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UNIVERSAL CALABI-YAU ALGEBRA: TOWARDS AN UNIFICATION OF GEOMETRY WITH SU(N) HOLONOMY

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Abstract

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We discuss some results in Calabi-Yau universal-gebra suitable for constructing and classifying the infinite series of the compact complex spaces with $SU(n)$ holonomy. This universal-gebraic approach includes natural extensions of reflexive weight vectors to higher dimensions. It includes a 'dual' construction based on the Diophantine decomposition of invariant monomials, which provides explicit recurrence formulae for the numbers of Calabi-Yau spaces in arbitrary dimensions.

Аннотация

Ансельмо Ф., Масликов А., Волков Г. Универсальная Калаби–Яу алгебра: к единой геометрии с $SU(n)$ голономией: Препринт ИФВЭ 2002–39. – Протвино, 2002. – 22 с., 9 рис., 3 табл., библиогр.: 12.

Мы обсуждаем некоторые результаты в области Калаби-Яу универсалгебры, которые могут быть использованы для построения и классификации бесконечных серий компактных комплексных пространств с $SU(n)$ голономией. Наш универсалгебраический подход включает естественные расширения рефлексивных весовых векторов в высших размерностях. Это включает дуальную конструкцию, основанную на Диофантовом разложении инвариантных мономов, что дает явные рекурентные формулы для множеств пространств Калаби-Яу в произвольной размерности.

Dedicated to the memories of Maria Volkova and Julia Fadeeva

1. Introduction: an Algebraic Way to Unify Calabi-Yau Geometry

The way to find the correct extension of the Standard Model based on the quantum field theory and principle of gauge invariance could be considered by taken into account both types of geometry, compactified and uncompactified spaces with some dualities between them. The compactified spaces and their singularities are responsible for the origin of the internal symmetries and can naturally explain the principle of the local gauge Young-Mills symmetry. Correspondingly, the uncompactified geometry gives an origin of the external symmetries which are really exist as the global gauge symmetries, like Lorentz symmetry and also some disrete gauge symmetries, like P,T,C unified in CPT-theorem. The advantage of such geometrical approach is that there should also exist the duality between these two types of of symmetries used in the SM, the duality between the external and internal symmetries. Thus in such approach one can naturally overcome the no-go Coleman-Mandula-theorem.

To explain the dynamics of an appearence of electromagnetism was enough to introduce only one extra dimension with very famous circle topology. In geometry the equivalence between the U(1) symmetry and circle is very famous fact. To give the geometrical correspondence for YM-symmetries of the Standard Model, $SU(3) \times SU(2) \times U(1)$, needs to consider the geometry with more extra dimensions having more complicated topology. Also this compactified geometry gives the direct explanation of the principle of local gauge invariance. For this we could consider, for example, the Calabi-Yau foam-vacuum. The symplest singularity of this vacuum-geometry is connected with $U(1)^{em}$ local gauge symmetry ($U(1)$-Coulomb phase). To produce the W, Z -boson in this geometrical foam-vacuum, needs to produce the singularities corresponding to the Higgs phase. So the main goal of the studying of the compactified geometry is to understand the correspondence between the geometrical objects with its singularities and YM symmetries. Of course, on this way one should understand how singularities can explain when appear Coulomb or Higgs phases. The foam-vacuum structure should also geometrically explain the principle of maximal velocity of electromagnetic wave expansion. To solve these questions one should consider the geometrical objects with some universal properties. One can suggest that the property of universality is general for all space dimensions. In Riemannian geometry one could consider such property of n-dimensional compact spaces connected with holonomy symmetry.

It is already well known that the simplest compactification on the symmetric spaces with some isometry and corresponding holonomy groups were intensively used to extend the idea of Kaluza-Klein in the supergravity approach and then in compactification of five superstrings and M/F-theories. It seems that the symmetric spaces with the corresponding holonomy symmetries it is better to use for constructing the uncompactified geometry of our world. One can use the space-expansion cycles of some dimension to construct Cosmology. For example, the standard cosmology is based on the one three-dimensional cycle S, the evolution of which is determined by Einstein-Gilbert equation.

The notion of the holonomy group was induced by E.Cartan for classification of all Riemannian locally symmetric spaces. The holonomy group H is one of the main characteristic of an affine connection on a manifold M. The definition of holonomy group is directly connected with parallel transport along the piece-smooth path joining two points $x \in M$ and $y \in M$. The parallel transport for a connected n-dimensional manifold M with Riemannian metric g and Levi-Civita connection using the connection defines the isometry between the scalar products on the tangent spaces T_xM and T_yM at the points x and y. So for any point $x \in M$ one can represent the set of all linear automorphisms of the associated tangent spaces T_xM which are induced by parallel translation along x-based loop.

If a connection is locally symmetric then its holonomy group equals to the local isotropy subgroup of the isometry group G. Hence, the holonomy group classification of these connections is equivalent to the classification of symmetric spaces which was known completely long ago [1]. The full list of symmetric spaces is given by the theory of Lie groups through the homogeneous spaces $M = G/H$, where G is a connected group Lie acting transitively on M and H is a closed connected Lie subgroup of G, what determines the holonomy group of M. Symmetric spaces have transitive groups of isometries. The known examples of symmetric spaces are R^n, spheres S^n, CP^n etc.

For the first time, in 1955, Berger presented the classification of irreducibly acting matrix Lie groups occured as the holonomy of a torsion free affine connection [2]. The Berger list of non-symmetric irreducible Riemannian manifolds with the list of holonomy groups H of M one can see, for example, in [3] (see the Table 1 and Fig. 1).

Table 1. *Some examples of symmetric Riemannian spaces and the list of Berger classification for non-symmetric Riemannian spaces.*

M	G_{ISOM}	H_{Hol}	Dim_R	$metrics$
$SO(n+1)/SO(n)$	$SO(n+1)$	$SO(n)$	n	S^n
$O(n+1)/O(n)xO(1)$	$O(n+1)$	$O(n)$	n	RP^n
$SU(n)/SU(n-1)$	$SU(n)$	$SU(n-1)$	$2n-1$	S^{2n+1}
$SU(n)/SU(n-1)xU(1)$	$SU(n)$	$SU(n-1)xU(1)$	$2n-1$	$CP^n = S^{2n+1}/U(1)$
$Sp(n)/Sp(n-1)$	$Sp(n)$	$Sp(n-1)$	$4n-1$	S^{4n+3}
$Sp(n)/Sp(n-1)xSp(1)$	$Sp(n)$	$Sp(n-1)xSp(1)$	$4n-1$	$HP^n = S^{4n+3}/Sp(1)$
$M - general$	$-$	$SO(n)$	n	
$Kahler$	$-$	$U(n) \subset O(2n)$	n	
$Calabi - Yau$	$-$	$SU(n) \subset SO(2n)$	$2n$	
$Hyper - Kahler$	$-$	$Sp(n) \subset SO(4n)$	$4n$	
$quaternion - -Kahler$	$-$	$Sp(n)xSp(1) \subset SO(4n)$	$4n$	
$exceptional$	$-$	$G(2) \subset SO(7)$	7	
$exceptional$	$-$	$spin(7) \subset SO(8)$	8	
$exceptional$	$-$	$spin(9) \subset SO(8)$	16	

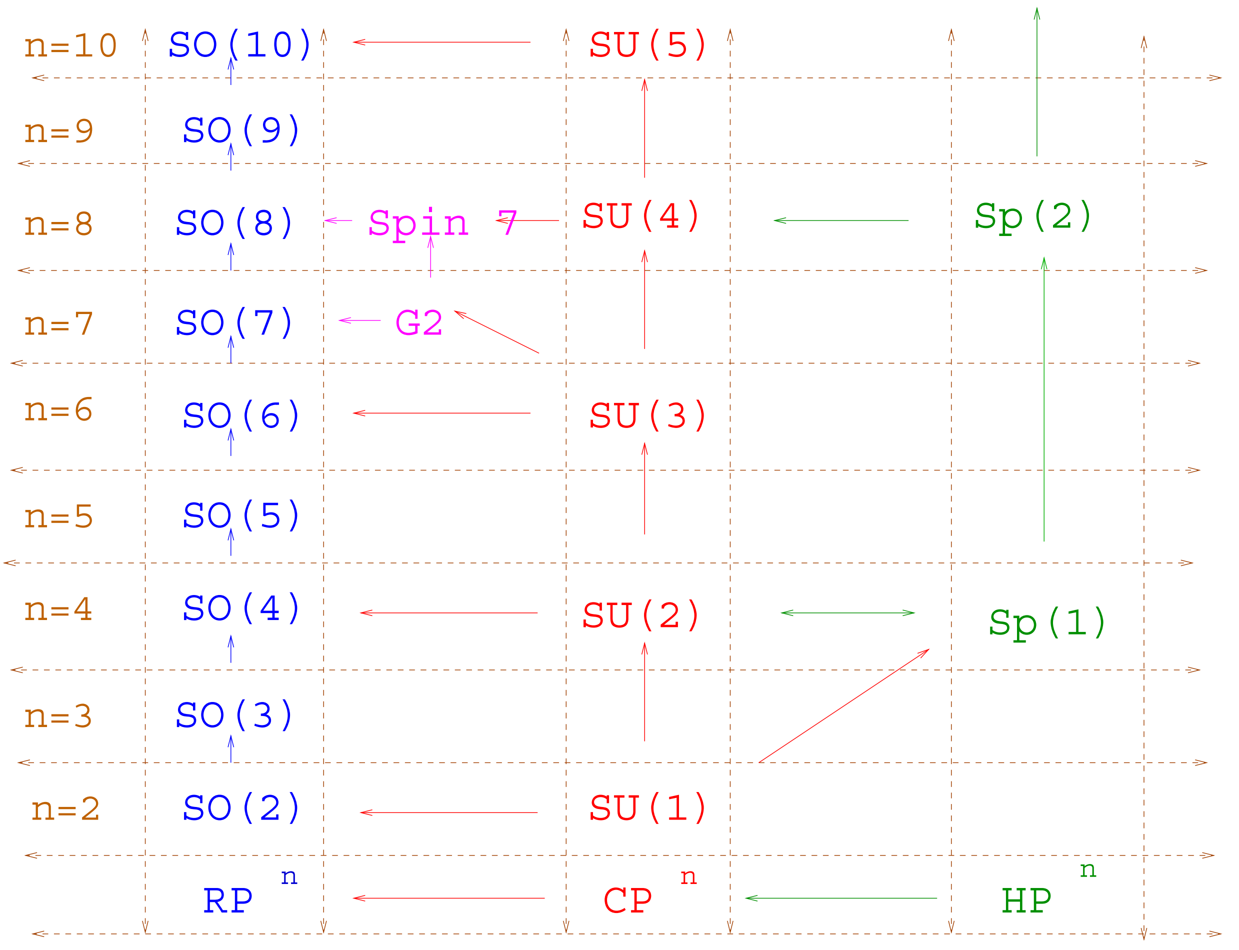

Fig. 1. *The three infinite series of the manifolds in RP^n, CP^n, HP^n with $O(n)$, $SU(n)$, $Sp(n)$ holonomy groups and two exceptional cases with holonomy groups, $G2$ and $Spin7$.*

