]$ of P and Q. The multiplicative basis of $F[P]$ is formed by $e^{\frac{\alpha}{2}n}$, $n \in \mathbb{Z}$. The $F[Q]$-module is split into $F[P] = F[P]_0 \oplus F[P]_1$ where $F[P]_n = F[Q]e^{\frac{\alpha}{2}n}$. The sl_2-module structure on the space $W = F[a_{-1}, a_{-2}, ...] \otimes F[P]$ is given by the action of the $a_k, k \in \{\mathbb{Z} - 0\}$ and e^α, $\partial_\alpha = a_0$ generators in accordance with the rules

$$
\begin{aligned}
&a_k(f \otimes e^\beta) = (a_k f \otimes e^\beta), \quad k < 0, \\
&a_k(f \otimes e^\beta) = ([a_k, f] \otimes e^\beta), \quad k > 0, \\
&e^\alpha(f \otimes e^\beta) = (f \otimes e^{\alpha+\beta}), \\
&\partial_\alpha(f \otimes e^\beta) = (\alpha, \beta)(f \otimes e^\beta), \\
&K = 1 \otimes q^{\partial_\alpha}, \\
&\gamma = q \otimes id.
\end{aligned}
\tag{B.18}
$$

Then W is a $U_q'(\widehat{sl_2})$-module. Its submodules are isomorphic to irreducible highest weight modules $V(\Lambda_n)$ with the highest vectors $|\Lambda_n >= |1 \otimes e^{\frac{\alpha n}{2}} >$, $n = 0, 1$.

C. Conformal Toda systems

Let us recall some known results concerning conformal Toda systems in the classical region. The conformal abelian Toda fields $\phi = \sum_{i=1}^{r} h_i \phi_i$ (here h_i, $i = 1, ..., r$ denote the Cartan elements of $\mathcal{G}$) are associated with a complex simple Lie algebra $\mathcal{G}$ of rank r endowed with the principal gradation, and satisfy the equations

$$
\partial_+\partial_-\phi + \frac{4\eta^2}{\beta} \sum_{i=1}^{r} m_i \frac{\alpha_i}{\alpha_i^2} e^{\beta\alpha_i \cdot \phi} = 0 \tag{C.1}
$$

(m_i are the labels on the Dynkin diagram of $\mathcal{G}$). In (C.1) β is a coupling constant and η is a length scale factor. The general solution to the system (C.1) is represented in a holomorphically factorisable form [2]

$$
e^{-\beta\lambda_j \cdot \phi} =< \Lambda_j|\gamma_+^{-1}\mu_+^{-1}\mu_-\gamma_-|\Lambda_j > , \tag{C.2}
$$

where $\gamma_\pm(z_\pm) : \mathcal{M} \to G_0$, and $\mu_\pm(z_\pm) : \mathcal{M} \to G_\pm$ are holomorphic and anti–holomorphic mappings respectively; $|\Lambda_j >$, $j = 1, ..., r$ is the highest vector of the j–th fundamental representation of $\mathcal{G}$ labelled by the fundamental weight λ_j. The mappings $\mu_\pm(z^\pm)$ satisfy

$$
\partial_\pm\mu_\pm = \kappa_\pm\mu_\pm, \tag{C.3}
$$

where $\kappa_\pm$ realizes the mappings $\mathcal{M} \to \mathcal{G}_{\pm 1}$, i.e.,

$$\kappa_\pm(z^\pm) = \sum_{i=1}^{r} \Psi^{(0)}_{\pm i} \cdot x_{\pm i}, \tag{C.4}$$

(x_i, $i = 1, ..., r$ are Chevalley generators of $\mathcal{G}$) and

$$\Psi^{(0)}_{\pm j} = m_j e^{\mp\beta \sum\limits_{i=1}^{r} k_{ji}\phi^{(0)}_{\pm i}}, \tag{C.5}$$

where $\phi^{(0)}_{\pm i}$ are free fields.

