A.A. Arkhipov

HARMONY OF THE FROISSART THEOREM
IN FUNDAMENTAL DYNAMICS OF PARTICLES AND NUCLEI

Abstract

Arkhipov A.A. Harmony of the Froissart Theorem in Fundamental Dynamics of Particles and Nuclei: IHEP Preprint 2002-42. – Protvino, 2002. – p. 21, figs. 4, refs.: 24.

It has been shown that the great ancient Pythagorean ideas have found themselves in the latest researches in high energy elementary particles and nuclear physics. In this respect we concern and discuss the mathematical, physical and geometrical aspects of the famous Froissart theorem and in this way one establishes a link of this theorem to the mathematics and ideas elaborated in the Pythagorean school. A harmony of the Froissart theorem in fundamental dynamics of particles and nuclei has been displayed. We argue that a harmony of the Froissart theorem allow us to hear the new notes of *"the music of the spheres"* just in the Pythagoreans sense.

Аннотация

Архипов А.А. Гармония теоремы Фруассара в фундаментальной динамике частиц и ядер: Препринт ИФВЭ 2002-42. – Протвино, 2002. – 21 с., 4 рис., библиогр.: 24.

В работе обсуждается проявление Пифагоровых гармонических структур в динамике частиц и ядер высоких энергий. Анализ математических, физических и геометрических аспектов известной теоремы Фруассара позволил нам установить глубокие связи этой теоремы с идеями и математикой, развитыми в школе Пифагора, а обнаруженные нами гармонические структуры в фундаментальной динамике частиц и ядер мы трактуем как гармонию, предоставленную нам теоремой Фруассара. Здесь мы настаиваем на том, что гармония теоремы Фруассара позволяет нам услышать новые звуки "музыки сфер" в том же смысле, о котором рассуждали древние пифагорейцы.

Introduction: Pythagoreanism

11. ... за неразумные помышления их неправды, по которым они служили бессловесным пресмыкающимся и презренным чудовищам, Ты в наказание послал на них множество бессловесных животных, чтобы они познали, что чем кто согрешит, тем и наказывается ... и без этого они могли погибнуть от одного дуновения, преследуемые правосудием и рассеиваемые духом силы твоей; но Ты все расположил мерою, числом и весом.

Книга премудрости Соломона

Once upon a time the great russian physicist-theorist and mathematician Bogoljubov said that the last line in the above written fragment from the Book of Proverbs which is a part of the Bible (non-canonical though!) "...; but You did all arrange by measure, number and weight" represents the definition of the Physics [1]. Probably this – "...; but You did all arrange by measure, number and weight" – was an earliest evidence of the principle that All in the World have to be in harmony with each other.

In fact, the word harmony (Syn: music: accord, concord, consonance) has the Greek origin from $\dot{\alpha}\rho\mu\acute{o}\nu\acute{\iota}\alpha$ which means orderliness (symmetry) of the whole, commensurability (proportionality) of its parts. The idea of harmony has intensively been elaborated by Pythagoras, the Greek philosopher and mathematician and founder of the Pythagorean school [2]. Originally from Samos, Pythagoras founded a society which was at once a religious community and a scientific school flourished at Kroton in Southern Italy about the year 530 B.C. Pythagoras was the first genius of western culture. He had a multifaceted magnetic personality – an intelligent mathematician and a religious thinker, both co-existed in him. His main contributions are in geometry, numbers, music, cosmology, astronomy, philosophy and religion. Pythagoras must have been one of the world's greatest men, but he wrote nothing though numerous works are attributed to him, and it is hard to say how much of the doctrine one knows as Pythagorean

1

is due to the founder of the society and how much is later development. It is also hard to say how much of what we are told about the life of Pythagoras is trustworthy. For a mass of legend gathered around his name: Sometimes he is represented as a man of science, and sometimes as a preacher of mystic doctrines, and we might be tempted to regard one or other of those characters as alone historical. Certainly, it's true that there is no need to reject either of the traditional views.