For $H = SO(n)$ the holonomy principle means that there are not parallel (constant) tensor fields apart from metric and orientation. The next example $H = U(n) \subset SO(2n)$ is preserving apart from metric the complex structure J on R^{2n} which is parallel (constant) and orthogonal ($J \in SO(2n), J^2 = -1$). These manifolds with holonomy contained in $U(n)$ are Riemannian manifolds with a complex structure J called as Kähler manifolds.

We will accent here on the infinite series of Calabi-Yau spaces with $SU(n)$ holonomy group [4]. Following Joyce [3] it is better here to define the Calabi-Yau n-folds as a quadruple (M, J, g, Ω) where (M, J) is a complex compact n-dimensional manifold with complex structure J, g is a Kähler metrics with $SU(n)$-holonomy group, and Ω is a non-zero constant (parallel) $\Omega = (n, 0)$-tensor called by the holomorphic volume form (see Fig. 2).

In principle, it is enough to define the Calabi-Yau n-folds a little shorter *i.e.* a Calabi-Yau n-fold is a compact Kähler manifold (M, J, g) of dimension n with $SU(n)$ holonomy group. And then one can prove for Calabi-Yau n-folds the existence of constant (parallel) holomorphic $\Omega = (n, 0)$ form. More exactly, using the holonomy principle one can choose for each point $x \in M$ the complex coordinates $(z_1, ..., z_n)$ in which

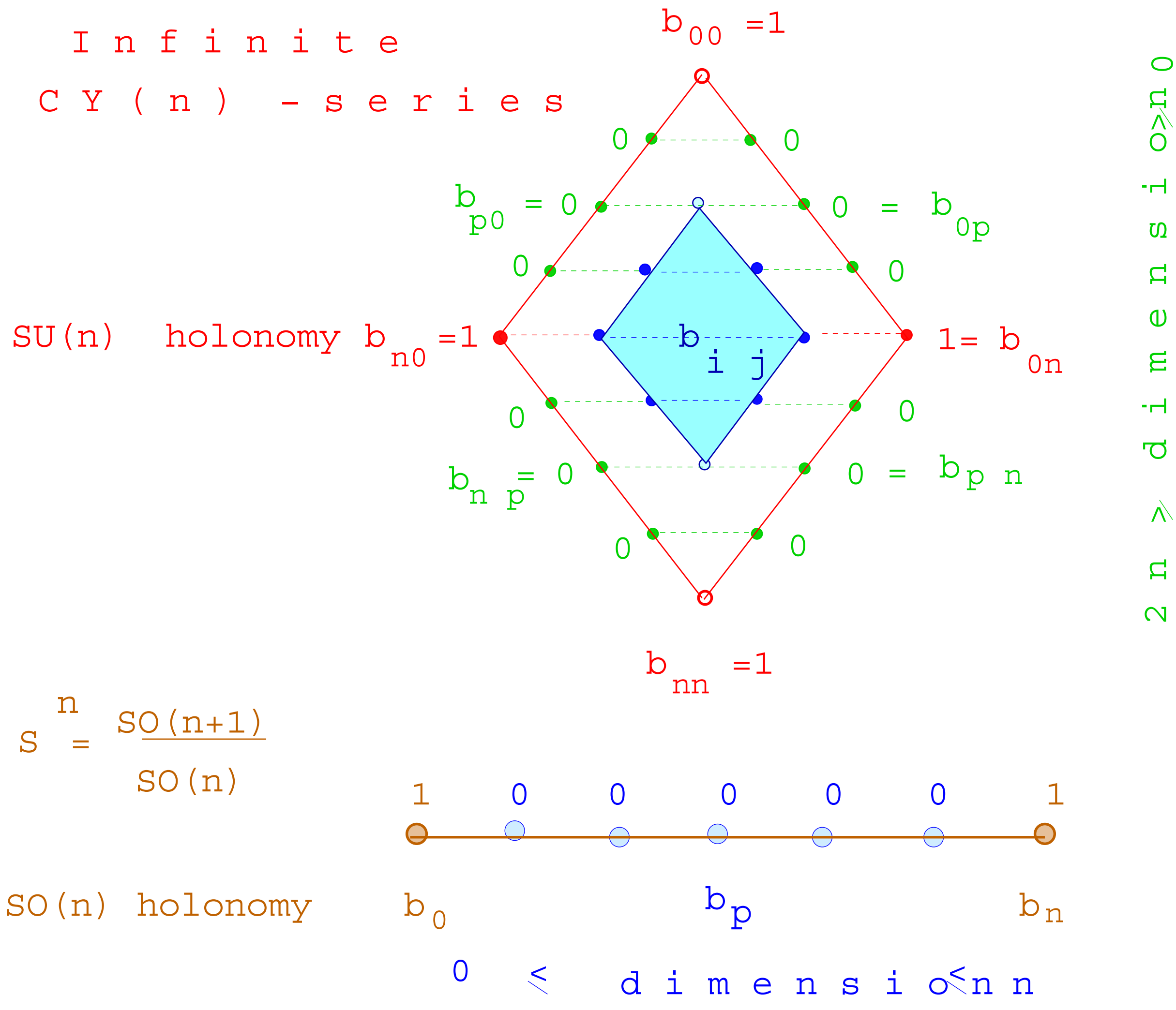

Fig. 2. *The general form of Betti-Hodge diamond defining the infinte series of CY spaces with SU(n) holonomy. For comparision we give the infinite series of symmetric spaces S^n with SO(n) holonomy.*

$$
\begin{aligned}
g &= |dz_1|^2 + \ldots |dz_n|^2, \\
\omega &= \frac{i}{2}(dz_1 \Lambda d\bar{z}_1 + \ldots + dz_n \Lambda d\bar{z}_n), \\
\Omega &= dz_1 \Lambda \ldots \Lambda dz_n, \qquad (1)
\end{aligned}
$$

where the form Ω is unique up to multiplication by $\exp(i\phi)$ for $\phi \in [0, 2\pi)$. The existence of a parallel form of type $(n, 0)$ means that the cannonical bundle $K_M := \Omega^n_M$ is flat. In other words, the Ricci curvature which for Kähler manifold is just the curvature of K_M is equal to zero. Due to Yau's proof of the Calabi conjecture one has the following: If (M, J) is a compact complex n-fold admitting Kähler metrics with trivial cannonical bundle then there exists a unique Ricci-flat metrics g for each Kähler class of M and the holonomy group is $H = SU(n)$. We would like to present the possibility for algebraic solution of this infinite Calabi-Yau series of compact complex n-folds with $SU(n)$ holonomy (see Fig. 3).

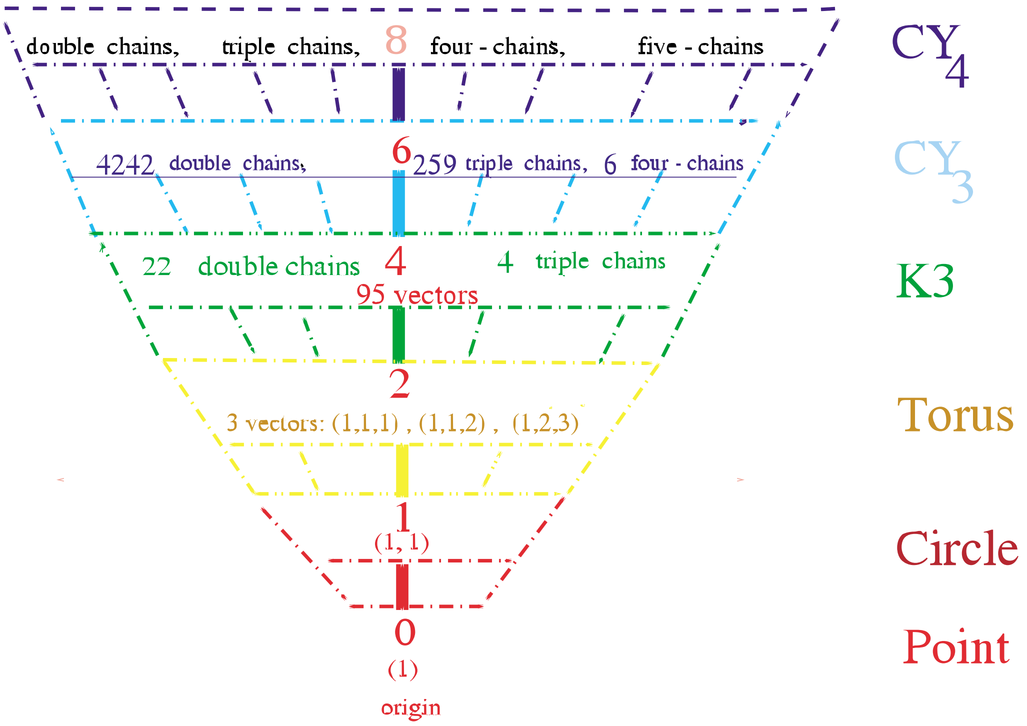

Fig. 3. *The genealogical tree for Calabi-Yau n-folds.*

2. The Arity-Dimension Structure of Universal Calabi-Yau Algebra (universal-gebra)

The starting point for our universal-gebraic approach to the classification of Calabi-Yau spaces has been the construction of 'reflexive' weight vectors $\vec{k}$, whose components specify the complex quasihomogeneous projective spaces $CP^n(k_1, k_2, ..., k_{n+1})$. These have $(n+1)$ quasihomogeneous coordinates $x_1, ..., x_{n+1}$, which are subject to the following identification:

$$(x_1, \ldots, x_{n+1}) \sim (\lambda^{k_1} \cdot x_1, \ldots, \lambda^{k_{n+1}} \cdot x_{n+1}). \tag{2}$$

In the case of CP^n projective spaces there exists a very powerful conjecture, called Chow's theorem, that each analytic compact (closed) submanifold in CP^n can be specified by a set of polynomial equations. The set of zeroes of quasihomogeneous polynomial equations, hereafter referred to as Calabi-Yau equations, define a projective algebraic variety in such a weighted projective space.