D. Soliton solution of the sine-Gordon in homogeneous gradation

The other way to construct soliton solutions to the sine-Gordon equation is to consider the general solution (2.2) in the homogeneous gradation and to use vertex operators [22] which are related to the homogeneous Heisenberg subalgebra of $\widehat{sl_2}$. Take the general solution (2.2) to the affine Toda system (2.1). In the homogeneous gradation the mappings $\gamma_\pm$ can be parametrized as

$$\gamma_\pm = e^{d\phi_d} e^{c\phi_c} e^{\phi_0^\pm x_0^\pm}, \tag{D.1}$$

where d is the grading operator, c is the center of $\widehat{sl_2}$ and $x_k^\pm$ are the generators of the subspaces $\widehat{\mathcal{G}}_k$ corresponding to the homogeneous gradation. The mappings $\mu_\pm$ satisfy

$$\partial_\pm \mu_\pm = \kappa_\pm \mu_\pm, \tag{D.2}$$

where

$$\kappa_\pm(z^\pm) = a_{\pm 1} + \phi^\pm x_1^\pm. \tag{D.3}$$

In order to obtain a soliton solution we put

$$\phi^\pm = 0, \qquad \phi_0^\pm = 0. \tag{D.4}$$

Then the general solution reduces to

$$e^{-\beta\phi(z^+,z^-)} = \frac{< \Lambda_1 | e^{a_{+1}z^+} \mu(0) e^{a_{-1}z^-} | \Lambda_1 >}{< \Lambda_0 | e^{a_{+1}z^+} \mu(0) e^{a_{-1}z^-} | \Lambda_0 >}. \tag{D.5}$$

The following group element $\mu(0)$ in (D.5) generates the N–soliton solution

$$\mu(0) = e^{-\frac{\alpha}{2}N} \prod_{n=1}^{N} \left[exp\left((-1)^{\partial_\alpha + 1} i Q_n \Phi(\zeta_n) \right) e^{\frac{\alpha}{2}} \zeta_n^{\frac{1}{2}\partial_\alpha} \right], \tag{D.6}$$

where the action of the operators $\frac{1}{2}\partial_\alpha$ and $e^{\frac{\alpha}{2}}$ on the highest vectors $|\Lambda_n >= |1 \otimes e^{\frac{\alpha}{2}n} >$, $n = 0, 1$ is the same as in the case of $U_q'(\widehat{sl_2})$ (see Appendix B) [22] when $q = 1$.

The operator $\Phi(\zeta)$ is given by

$$\Phi(\zeta) = exp\left(\sum_{n=1}^{\infty} \frac{a_{-n}}{n} \zeta^n \right) exp\left(-\sum_{n=1}^{\infty} \frac{a_{+n}}{n} \zeta^{-n} \right), \tag{D.7}$$

and diagonalizes the action of $a_{\pm k}$, $k \in \mathbb{N}$, i.e.,

$$[a_{\pm k}, \Phi(\zeta)] = \zeta^{\pm k}\Phi(\zeta). \tag{D.8}$$

Also note that a product of two vertex operators can be normally ordered as

$$\Phi(\zeta_1)\Phi(\zeta_2) = X(x) : \Phi(\zeta_1)\Phi(\zeta_2) :, \tag{D.9}$$

where

$$X(x) = exp\left(-\sum_{n=1}^{\infty}\frac{x^{2n}}{n}\right) = exp\left(\log(1-x^2)\right). \tag{D.10}$$

When $x = 1$, $X(x)$ vanishes which has a consequence that

$$\Phi(\zeta)\cdot\Phi(\zeta) = 0. \tag{D.11}$$

Therefore, an exponentiation of the $\Phi(\zeta)$ operator terminates after the first order.