Even though many wonderful things related to Pythagoras, belong to legend, and seem to have no historical foundation, similarly the description of the learned works which he wrote is not attested by reliable historians and also belongs to the region of fable, nevertheless it is no doubt however, that he founded a school, or, rather, a religious philosophical society, which exerted great influence on the intellectual development of human civilization and had a fundamental importance all the time. Of great influence were the Pythagorean doctrines that numbers were the basis of all things and possessed a mystic significance, in particular the idea that the cosmos is a mathematically ordered whole. Aristotle wrote: "Pythagorean having been brought up in the study of mathematics, thought that things could be represented by numbers ... and that the whole cosmos consists of a scale and a number". Briefly stated, the doctrine of Pythagoras was that all things are numbers. Pythagoras was led to this conception by his discovery that the notes sounded by stringed instrument are related to the length of the strings. He conducted remarkable investigation in "music" as he was a musician. Harmonies correspond to most beautiful mathematical ratio, he stated. Melodious musical tunes could be produced on a stringed instrument by plucking the string at particular points, which correspond to mathematical ratios. Such beautiful mathematical ratios are $1 : 2$ (an octave), $2 : 3$ (a fifth), and $3 : 4$ (a fourth). Pythagoras recognized that first four numbers $1, 2, 3, 4$ known as "tetractys", whose sum equals Ten $(1 + 2 + 3 + 4 = 10)$, contained all basic musical intervals: the octave, the fifth and the fourth. In fact, all the major consonances, that is, the octave, the fifth and the fourth are produced by vibrating strings whose lengths stand to one another in the ratios of $1 : 2$, $2 : 3$ and $3 : 4$ respectively. Recent major scale in according to Pythagoras tune looks like

$$1; \frac{8}{9}; \frac{64}{81}; \frac{3}{4}; \frac{2}{3}; \frac{16}{27}; \frac{128}{243}; \frac{1}{2},$$

where $\frac{8}{9} = \frac{2}{3} \cdot \frac{2}{3} \cdot 2$ is major second (fifth of fifth with octave lowering); $\frac{16}{27} = \frac{2}{3} \cdot \frac{8}{9}$ is major sixth (fifth of major second); $\frac{64}{81} = \frac{2}{3} \cdot \frac{16}{27} \cdot 2$ is major third (fifth of sixth with octave lowering); $\frac{128}{243} = \frac{2}{3} \cdot \frac{64}{81}$ is major seventh (fifth of third).

The resemblance which Pythagoras perceived between the orderliness of music, as expressed in the ratios which he had discovered and the idea that cosmos is an orderly whole, made up of parts harmoniously related to one another, led him to conceive of the cosmos too as mathematically ordered. Pythagoras compared the eight planets (there were seven planets known the Babylonians: Moon, Mercury, Venus, Sun, Mars, Jupiter and Saturn), including the Earth, with the musical octave and the seven planets, excluding the Earth, as seven strings of the musical instrument Lair. The planets situated at different distances and moving at different speed correspond to different notes on musical octave. The planets moving with higher speed produce higher notes and those with lower speed produce lower notes. The celestial harmony of moving planets produces heavenly music ("the music of the spheres") analogous to different notes of musical octave.

According to Pythagoras, the sphere was the most beautiful solid and the circle the most beautiful shape. Thus, a spherical planet moving in circular orbit would form a harmonious

constellation. Pythagoras worked out the distances of the planets from the Earth. He arranged the planets in order of increasing distances of the planets from the Earth. The order given by him was the Moon, Mercury, Venus, the Sun, Mars, Jupiter and Saturn. Some Pythagoreans believed that the Earth moved round a central fire. The Earth did not always face the central fire. This accounted for day and night on the Earth. They also believed that the Moon as well as the Sun shone because they reflected light from their surfaces received from the central fire. Perhaps the idea of central fire later on led to the heliocentric (Sun at the centre of the Solar system) configuration of Solar system. Pythagoras observation of heavens suggested to him that the motion of the heavenly bodies was cyclic and that the heavenly bodies returned to the place from which they had started. From this, Pythagoras concluded that there must be a cycle of cycles, a greater year and on its completion the heavenly bodies returned to the original position and the same heavenly constellation would be observed again and again. He called this the eternal recurrence.