A d-dimensional Calabi-Yau space X_d can be given by the locus of zeroes of a transversal quasihomogeneous polynomial $\wp$ of degree $deg(\wp) = [d] : [d] = \sum_{j=1}^{n+1} k_j$ in a complex projective space $CP^n(\vec{k}) \equiv CP^n(k_1, ..., k_{n+1})$ [5]:

$$X \equiv X^{(n-1)}(k) \equiv \{\vec{x} = (x_1, ..., x_{n+1}) \in CP^n(k) | \wp(\vec{x}) = 0\}. \tag{3}$$

The general quasihomogeneous polynomial of degree $[d]$ is a linear combination

$$\wp = \sum_{\vec{\mu}_\alpha} c_{\vec{\mu}_\alpha} x^{\vec{\mu}_\alpha} \tag{4}$$

of monomials $x^{\vec{\mu}_\alpha} = x_1^{\mu_{1\alpha}} x_2^{\mu_{2\alpha}} ... x_{r+1}^{\mu_{(r+1)\alpha}}$ with the condition:

$$\vec{\mu}_\alpha \cdot \vec{k} \, = \, [d]. \tag{5}$$

This algebraic projective variety is irreducible if and only if its polynomial is irreducible. A hypersurface will be smooth for almost all choices of polynomials. To obtain Calabi-Yau d-folds one should choose reflexive weight vectors (RWVs), related to Batyrev's reflexive polyhedra or to the set of IMs. Other examples of compact complex manifolds can be obtained as the complete intersections (CICY) of such quasihomogeneous polynomial constraints:

$$X_{CICY}^{(n-r)} = \{\vec{x} = (x_1, \ldots x_{n+1}) \in CP^n \mid \wp_1(\vec{x}) = \ldots = \wp_r(\vec{x}) = 0\}, \tag{6}$$

where each polynomial $\wp_i$ is determined by some weight vector $\vec{k}_i$, $i = 1, \ldots, r$.

A useful technique for constructing Calabi-Yau spaces in any number of dimensions is to visualize the various possible monomials $(x_1^{\mu_1} x_2^{\mu_2} ... x_n^{\mu_{n+1}})_\alpha$ as the $m_\alpha = (\mu_1, ..., \mu_{n+1})_\alpha$ points in the Z_{n+1} integer lattice of an n-dimensional polyhedron. Using this technique, Batyrev [6] demonstrated how to associate by explicit construction a mirror polyhedron to each Calabi-Yau space. This approach also established in a very elegant way the corresponding mirror duality among Calabi-Yau spaces.

The Universal Calabi-Yau Algebra (UCYA) structure of reflexive weight vectors in different dimensions depends on two integer parameters: the 'arity' r of the combination operation ω_r, and the dimension n of the reflexive weight vectors (RWVs), that are connected one-to-one with Batyrev's reflexive polyhedra. These weight-vectors could be classified using the natural extensions of lower-dimensional vectors and their combination via binary, etc., operations (see Fig. 4). The innovation is the introduction of a complementary universal-gebraic approach to the construction of Calabi-Yau spaces, based on the construction of suitable monomials $\vec{\mu}$ obeying the 'duality' condition: $\vec{k} \cdot \vec{\mu}_\alpha = d$. This construction supplements the previous geometrical method related to Batyrev polyhedra, and enables one to calculate the numbers of eldest vectors, and hence chains, in arbitrary dimensions. We verify explicitly that the eldest vectors found in the two different ways agree in several instances for both CY_3 and CY_4 spaces, providing increased confidence in our results. The study of the Calabi-Yau equations and the associated hypersurfaces via the remarkable composite properties of IMs provides an alternative algebraic route to reflexive polyhedron techniques. Central rôles are played in our approach by the composite structures in lower dimensions $\leq (d-1)$ of CY d-folds, and the algebraically dual ways of expansions using weight vectors $\vec{k}$ and invariant monomials (IMs). By analogy with the Galois normal extension of fields, we term the first way of expanding weight vectors a *normal* extension, and the dual decomposition in terms of IMs we call the *Diophantine* expansion. These two expansion techniques are consistently combined in our universal-gebraic approach, whose composition rules exhibit explicitly the internal structure of the Calabi-Yau universal-gebra. Our method is closely connected to the well-known Cartan method for constructing Lie algebras, and reveal various structural relationships between the sets of Calabi-Yau spaces of different dimensions. We interpret our approach as revealing a 'Universal Calabi-Yau Algebra' (universal-gebra) [7] for the following reasons: 'Universal' because it may, in principle, be

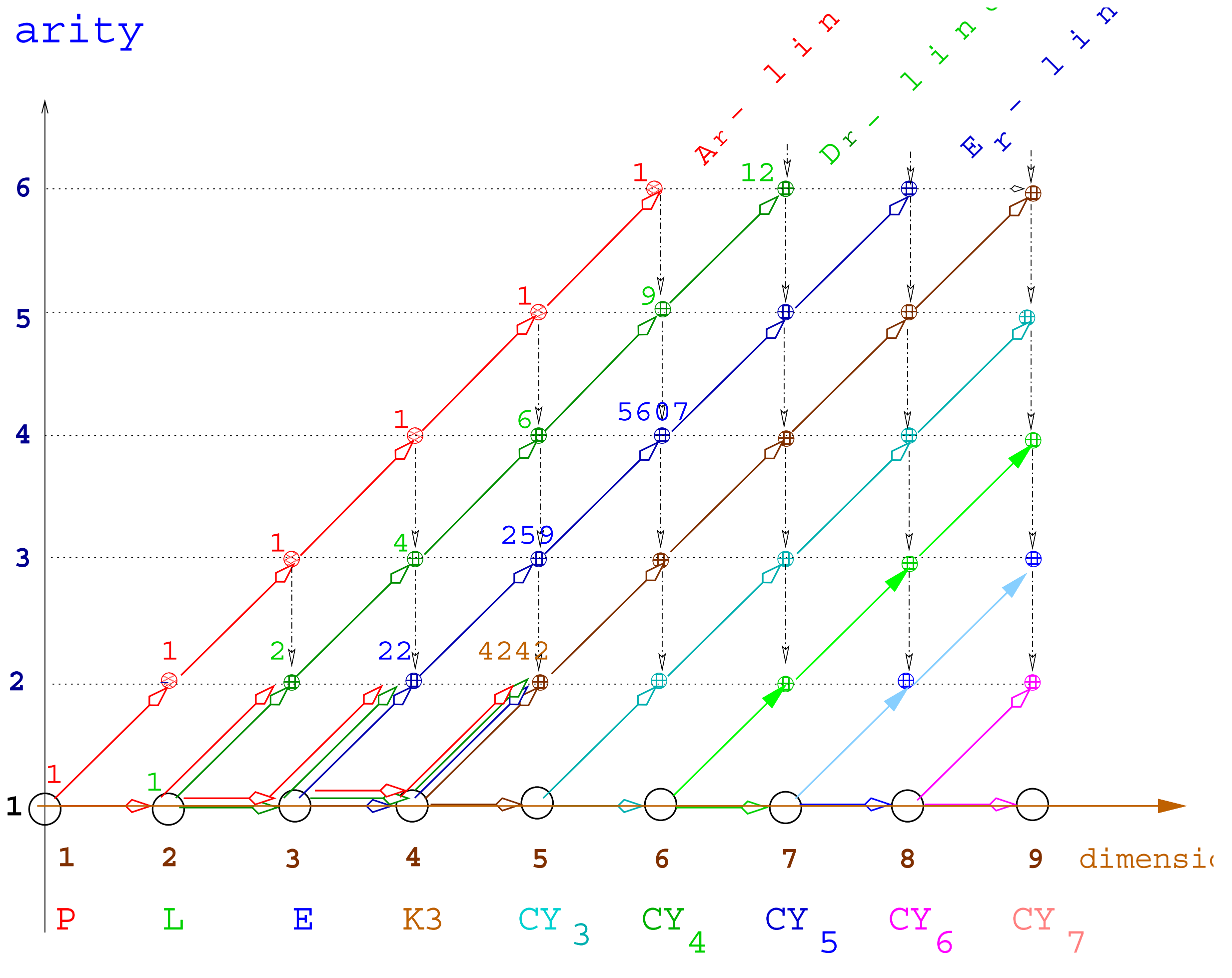

Fig. 4. *The arity-dimension plot, showing the numbers of eldest vectors/chains obtained by normal extensions of RWVs, including previous results for* CY_3 *and lower-dimensional spaces, and new results for* CY_4 *and* CY_5 *spaces.*

used to generate all Calabi-Yau manifolds of any dimension with all possible substructures, and 'Algebra' (n-gebra) because it is based on a sequence of binary and higher n-ary operations on weight vectors and monomials. We should stress that in this construction the n-ary composition does not reduce to the $(n-1)$-,...,2-ary composition rules.