In the limit $q \longrightarrow 1$ soliton-soliton, antisoliton-antisoliton and soliton-antisoliton scattering reduce to the classical case. I.e., (E.4)-(E.6) (see Appendix E), degenerate to

$$F^a(\zeta_1)F^b(\zeta_2) = \frac{1}{x}\frac{X(x)}{X(x^{-1})}F^b(\zeta_2)F^a(\zeta_1), \tag{D.12}$$

where $x^2 = \frac{\zeta_2}{\zeta_1}$, a, b denote soliton (antisoliton), and the factor $\frac{1}{x}$ comes from the commutation of $e^{\frac{\alpha}{2}}\zeta_1^{\frac{1}{2}\partial_\alpha}$ and $e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}$ operators (see Section 3). Therefore, the vertex operator generating a classical soliton solution is

$$F(\zeta) = Q\Phi(\zeta)e^{\frac{\alpha}{2}}\zeta_2^{\frac{1}{2}\partial_\alpha}. \tag{D.13}$$

Taking into account the properties of the operator $F(\zeta)$, we rewrite the solution (D.5) as

$$e^{-\beta\phi(z^+,z^-)} = \frac{<\Lambda_1|\left(1+(-1)^{\partial_\alpha+1}iQ\Phi(\zeta)\right)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_1>}{<\Lambda_0|\left(1+(-1)^{\partial_\alpha+1}iQ\Phi(\zeta)\right)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}|\Lambda_0>}. \tag{D.14}$$

The final form of (D.14) is

$$e^{-\beta\phi(z^+,z^-)} = \frac{1+iQe^{\zeta z^+-\zeta^{-1}z^-}}{1-iQe^{\zeta z^+-\zeta^{-1}z^-}}\cdot\zeta. \tag{D.15}$$

The antisoliton solution can be associated with the vertex operator

$$\bar{F}(\zeta) = -Q\Phi(\zeta)e^{\frac{\alpha}{2}}\zeta^{\frac{1}{2}\partial_\alpha}. \tag{D.16}$$

E. Exact factorized S-matrix of quantum sine-Gordon model

The quantum sine-Gordon model has been considered in the framework of the S-matrix approach in [12]. The model is known as an example of the relativistic quantum field theory leading to the factorized scattering. In the special construction given in [12] some associative non-commuting operators $F_i(\theta_j)$ correspond to particles in the theory. The variable θ_j is associated with the rapidity and i in $F_i(\theta_j)$ denotes the kind of a particle. Asymptotic *in* and *out* states in the scattering theory consist of $F_i(\theta_j)$ operators product. Particles of the same kind are represented by identical operators; in this the statistics is not taken into account. The scattering of particles is described by the factorized S–matrix, i.e., the commutation of two operators depends on the S–matrix elements. Namely, for two particles of different masses one has (the reflection is forbidden in this case)

$$F_1(\theta_1)F_2(\theta_2) = S_T(\theta_{12})F_2(\theta_2)F_1(\theta_1). \tag{E.1}$$

When particles are of a different kind but have equal masses, we should also include the reflection process

$$F_1(\theta_1)F_2(\theta_2) = S_T(\theta_{12})F_2(\theta_2)F_1(\theta_1) + S_R^{1,2}(\theta_{12})F_1(\theta_2)F_2(\theta_1), \tag{E.2}$$

which gives

$$F_1(\theta_1)F_2(\theta_2) = S(\theta_{12})F_2(\theta_2)F_1(\theta_1), \tag{E.3}$$

when particles are identical. Thus, any product of operators identified with a state can be rearranged by means of rules (E.1)-(E.3). Each commutation of two operators denotes a two-particles collision.

The spectrum of the quantum sine-Gordon model consists of solitons, antisolitons and soliton-antisoliton bound states (quantum doublets). In terms of operators, the commutation rules for the soliton-soliton, antisoliton-antisoliton and soliton-antisoliton scattering can be written as

$$A(\theta_1)A(\theta_2) = S(\theta_{12})A(\theta_2)A(\theta_1), \tag{E.4}$$

$$\bar{A}(\theta_1)\bar{A}(\theta_2) = S(\theta_{12})\bar{A}(\theta_2)\bar{A}(\theta_1), \tag{E.5}$$

$$A(\theta_1)\bar{A}(\theta_2) = S_T(\theta_{12})\bar{A}(\theta_2)A(\theta_1) + S_R(\theta_{12})A(\theta_2)\bar{A}(\theta_1), \tag{E.6}$$

where $A(\theta_i)$ and $\bar{A}(\theta_i)$ denote a soliton and an antisoliton states, respectively. Soliton-antisoliton bound states and the soliton scattering have been investigated in the semiclassical approach, see references in [12].