Pythagoras doctrine that mathematics contains the key to all philosophical knowledge, an idea, which was by his followers afterwards developed into an elegant number-theory. The Pythagorean philosophy in its later elaboration is dominated by the number-theory. Being the first, apparently, to observe that natural phenomena, especially the phenomena of the astronomical world, may be expressed in mathematical formulas, the Pythagoreans held that numbers are not only the symbols of reality, but the very substance of real things. Pythagoras associated numbers with geometrical notions and numerical ratios with shapes. He associated number one with a point, too with a line, three with a triangle (the surface) and four with a tetrahedron (the solid). Thus, one point generates dimensions, two points generate a line of one dimension, three points generate a surface of two dimensions, and four points generate three-dimensional solid figures. In geometry, numbers represent lengths, their squares represent areas, their cubes represent volumes. Starting from numbers, numerical ratios and their powers, one can construct geometrical figures of different shapes and geometrical solids of different sizes. Using distance the arrangement of planets, their motion, their orbital path, their distances from the center and their interrelations with each other can be worked out. Thus, according to Pythagoras all relations could be reduced to number relations and hence, the whole cosmos is a scale and a number based phenomenon.

According to Pythagoras, Ten is the perfect number, because it is the sum of one, two, three, and four – the point, the line, the surface, and the solid. There are the second type of perfect numbers : According to Pythagoras the second type of perfect numbers are those were the numbers equal to sum of their factors. For instance 28 has factors 1, 2, 4, 7, 14 and $1 + 2 + 4 + 7 + 14 = 28$.

From perfect numbers, Pythagoras was led to amicable numbers like 220 and 284. Amicable numbers form a pair of numbers where each number is equal to the sum of the factors of the other numbers. For instance 220 has factors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110. The sum of these factors is $1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 = 284$. Moreover, 284 has factors 1, 2, 4, 71 , 142 . The sum of these factors is $1 + 2 + 4 + 71 + 142 = 220$.

Triangular numbers have been introduced by Pythagoras: Pythagoras called numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66 as triangular numbers because these numbers can be arranged so as to form triangles.

If a, b, c are sides of a right-angled triangle and c is the hypotenuse then according to Pythagoras theorem $c^2 = a^2 + b^2$. The triad of positive integers (a, b, c) satisfying the relation $c^2 = a^2 + b^2$ is called the Pythagorean triad of numbers. About fifteen such triads were previously known like

(3,4,5), (5,12,13), (7,24,25), (9,12,15), (15,36,39). The Pythagorean triads in which the numbers a, b, c do not have a common factor are called primitive Pythagorean triads. For example (3,4,5), (5,12,13), (7,24,25) etc. are primitive Pythagorean triads. But (9,12,15), (15,36,39) are not primitive triads. It is believed that Pythagoras himself discovered the formula for determining triads of numbers satisfying the relation $c^2 = a^2 + b^2$. In fact, all Pythagorean triads can be expressed via formulae

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2, \tag{1}$$

where m, n are any positive integers ($m > n > 0$).

From his observations in music, mathematics and astronomy, Pythagoras generalized that everything could be expressed in terms of numbers and numerical ratios. Numbers are not only symbols of reality, but also substances of real things. Hence, he claimed - All is number. The importance of this conception is very great, for example, it is the ultimate source of Galileo's belief *"Il libro della natura é scritto in lingua matematica"* that the book of nature is written in mathematical symbols and hence the ultimate source of modern physics in the form in which it came to us from Galileo.

It may be taken as certain that the union of mathematical genius and mysticism is common enough[1]. Pythagoras himself discovered the numerical ratios which determine the concordant intervals of the musical scale. Similar to musical intervals, in medicine there are opposites, such as the hot and the cold, the wet and the dry, and it is the business of the physician to produce a proper "blend" of these in the human body. The Pythagoreans contended that the opposites are found everywhere in Nature, and the union of them constitutes the harmony of the real world. They also argued for the notion that virtue is a harmony, and may be cultivated not only by contemplation and meditation but also by the practice of gymnastics and music.

Pythagoras held the theory that what gives form to the Unlimited is the Limit. That is the great contribution of Pythagoras to philosophy, and we must try to understand it. It was natural for Pythagoras to look for something of the same kind in the world at large. Musical tuning and health are alike means arising from the application of Limit to the Unlimited.

In their psychology and their ethics the Pythagoreans used the idea of harmony and the notion of number as the explanation of the mind and its states, and also of virtue and its various kinds. Pythagoras argued that there are three kinds of men, just as there are three classes of strangers who come to the Olympic Games. The lowest consists of those who come to buy and sell, and next above them are those who come to compete. Best of all are those who simply come to look on. Men may be classified accordingly as lovers of wisdom, lovers of honour, and lovers of gain. That seems to imply the doctrine of the tripartite soul, which is also attributed to the early Pythagoreans on good authority.