Our objective is to construct an universal-gebra [7] acting on the set of reflexive weight vectors in all dimensions, $A_n \equiv \{\text{RWV}(n)\}$, and the corresponding set of invariant monomials, $\{IMs(n)\}$, which is 'dual' to A_n in the sense of (5). We note that the number of IMs is much less the full set of monomials $\vec{m}_\alpha : 1 \leq \alpha \leq \alpha_{max}$ which determine the Calabi-Yau equation. Through the IMs one can determine the highest vectors of the chains and also the full list of weight vectors in the corresponding chain. To see this, we start from the unit IM in some dimension n and then, via a Diophantine expansion, can go on to determine the conic IMs, the cubic IMs, the quartic IMs, etc.. Similarly, one can continue this process of studying the set of IMs via the Diophantine expansions of conic IMs, of cubic IMs, etc..

The RWVs and IMs provide independent routes for constructing explicitly Calabi-Yau spaces of arbitrary dimension (including CICYs). The resulting UCYA structure of RWVs in different dimensions depends on two integer parameters, including the 'arity' r defined below, as well as the dimension n. An overview in the (n, r) plane is shown in Fig. 4, where the entries $A_n^{(r)}$ label the types of possible eldest vectors, corresponding to 'chains' of related Calabi-Yau spaces.

The algebraic-geometry realization [5,8] of Coxeter-Dynkin diagrams provides a general characterization of the possible structures in singular limits of Calabi-Yau hypersurfaces. Thus, a deeper understanding of the origins of gauge invariance provides an additional motivation for studying string vacua via our unification of the complex geometry of $d = 1$ elliptic curves, complex tori, $K3$ manifolds, CY_3, CY_4, etc. This point is illustrated in Fig. 5, where the points on the the first three sloping lines, labelled A_r (red), D_r (green) and E (blue), correspond to those d-folds that are characterized by the 'maximal' quotient A, D, E singularities, respectively[1]. As we discuss later in more detail, this characterization of the types of singularities is directly connected to the degrees of the associated monomials - linear, conics, cubics, quartics, etc., that appear along the corresponding sloping lines.

3. Some Results

We have presented a Universal Calabi-Yau Algebra (UCYA) which provides a two-parameter classification of $CY - d$ spaces in terms of arity and dimension. This universal-gebra is based on the following ingredients:

- Universal composition rules
- Normal expansions and Diophantine decompositions
- Mirror symmetry
- McKay and UCYA- correspondences
- Singularities and link with Cartan-Lie algebras

We have shown that this universal-gebraic approach leads us to a natural formalism for a unified description of complex geometry in all dimensions, including $K3$ spaces and Calabi-Yau d-folds for any d.

Our construction of a Universal Calabi-Yau algebra (UCYA) is based on the two integer parameters, *arity* and the *dimension* of the reflexive weight vectors (RWVs). We discussed previously how these could be classified using the natural extensions of lower-dimensional vectors and their combination via binary, ternary, etc., operations. It was the reason why it is better to call this structure by universal-gebra. The main innovation in this paper is the introduction of a complementary universal-gebraic approach to the construction of Calabi-Yau spaces, based on the construction of suitable monomials $\vec{\mu}$ obeying the 'duality' condition: $\vec{k} \cdot \vec{\mu}_\alpha = d$. This 'dual' approach is based on suitable decompositions of invariant monomials (IMs) of given dimensionality, yielding eldest vectors that could only be obtained by higher-order n-ary operations in the previous approach. This construction supplements the previous geometrical method related to Batyrev polyhedra, and enables one to calculate the numbers of eldest vectors, and hence chains, in arbitrary dimensions. We verify explicitly that the eldest vectors found in the two different ways agree in several instances for both CY_3 and CY_4 spaces, providing increased confidence in our results.

[1] To be more precise, the D line includes also A-type singularities, and the E line includes also D-type and A-type singularities.

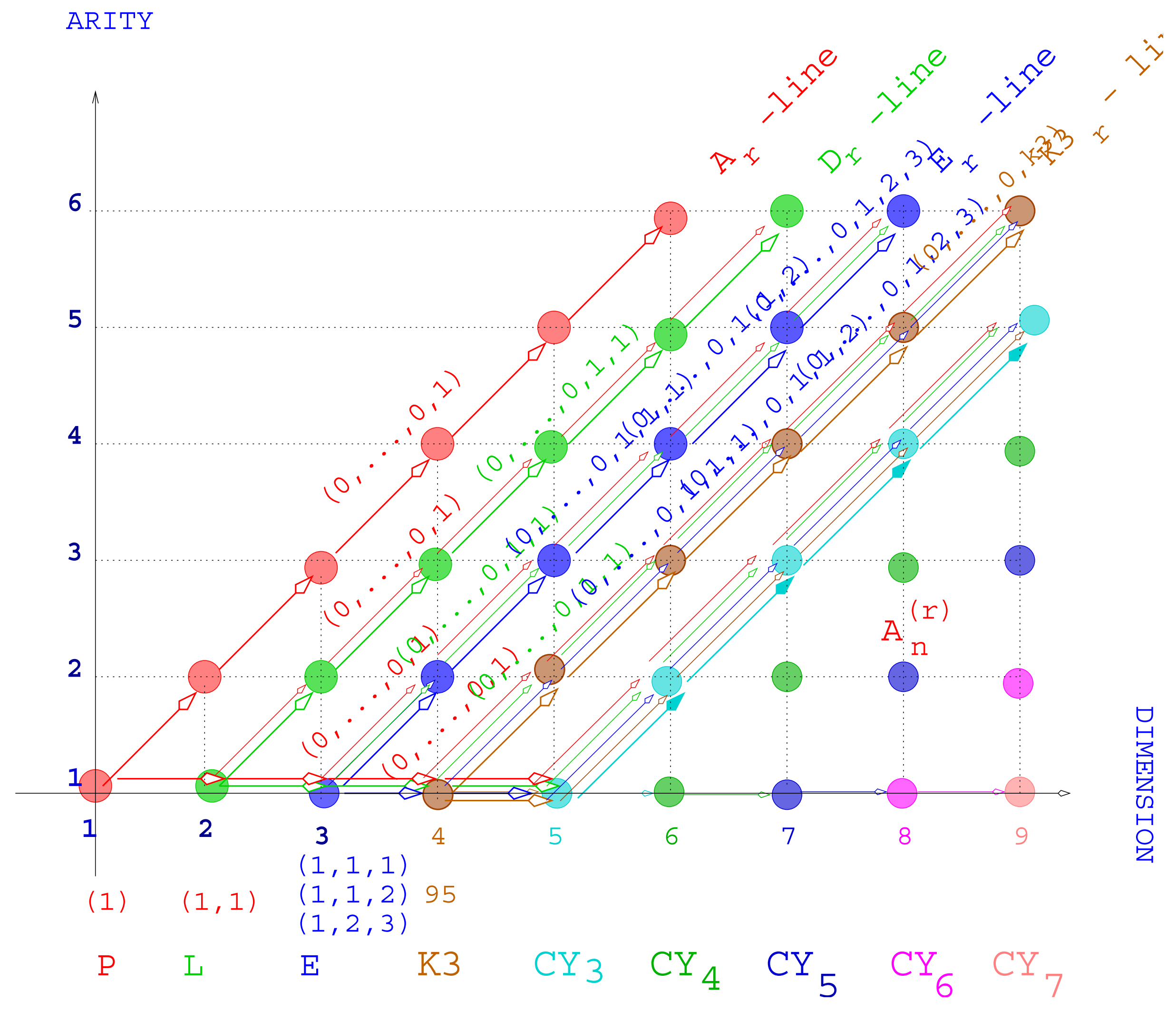

Fig. 5. *The arity-dimension plot illustrating the composite structure from the types of the extended weight-vectors for the all slope-lines.*

The study of the Calabi-Yau equations and the associated hypersurfaces via the remarkable composite properties of IMs provides an alternative algebraic route to reflexive polyhedron techniques. We recall that the arity-dimension parameter structure is directly connected to the singularity properties of Calabi-Yau hypersurfaces, and thereby to the types of Cartan-Lie algebras. Using these remarkable properties one can hope to decypher the Calabi-Yau genome in any dimension.

Since the description of the UCYA is based on structures with two integer parameters, the arity and dimension of the reflexive weight vectors (RWVs), we have classified the structures of CY_d spaces along the diagonal $A_r, D_r, E_r, ...$ lines in this plane. In this article we have studied only the d-folds along the first three lines, presenting new results for low d and some recurrence formulae valid for all d.

As an alternative to the Batyrev reflexive polyhedron method, we have proposed a new description of CY_d spaces based on the structures of the set of invariant monomials (IMs). We have shown that the IM approach, which is based on Diophantine decompositions, is a valuable alternative to the normal RWV expansion approach. We have demonstrated this by comparing the results of both approaches for the first three diagonal lines, A_r, D_r and E_r, in the arity-dimension plot for CY_3, CY_4 cases.

We have shown that recurrence relations for conic, cubic and quartic monomials give us the formulae for the numbers of IMs in arbitrary dimensions. This was illustrated in three cases, for CY_d spaces with $\{10\}_\Delta$, $\{9\}_\Delta$ and $\{7\}_\Delta$fibres. This confirms that, in the framework of the UCYA, the Calabi-Yau 'genome' can in principle be solved completely.

As an example of the extension procedure in the case of $K3$ manifolds, we classified [9] the 95 different possible weight vectors $\vec{k}$ in 22 binary chains generated by pairs of extended vectors, which included 90 of the total, and 4 ternary chains generated by triplets of extended vectors, which yielded 91 weight vectors of which 4 were not included in the binary chains. The one remaining $K3$ weight vector was found in a quaternary chain [9]. This algebraic construction provides a convenient way of generating all the $K3$ weight vectors, and arranging them in chains of related vectors whose overlaps yield further indirect relationships.

Moreover, our construction builds higher-dimensional Calabi-Yau spaces systematically out of lower-dimensional ones, enabling us to enumerate explicitly their fibrations. As examples, we showed previously [9,10] how our construction reveals elliptic and $K3$ fibrations of CY_3 manifolds. Our approach may also be used to obtain the projective weight vector structure of a mirror manifold, starting from those of a given Calabi-Yau manifold.