The calculation of the S-matrix elements in [12] was based on the S-matrix analytical properties following from general principles of the quantum field theory and on the semiclassical data analysis. The S-matrix elements $S(\theta)$, $S_T(\theta)$ and $S_R(\theta)$ are given by

$$S_T(\theta) = -i\frac{\sinh\left(\frac{8\pi}{\gamma}\theta\right)}{\sin\left(\frac{8\pi^2}{\gamma}\theta\right)}S_R(\theta), \tag{E.7}$$

$$S(\theta) = -i\frac{\sinh\left(\frac{8\pi}{\gamma}(i\pi-\theta)\right)}{\sin\left(\frac{8\pi^2}{\gamma}\theta\right)}S_R(\theta), \tag{E.8}$$

$$S_R(\theta) = \frac{1}{\pi}\sin\left(\frac{8\pi^2}{\gamma}\theta\right)U(\theta), \tag{E.9}$$

where $\gamma = \frac{|\beta|^2}{1-\frac{|\beta|^2}{8\pi}}$ is the renormalized coupling constant

$$U(\theta) = \Gamma\left(\frac{8\pi}{\gamma}\right)\Gamma\left(a + i\frac{8\theta}{\gamma}\right)\Gamma\left(1 - \frac{8\pi}{\gamma} - i\frac{8\theta}{\gamma}\right)\prod_{k=1}^{\infty}\frac{R_n(\theta)R_n(i\pi-\theta)}{R_n(0)R_n(i\pi)}, \tag{E.10}$$

$$R_n(\theta) = \frac{\Gamma(2n\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma})\Gamma(1 + 2n\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma})}{\Gamma((2n+1)\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma})\Gamma(1 + (2n-1)\frac{8\pi}{\gamma} + i\frac{8\theta}{\gamma})}. \tag{E.11}$$

F. Quantum dilogarithm

The regularized version of the quantum dilogarithm was introduced in [25]

$$S_q^{reg}(w) = exp\left(\frac{1}{4}\int_{-\infty}^{+\infty}\frac{dx}{x}\frac{(w)^{-ix}}{\sinh(\pi x)\sinh(\mu x)}\right), \tag{F.1}$$

where $q = e^{i\mu}$, and a contour of integration in (F.1) goes above the pole at the origin. The quantum dilogarithm satisfies the defining property

$$\frac{S_q^{reg}(qw)}{S_q^{reg}(q^{-1}w)} = \frac{1}{1+w}. \tag{F.2}$$

The unregularized version of the quantum dilogarithm [24,25]

$$S_q^{unreg}(w) = \prod_{k=0}^{\infty}(1+q^{2k+1}w) = exp\left(\sum_{k=1}^{\infty}\frac{(-w)^k}{k(q^k-q^{-k})}\right), \tag{F.3}$$

satisfies the same defining property (F.2), but the first expression in (F.3) one converges when $|q| < 1$, while the second converges when $|q| \neq 1$, and $|w| < 1$. The unregularized quantum dilogarithm is related to the regularized one by the formula

$$S_q^{reg}(x) = S_q^{unreg}(x) \cdot S_{\widetilde{q}}^{unreg}(\widehat{x}), \tag{F.4}$$

where $\widetilde{q} = q^{\frac{\pi^2}{\mu^2}}$ and $\widehat{x} = x^{-\frac{\pi}{\mu}}$. Equation (F.4) can be easily verified by means of the residue calculation. Note that the unregularized dilogarithm divergencies at $|q| = 1$ cancel each other in (F.4).

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