Table 2. *The invariant monomials (linear and quadratic) for the (1)+(11) RWVs extended to Weierstrass, K3, CY_3 and CY_4 spaces, corresponding to the D_r line on the arity-dimension Fig. 4. In the case of the A_r line, there can be only linear invariant monomials.*

P	*L*	*W*	*K3*	*K3*	*Qu*	*in*	*tic*	*Se*	*x*	*t*	*ic*
(1)	(11)	(111)	(1111)		(11111)			(111111)			
	(20)	(200)	(2000)		(20000)			(200000)			
		(210)	(2100)	(2110)	(21000)	(21100)	(21110)	(210000)	(211000)	(211100)	(211110)
		(220)	(2200)	(2210) (2220)	(22000)	(22100) (22200)	(22110) (22210) (22220)	(220000)	(221000) (222000)	(221100) (222100) (222200)	(221110) (222110) (222210) (222220)

We first explain the mechanism of Diophantine expansion for the invariant unit monomials $E_n = (1, ..., 1)_n$, which is relevant for the CY_d spaces with substructures corresponding to the second D_r line with arity $r = (n-1)$ in Fig. 5.

Due to our universal-gebraic description the second slope D_r line is connected to the almost trivial substructure of CY_d, i.e. to those CY_d having "circles" or whose reflexive polyhedra have line-reflexive polyhedra. This structure is determined by conics monomials which we can find through the Diophantine expansion of unit monom to the two conics, C_1 and C_2. For illustration we can give also here the table of conics monoms, which before we gave in the long last article [9]. Let us remind that first A_r-line is determined by unit monoms $E_n = (1, ...,)$ The structure of the CY_d on the next third E_r-line already will be determined by cubics and quartics IM and etc. (see Fig. 5).

We can determine all the possible conic IMs for this line, starting from the unit monomial E_n and two conic monomials, $C_{i(n)}$ and $C_{j(n)}$. These monomials should satisfy the following Diophantine property:

$$\frac{1}{2}(C_{i(n)} + C_{j(n)}) = E_n, \tag{7}$$

where the index n notes the dimension being considered. This Diophantine expansion yields the following numbers of possible different types of conic monomials in any dimension n,

$$N_{conics} = \frac{(n)(n-1)}{2}. \tag{8}$$

In order to enumerate the IMs and the corresponding chains of Calabi-Yau spaces, one starts from all possible pairs of conic monomials with the required Diophantine property, $r = (n-1)$, and solves the following equations:

$$\vec{k}^{i(ex)} \cdot C_{1(n)} = \vec{k}^{i(ex)} \cdot E_n = d(\vec{k}^{i(ex)}). \tag{9}$$

To give sense to these equations and, consequently, to evaluate the finite numbers of chains and their eldest vectors in the case of arity $r = (n-1)$, we first recall that, in the UCYA, the points on this line in the arity-dimension plane are determined by n-dimensional extensions of the two eldest vectors $\vec{k}_1 = (1)$ and $\vec{k}_2 = (1,1)$. This means that the possible values of $d(\vec{k}^{i(ex)})$ in these equations are only 1 and Also, the components of the extended vectors can only be 0 and 1. Due our algebra this second slope-line is determined only by extension of the weight vectors (1) and (1,1). So their dimensions can be only 1 or 2, respectively.

It is then simple to verify the following recurrence formula for the numbers of chains along this line:

$$\begin{aligned} N_{chains} &= k \cdot (k+1), \qquad if \;\; n = (2k+1), \\ N_{chains} &= k^2, \qquad if \;\; n = (2k). \end{aligned} \tag{10}$$

Thus, along the line $r = (n-1)$, the numbers of the eldest vectors and chains in dimensions $n = 2, 3, 4, ...$ are the following: 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, 36, 42, 49, 56, 64, 72, 81, 90, 100, 110, 121, 132, 144, ... Thus, if n=2, k=1—- $N_{chains} = k^2 = 1$, then for n=3 k=1, but $N_{chains} = k \cdot (k+1) = 2$. similarly, for n=4, k=2 and the corresponding formula is $N_{chains} = k^2 = 4$, for CY_3 n=5 and k=2, $N_{chains} = k \cdot (k+1) = 6$ etc.

Extending our previous approach to the third line in Fig. 5, the first step is to enumerate the cubic and quartic monomials, from which we can find all the IMs along this E_r line.

The appearance of cubic monomials is connected with the following new Diophantine condition for the expansion of the unit monomials E_n of the A_r line:

$$E_n \mapsto \{P_1, P_2, P_3 | \frac{1}{3}(P_1 + P_2 + P_3) = E_n\}. \tag{11}$$

However, the set of appropriate cubic monomials is somewhat more restricted. Similarly, the appearance of quartic monomials is connected with the possible Diophantine expansion of the conic monomials $C_{i(n)}$ of the second D_r line:

$$C_{i(n)} \mapsto \{P_1, P_2 | \frac{1}{2}(P_1 + P_2) = C_{i(n)}\}. \tag{12}$$

We would like explain the structure of all IMs through the compostion structure and here and everywhere we use the Diophantine mechanism for appearence of all higher dimension IM through the IM of the lower dimensions.

Table 3. *The invariant monomials (cubics and quartics) for the five weight vectors (1)+(11)+(111)+(112)+(123), extended to Weierstrass, K3, CY_3 and CY_4 spaces, corresponding to the E_r line on the arity-dimension plot Fig. 4.*

W	*K3*	*K3*	*N*	*Quin*	*ti*	*c*	*N*	*Se*	*xt*	*i*	*c*	*N*
(300)	(3000)		1	(30000)			1	(300000)				1
(310)	(3100)	(3110)	2	(31000)	(31100)	(31110)	3	(310000)	(311000)	(311100)	(311110)	4
	(3200)	(3210)		(32000)	(32100)	(32110)		(320000)	(321000)	(321100)	(321110)	
					(32200)	(32210)			(322000)	(322100)	(322110)	
			2				5			(322200)	(322210)	9
	(3300)	(3310)		(33000)	(33100)	(33110)		(330000)	(331000)	(331100)	(331110)	
					(33200)	(33210)			(332000)	(332100)	(332110)	
										(332200)	(332210)	
					(33300)	(33310)			(333000)	(333100)	(333110)	
										(333200)	(333210)	
			2				7			(333300)	(333310)	16
(400)	(4000)		1	(40000)			1	(400000)				1
	(4100)	(4110)	2	(41000)	(41100)	(41110)	3	(410000)	(411000)	(411100)	(411110)	4
	(4200)	(4210)		(42000)	(42100)	(42110)		(420000)	(421000)	(421100)	(421110)	
					(42200)	(42210)			(422000)	(422100)	(422110)	
			2				5			(422200)	(422210)	9
	(4300)	(4310)		(43000)	(43100)	(43110)		(430000)	(431000)	(431100)	(431110)	
					(43200)	(43210)			(432000)	(432100)	(432110)	
										(432200)	(432210)	
					(43300)	(43310)			(433000)	(433100)	(433110)	
										(433200)	(433210)	
			2				7			(433300)	(433310)	16
	(4400)	(4410)		(44000)	(44100)	(44110)		(440000)	(441000)	(441100)	(441110)	
					(44200)	(44210)			(442000)	(442100)	(442110)	
										(442200)	(422210)	
					(44300)	(44310)			(443000)	(443100)	(443110)	
										(443200)	(443210)	
										(433300)	(433310)	
					(44400)	(44410)			(444000)	(444100)	(444110)	
										(444200)	(444210)	
										(444300)	(444310)	
			2				9			(444400)	(444410)	25

As shown in Fig. 6, there are recurrence formulae for the numbers of IMs in any dimension, which are obvious for the leading (red and green) lines in the arity-dimension plane. The resulting expressions for the numbers of cubic and quartic monomials are, respectively:

$$\begin{aligned} N_{cubics} &= \frac{1}{6}(n-2)(n-1)(n+3), \\ N_{quartics} &= \frac{1}{24}(n-2)(n-1)(n)(n+5). \end{aligned} \tag{13}$$

There are remarkable links between the numbers of conics, cubics and quartics. For example, to obtain the number of quartics in dimension n, one should sum over all the cubics in dimensions $3, 4, ..., n$, i.e., $N^{(n)}_{Quart} = \sum_{i=3}^{i=n} N^{(i)}_{Cub}$. Thus, as seen in Fig. 6, the number 105 of quartic monomials in the septic Calabi-Yau case can be represented as follows: $2_{dim=3} + 7_{dim=4} + 16_{dim=5} + 30_{dim=6} + 50_{dim=7}$.

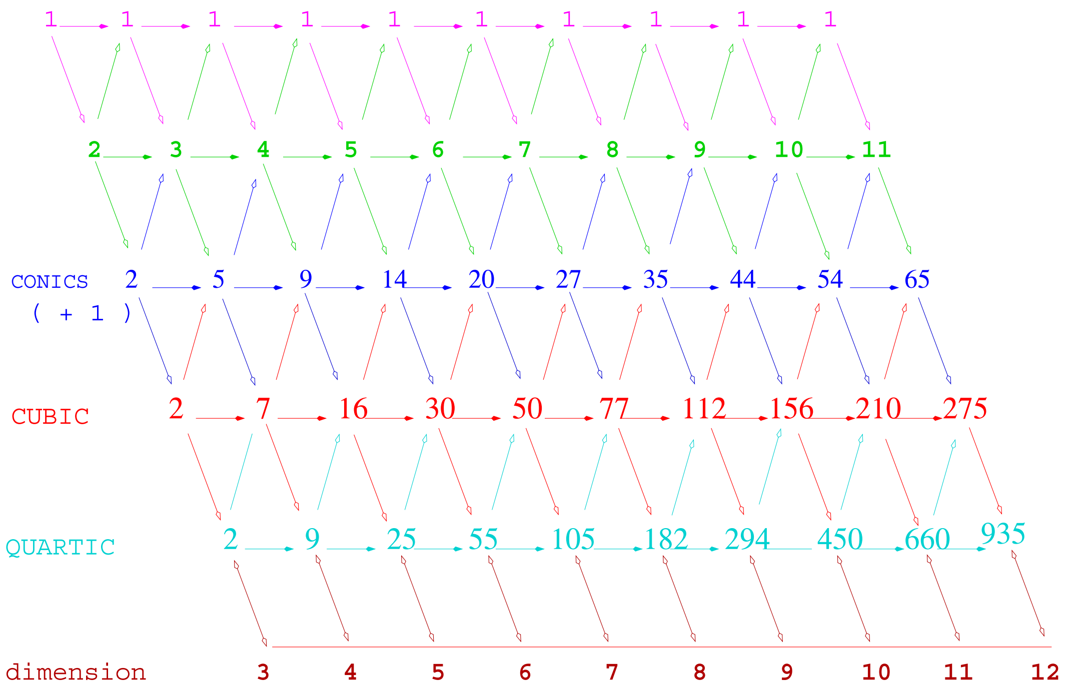

Fig. 6. *Lattice illustrating recurrence relations for the numbers of conic, cubic and quartic monomials.*

Based on Fig. 6, one can convince oneself that there also exist n-dimensional recurrence formulae for the numbers of IMs along other diagonal lines in any dimension, as we have found for the first two lines on the arity-dimension plot in Fig. 4. However, the situation can become complicated, because, in the construction of the cubic and quartic IMs, one must also take into account conic and conic + cubic monomials, respectively. In the case of Calabi-Yau spaces with Weierstrass fibres, it is also important to know the list of sextic monomials. In Weierstrass case there should always appear six degree monomial which is not IM. We would like to note that to deduce the recurrence it is much easier to look for not for the reccurence of three cubics IMs (or two quartics and one conics IMs) with Diophantine condition but only for the form of one sectic monomial. And actually, to find this recurrence one can use the recurrence of such sextic monoms. Therefore, we firstly give the recurrence for the sextic monomials:

$$C_{n+2}^{n-3} = \frac{(n+2)!}{(n-3)!5!}, \tag{14}$$

where $n \geq 3$ is the dimension of the weight-vector space. Using this formula for the sextic monomials we obtain the following expression for the number of Weierstrass IMs, $\{3\}$ and $\{4\}$:

$$N_{7\Delta}(n) = N_W(n) = C_{n+3}^{n-3} = \frac{(n+3)!}{(6!)(n-3)!}, \tag{15}$$

valid for all dimensions.

One can see from the Fig. 6 and the corresponding tables that the IMs determine completely the fibration structures of the 22 $K3$ chains:

$$
\begin{aligned}
\{IM\}_4 \;\; \mapsto \;\; & \Big(1\cdot\{4\}_{\triangle}\Big)+\Big(\mathbf{2}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big) \\
+ \;\; & \Big(2\cdot\{5\}_{\triangle}+1\cdot\{5\}_{\Box}\Big) \\
+ \;\; & \Big(\mathbf{4}\cdot\{\mathbf{9}\}_{\boldsymbol{\triangle}}+2\cdot\{9\}_{\Box}\Big) \\
+ \;\; & \Big(\mathbf{7}\cdot\{\mathbf{7}\}_{\boldsymbol{\triangle}}+1\cdot\{7\}_{\Box}\Big) \\
+ \;\; & \Big(1\cdot\{6\}_{\Box}\Big)+\Big(1\cdot\{8\}_{\Box}\Big) \\
\mapsto \;\; & \{22\}.
\end{aligned} \tag{16}
$$

This expansion in terms of fibration structures is very helpful for extending these $K3$ results to more general CY_d spaces, via recurrence relations. As we show later, each of the terms $\{10,4,...\}_{\triangle,\Box,...}$ in the expansion has its own recurrence relation, of which we later derive several examples, indicated in bold script: $\mathbf{2}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}$, etc., providing complete results in any number of dimensions for the numbers of CY_d spaces with these particular fibrations. A similar recurrence formula could be derived for any analogous fibration.

There are fixed types and numbers of IMs which determine the structures of the full 259 (irreducible 161) chains, and they are similar to those we already indicated for the $K3$ case, as seen, for example, in the following Fig. 6.

$$
\begin{aligned}
\{IM\}_5 \;\; \mapsto \;\; & \Big(9\cdot\{4\}_{\triangle}+\mathbf{4}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big) \\
+ \;\; & \Big(16\cdot\{5\}_{\triangle}+5\cdot\{5\}_{\Box}+1\cdot\{5\}_{\Box'}\Big) \\
+ \;\; & \Big(\mathbf{11}\cdot\{\mathbf{9}\}_{\boldsymbol{\triangle}}+5\cdot\{9\}_{\Box}+1\cdot\{9\}_{\Box'}\Big) \\
+ \;\; & \Big(\mathbf{28}\cdot\{\mathbf{7}\}_{\boldsymbol{\triangle}}+7\cdot\{7\}_{\Box}+1\cdot\{7\}_{Quint}\Big) \\
+ \;\; & \Big(8\cdot\{6\}_{\Box}+1\cdot\{6\}_{Quint}\Big) \\
+ \;\; & \Big(6\cdot\{8\}_{\Box}+1\cdot\{8\}_{Quint}\Big) \\
\mapsto \;\; & \{161\}.
\end{aligned} \tag{17}
$$

A further reduction in the number of chains has to be considered, from the 5,607 6-dimensional 4-vector chains to 2111 independent chains. We have already mentioned that there are different types of IMs even among the cubics $\{3\}$ and quartics $\{4\}$, and the number of different conics grows monotonically with increasing dimension n. We have also already remarked that there exists a recurrence formula for all types of IMs with arbitrary dimension n, and have already discussed the reccurences of the Weierstrass IMs $\{3_W\}$ and $\{4_W\}$. The possible types of cubic $\{3\}$, quartic $\{4\}$ and double conic IMs which describe the 2111 irreducible CY_3 chains have different structures, corresponding to the different types of intersections, that we can illustrate by the following expression:

$$
\begin{aligned}
\{IM\}_6 \ \mapsto\ & \Big(37\cdot\{4\}_{\triangle} + \mathbf{7}\cdot\{\mathbf{10}\}_{\boldsymbol{\triangle}}\Big)\\
+\ & \Big(66\cdot\{5\}_{\triangle} + 27\cdot\{5\}_{\Box} + 6\cdot\{5\}_{\Box'}\Big)\\
+\ & \Big(\mathbf{24}\cdot\{\mathbf{9}\}_{\boldsymbol{\triangle}} + 11\cdot\{9\}_{\Box} + 5\cdot\{9\}_{\Box'}\Big)\\
+\ & \Big(\mathbf{84}\cdot\{\mathbf{7}\}_{\boldsymbol{\triangle}} + 28\cdot\{7\}_{\Box} + 5\cdot\{7\}_{Quint} + 1\cdot\{7\}_{Sixt}\Big)\\
+\ & \Big(36\cdot\{6\}_{\Box} + 5\cdot\{6\}_{Quint}\Big)\\
+\ & \Big(21\cdot\{8\}_{\Box} + 5\cdot\{8\}_{Quint}\Big)\\
\mapsto\ & \{2111\}. \qquad (18)
\end{aligned}
$$

The recurrence relation for Calabi-Yau spaces with elliptic fibres $\{10\}_{\triangle}$ can be extended to the cases of CY_d spaces with $K3$ fibres, described by $\vec{k}_4 = (1,1,1,1)[4]$, whose algebraic equation includes the 35-point monomial and its mirror with 5 points. The IM_4 for this $K3$ space contains the four quartic monomials P_1, P_2, P_3, P_4 obeying the Diophantine equation: $(P_1 + P_2 + P_3 + P_4)/4 = E_4$. These monomials have in addition one very important condition: $P_i - P_j$ should be divisible by 4 for each choice of $i, j = 1,2,3,4, i \neq j$. The types of different n-dimensional $\{IM\}_4$, describing the $CY_d : n = d + 2 \geq 4$ spaces with $\{35\}_{\triangle}$ fibres are constructed only from the numbers 4 and 0. The number 1 will play an additional role. Therefore, similarly to the case of the third E_r line, the recurrence formulae for these IMs will be determined from the expansions of positive integer numbers in terms of four positive integers, i.e., (see Fig. 7).

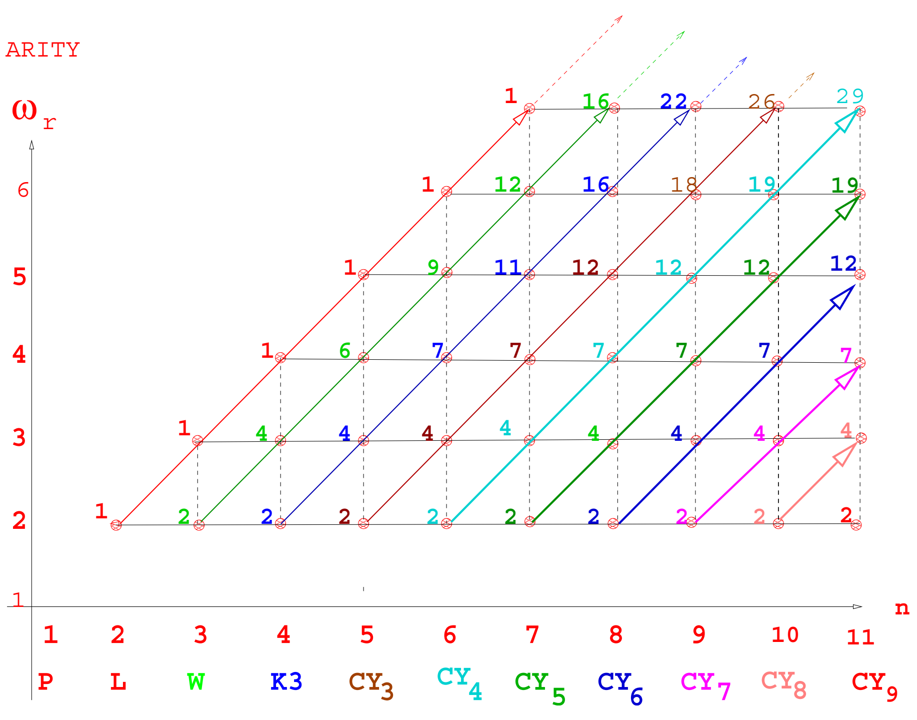

Fig. 7. *The numbers of recurrences of Calabi-Yau hypersurfaces with a* $(1,\ldots,1)_n$ *fibre are calculable along all lines* $n = r + p - 1$ *in the arity-dimension plot.*

The recurrence formula for Calabi-Yau spaces with elliptic fibres $\{10\}_\Delta$ can be extended to the cases of CY_d spaces with $K3$ fibres, described by $\vec{k}_4 = (1,1,1,1)[4]$, whose algebraic equation includes the 35-point monomial and its mirror with 5 points. The IM_4 for this $K3$ space contains the four quartic monomials P_1, P_2, P_3, P_4 obeying the Diophantine equation: $(P_1 + P_2 + P_3 + P_4)/4 = E_4$. These monomials have in addition one very important condition: $P_i - P_j$ should be divisible by 4 for each choice of $i, j = 1, 2, 3, 4, i \neq j$. The types of different n-dimensional $\{IM\}_4$, describing the $CY_d : n = d + 2 \geq 4$ spaces with such $\{35\}_\Delta$ fibres are constructed only from the numbers 4, 1 and 0. Therefore, similarly to the case of the third E_r line, the recurrence formulae for these IMs will be determined from the expansions of positive integer numbers in terms of four positive integers, as illustrated in Fig. 7.

$$
\begin{aligned}
& && K3 - line \\
n = 4(4) &= && 1+1+1+1 \\
n = 5(4) &= && 2+1+1+1 \\
& && N(5) = 2 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 6(4) &= && 3+1+1+1 = 2+2+1+1 \\
& && N(6) = N(5) + 2 = 4 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 7(4) &= && 4+1+1+1 = 3+2+1+1 = 2+2+2+1 \\
& && N(7) = N(6) + 3 = 7 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 8(4) &= && 5+1+1+1 = 4+2+1+1 = 3+3+1+1 = \\
&= && 3+2+2+1 = 2+2+2+2 \\
& && N(8) = N(7) + 5 = 12 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 9(4) &= && 6+1+1+1 = 5+2+1+1 = 4+3+1+1 = \\
&= && 4+2+2+1 = 3+3+2+1 = 3+2+2+2 \\
& && N(9) = N(8) + 6 = 18 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 10(4) &= && 7+1+1+1 = 6+2+1+1 = 5+3+1+1 = 5+2+2+1 = \\
&= && 4+4+1+1 = 4+3+2+1 = 3+3+3+1 = 3+3+2+2 \\
& && N(10) = N(9) + 8 = 26 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 11(4) &= && 8+1+1+1 = 7+2+1+1 = 6+3+1+1 = 6+2+2+1 = \\
&= && 5+4+1+1 = 5+3+2+1 = 5+2+2+2 = 4+4+2+1 \\
&= && 4+3+3+1 = 4+3+2+2 = 3+3+3+2 \\
& && N(11) = N(10) + 11 = 37 \\
& && \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{19}
$$

Similarly, this example can be extended to CY_4 (CY_d) spaces with a CY_3 (CY_{d-1}) fibre described by the RWV $\vec{k}_5 = (1,1,1,1,1)[5]$ ($\vec{k}_n = (1,...,1)_n$):

$$\begin{aligned}
& CY_3 - line \\
n = 5(5) &= 1+1+1+1+1 \\
n = 6(5) &= 2+1+1+1+1 \\
& N(6) = 2 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 7(5) &= 3+1+1+1+1 = 2+2+1+1+1 \\
& N(7) = N(6)+2 = 4 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 8(5) &= 4+1+1+1+1 = 3+2+1+1+1 = 2+2+2+1+1 \\
& N(8) = N(7)+3 = 7 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 9(5) &= 5+1+1+1+1 = 4+2+1+1+1 = 3+3+1+1+1 \\
&= 3+2+2+1+1 = 2+2+2+2+1 \\
& N(9) = N(8)+5 = 12 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 10(5) &= 6+1+1+1+1 = 5+2+1+1+1 = 4+3+1+1+1 \\
&= 4+2+2+1+1 = 3+3+2+1+1 = 3+2+2+2+1 = 2+2+2+2+2 \\
& N(10) = N(9)+7 = 19 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
n = 11(5) &= 7+1+1+1+1 = 6+2+1+1+1 = 5+3+1+1+1 \\
&= 5+2+2+1+1 = 4+4+1+1+1 = 4+3+2+1+1 = 4+2+2+2+1 \\
&= 3+3+3+1+1 = 3+3+2+2+1 = 3+2+2+2+2 \\
& N(11) = N(10)+10 = 29 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned} \tag{20}$$

The same approach can clearly be extended to establish the numbers of any other desired IMs. Indeed, for each slope line there is a recurrence formula of the type

$$N_{\ldots\Delta}(n) = N_{\ldots\Delta}(n-1) + n(p), \tag{21}$$

where p is the number of the sloping line, and $N_{\ldots\Delta}(n_{min}) = 2$, $n_{min} \geq 3$, enabling one to establish the numbers of any other desired IMs with fibre $(1,...,1)_n$.

We can also consider some other examples of CY_d on the fourth K3-line with given intersection (actually fibre is determined through the mirror manifold). For example, consider the intersection which is determined by $\vec{k} = (1,1,4,6)[12]$. We should stress that all these CY_d can be constructed through the Diophantine expansion of unit monomials $E_n = (1,...,1) \rightarrow \{P_1,P_2,P_3,P_4|1/4(P_1+P_2+P_3+P_4) = E_n\}$, where n=d+2. Also they can be got through the Diophantine expansion of conics monomial of the second slope-line $C_2 = (2,2,...,2,0)$, where

$C_1 = (0, ..., 0, 2)$. This expansion is the following: $C_2 \to \{P_1, P_2, P_3|1/3(P_1 + P_2 + P_3) = C_2\}$. And at last all these CY_d can be got through the Diophantine expansion of the cubics or quartics monomials of the third E_r line. The corresponding Diophantine expansion on the last stage is: $(Cub) \to \{P_1, P_2|1/2(P_1 + P_2) = (Cub)\}$. These results can be easily seen from the list of monomials corresponding to the reflexive weight-vector $\vec{k} = (1, 1, 4, 6)[12]$. The list of possible such CY_d can be determined similarly how it was determined the list of CY_d with Weierstrass fibre. There is only one discreapence, that instead of sextic monomials we should take the IMs of twelve degree, i.e. the degree of such monomials with respect to one fixed variable should be of the 12-th degree. Taking into account the list of possible such monomials the corresponding formula for CY_d with the (1,1,4,6)[12]- intersection will be the next:

$$N_{39\Delta}(n) = C_{n+3}^{n-4} = \frac{(n+3)!}{(7!)(n-4)!}, \tag{22}$$

where $n \geq 5$ and the number (39) corresponds to the all possible monommials of such intersection.

At last we note [9] that the lattice structure of the $K3$ projective vectors obtained by a binary construction exhibits a very interesting UCYA-correspondence between the Dynkin diagrams for Cartan-Lie groups in the A, D series and $E_{6,7,8}$ and particular reflexive weight vectors (see also Fig. 8):

$$\begin{aligned} \vec{k}_1 = (1) &\leftrightarrow A_r; \\ \vec{k}_2 = (1,1) &\leftrightarrow D_r; \\ \vec{k}_3 = (1,1,1) &\leftrightarrow E_6; \\ \vec{k}_3 = (1,1,2) &\leftrightarrow E_7; \\ \vec{k}_3 = (1,2,3) &\leftrightarrow E_8. \end{aligned} \tag{23}$$

This appearance in Calabi-Yau geometry of the A, D and E series of Cartan-Lie algebras is connected [9] with specific quotient singular structures of considered geometry like as Kleinian-Du-Val singularities C^2/Z_n.

For example, resolving the C^2/Z_n singularity gives for rational, i.e., genus zero, (-2)-curves an intersection matrix that coincides with the A_{n-1} Cartan matrix. For a general form of the C^2/G singularity, one can see [5]

One can try to explain this problem better reffering at the McKay correspondence [11]. So for $G = SL(2, C)$ a finite group, the quotient variety $X/C^2/G$ is called a Klein-Du-Val quotient singularity. The minimal resolution of such singularities $Y \to X$ has been well studied by Klein and Du Val later for the cases of subgroup G which are classified as cyclic, binary dihedral or binary group corresponding to one of the platonic solids, tetrahedron, cub-octahedron, and icosahedron-dodecahedron.

The quotient singularity can be defined as a hypersurface $X \subset C^3$ describing equation of known functions. The resolution $\phi : Y \to X$ is a surface Y with $K_Y = \phi^* K_X$. The exceptional locus $\phi^{-1}(0) \subset Y$ of the resolution consists of a set of rational (-2)-curves E_i, where E_i is a sphere, or $E_i \approx CP^1$ and with its selfintersection i $E_i^2 = -2$. The intersections $E_i E_k$ are given by one of the Coxeter-Dynkin diagrams, A_n, D_n, E_6, E_7, E_8.

Let consider the following space C^2/Z_2 and embedd its into C^3. Consider the hypersurface given by $f(x_0, x_1, x_2) = x_0 x_1 - x_2^2 = 0$. This hypersurfaces is smooth if and only if $f = \partial f / \partial x_i =$

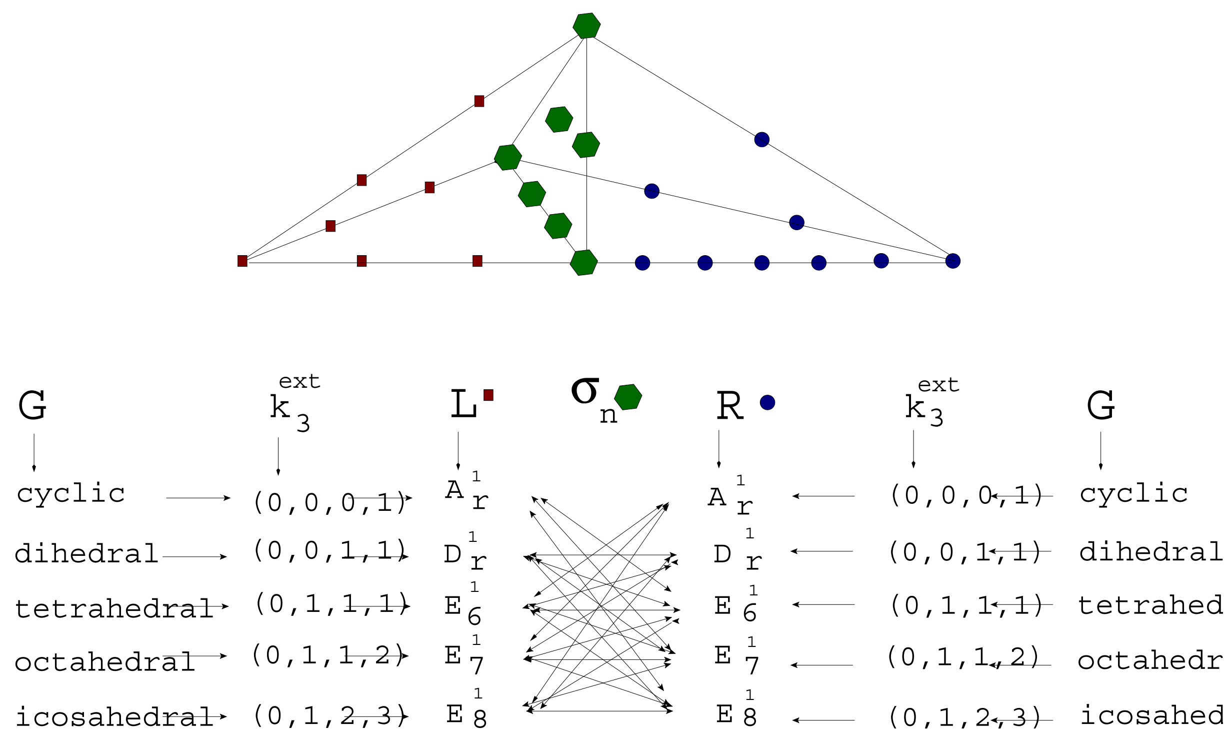

Fig. 8. *The $K3$ polyhedron determined by the reflexive weight vector $\vec{k}_4 = (1,1,3,4)$[9], which illustrates the appearance of Coxeter-Dynkin diagrams. The intersection σ is determined by 7 point monomials that correspond to the elliptic fibre $\{7\}_\Delta$, and divides the polyhedron into 7(+ 3) points on the left and 9(+7) on the right. These reproduce the Coxeter-Dynkin diagrams for affine E_6 and E_8, respectively. Underneath, we also show schematically the general nature of the highest-weight vectors obtained by arity-2 construction in the UCYA, displaying the one-to-one link between the 5-dimensional weight vectors and the ADE series of Cartan-Lie algebra in $K3$ hypersurfaces. The rôles of the discrete symmetry groups were discussed in [9].*

0 (i=0,1,2) has no solution. So f has only one singularity at the origin $x_0 = x_1 = x_2 = 0$. One can parametrize $f = 0$ by putting $x_0 = u^2, x_1 = v^2, x_2 = uv$. One can see immediately that the points (u,v) and $(-u,-v)$ are the same. Thus $f = 0$ in C^3 is really the orbifold C^2/Z_2, and x_0, x_1, x_2 can be considered as Z_2 invariant polynomials. Similarly, one can consider the singularity C^2/Z_n in C^3 space.

Also, similarly, one can consider the example for resolution of the singular variety $X = C^2/G$ where G is the dihedral group BD_8 [11]. The minimal resolution $Y \to X$ with $K_Y = K_X = 0$ can be built from McKay correspondence considering the conjugacy classes of group $G = BD_8$. After identifying the equal and conjugate group elements one can get the McKay graph coinciding with Dynkin D_4 diagram (see Fig. 9).

Similarly, the McKay correspondence can help us to find resolution of singularities making by the binary tetrahedral, binary octahedral and binary icosahedral finite groups and leading to the E_6, E_7 and E_8 Dynkin diagrams, respectively. The McKay correspondence was also well understood for $G \subset SL(3,C)$ and $X = C^3/G$ the quotient space, an affine variety with $K_X = 0$.

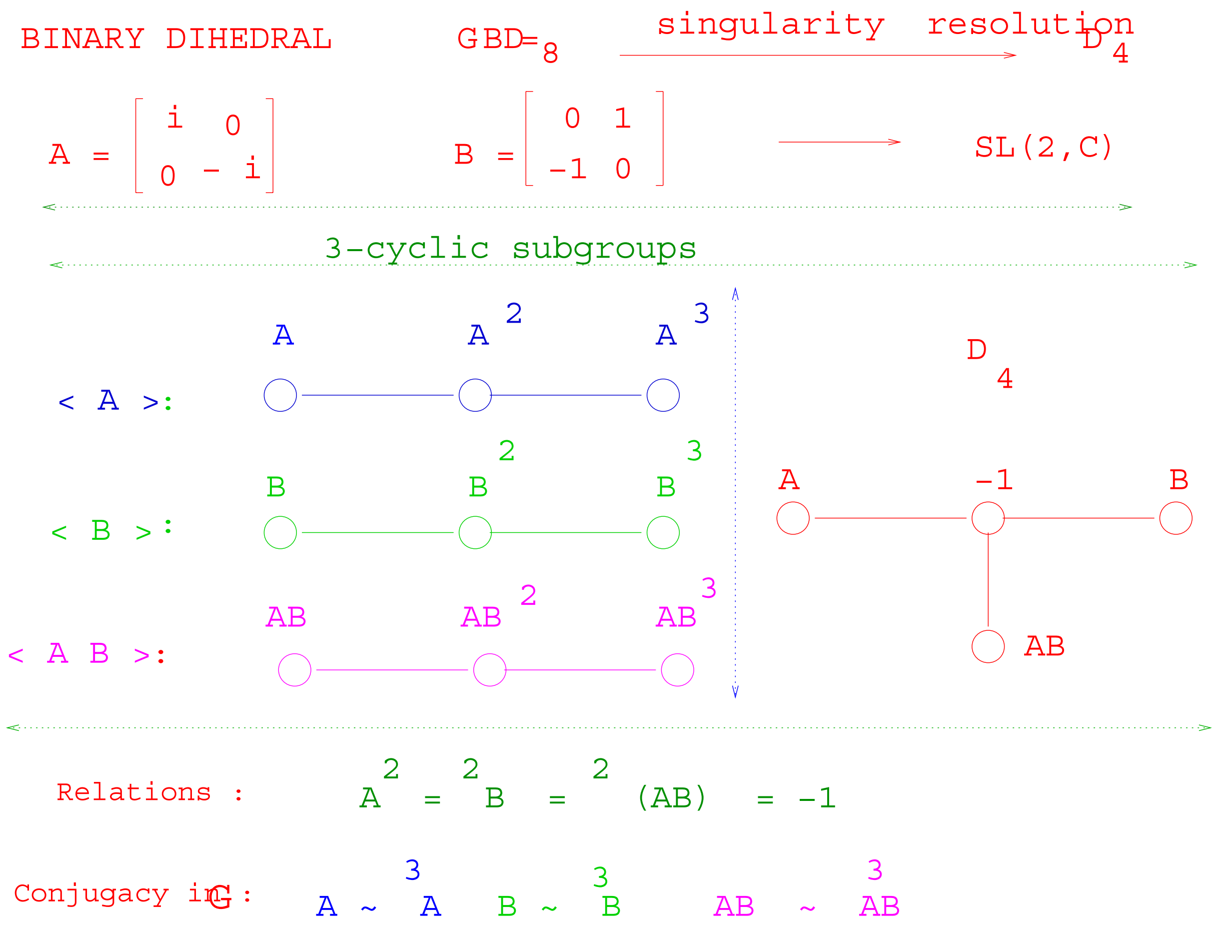

Fig. 9. *The D_4-structure of McKay-Quiver = Dynkin diagram for the quotient C^2/BD_8.*

Any discrete subgroup of holonomy group $SU(2)$ can be projected into a subgroup of $SO(3)$, and thus can be related to the finite symmetry classification of three-dimensional space. Thus, resolving the orbifold singularities yields a beautiful interrelation between the classification of finite group rotations in three-space and the ADE classification of Cartan-Lie algebras. Correspondingly, due these singularities in UCYA one can see that the CY_n- polyhedra with $(n \geq 3)$ can be also constructed from n-copies of Coxeter-Dynkin diagrams. But there is standing new interesting question to understand the list of quotient singularities for CY_3, connected with the finite subgroups of SU(3)-holonomy [11].

4. Discussion

We would like to say that the basic ideas of UCYA are connected with the n-gebra. Therefore it will be interesting to compare UCYA to theory of operads [12]. The notion of operad was connected to the idea of substition: given r functions $F_1, \ldots, F_r$ of $n_1, ..., n_r$ variables and a given function G_r in r-variables to yield a function H_n of $\{n = n_1 + \ldots n_r\}$ variables

$$H_n(x_1, \ldots, x_n) = g_r(F_1(x_1, \ldots, x_{n_1}), \ldots, F_{n_r}(x_{n_j}, \ldots, x_n)) \tag{24}$$

with $j = 1 + \sum_{i=1}^{i=n-1} n_i$. The operad is intended tp build a system of such functions and substitions(extensions). The operad consisys of the of a sequence $P(0), P(1), \ldots$ of objects and for every r-tuple $(n_1, ..., n_r)$ of a natural numbers $n_1, ..., n_r \geq 0$ there is a structure morphism

$$\gamma : P(r) \otimes P(n_1) \otimes \ldots P(n_1) \otimes P(n_r) \mapsto P(n_1 + \ldots + n_r). \tag{25}$$

Here the $\otimes$ denotes the symmetric monoidal structure. Operads are more important of their representations. There is a unit element $P(1)$. $P(2)$ encodes a binary operations on X. and P(3) is for ternary operations etc. In UCYA all $n_i = 1$ and the extension is going step by step depending on the arity r.

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