The Pythagoreans were religiously and ethically inclined, and strove to bring philosophy into relation with life as well as with knowledge. The Pythagoreans believed also in reincarnation or transmigration (doctrine of Rebirth), that is, the soul, after death, passes into another living thing, which presupposes the ability of the soul to survive the death of the body, and hence some sort of belief in its immortality.

The above detailed introduction is made so as to show in the next sections that the great ancient Pythagorean ideas have found themselves in the latest researches in high energy elementary particle and nuclear physics. In this respect we will concern and discuss the mathematical,

---

[1]One up-to-date outstanding mathematician contended that all scientists, working in the number-theory, have a conversation with the God.

physical and geometrical aspects of the famous Froissart theorem and in this way we will easily establish a link of this theorem to the mathematics and ideas elaborated in the Pythagorean school. In other words, we would like to show a harmony of the Froissart theorem just in the Pythagoreans sense.

1. Froissart theorem: mathematical, physical and geometrical aspects

In the year 1961 french physicist Marcel Froissart discovered and proved a remarkable theorem, which stated that two-body reaction $a + b \to c + d$ amplitude, satisfying Mandelstam representation, is bounded by expressions of the form $C s \ln^2 s$ at the forward and backward angles, and $C s^{\frac{3}{4}} \ln^{\frac{3}{2}} s$ at any fixed angle in the physical region, C being a constant, s being the total squared c.m. energy (one of the Mandelstam invariant variables s, t, u). This corresponds to the total cross sections increasing at most like $\ln^2 s$ [3]. A little bit later it was shown that the analytical properties of two-particle scattering amplitude, which may be established strictly in the framework of axiomatic Quantum Field Theory, bring us to the Froissart statements as well. Up-to-date derivation of the Froissart theorem can be realized in a few steps, and we briefly sketch out it here.

For simplicity we consider a reaction of elastic scattering $a + b \to a + b$ for two scalar particles. The scattering amplitude of the two-body reaction may be considered as a function of the invariant variable $s = (p_a + p_b)^2$ and two unit vectors $\mathbf{n}$ and $\mathbf{n}'$ on two-dimensional sphere S_2, which characterise the initial and final states of two-particle system: $\mathcal{F}_2(s; p_a' p_b', p_a p_b) = \mathcal{F}_2(s; \mathbf{n}', \mathbf{n}), \mathbf{n} = \mathbf{q}/|\mathbf{q}|, \mathbf{q}$ is c.m. momentum of particles in an initial state

$$\mathbf{q} = \overset{\circ}{\mathbf{P}}_a = -\overset{\circ}{\mathbf{P}}_b, \quad \overset{\circ}{\mathbf{P}}_{a,b} = L^{-1}(\overrightarrow{P_{ab}}) p_{a,b}, \quad P_{ab} = p_a + p_b,$$

$L(P)$ is Lorentz boost, and the same with the primes in a final state. In the first step we wright the partial wave expansion

$$\mathcal{F}_2(s; \mathbf{n}', \mathbf{n}) = \mathcal{F}_2(s; \mathbf{n}' \cdot \mathbf{n}) = \mathcal{F}_2(s; \cos\theta) =$$

$$= \frac{1}{\pi A_2(s)} \sum_{lm} Y_{lm}(\mathbf{n}') f_l(s) \overset{*}{Y}_{lm}(\mathbf{n}) = \frac{1}{\pi \Gamma_2(s)} \sum_l (2l+1) f_l(s) P_l(\mathbf{n}' \cdot \mathbf{n}), \quad (2)$$

where $A_2(s) = \Gamma_2(s)/S_2$, $\Gamma_2(s)$ is two-particle phase space volume, S_2 is a surface of two-dimensional unit sphere, $\cos\theta = \mathbf{n}' \cdot \mathbf{n}$, and an addition theorem for the spherical harmonics in second line of Eq. (2) has been used. The second invariant Mandelstam variable t (momentum transfer) is related to $\cos\theta$ by the following Equation

$$\cos\theta = 1 + \frac{t}{2\mathbf{q}^2}. \quad (3)$$

A remarkable analytic properties of scattering amplitudes as functions of momentum transfer have been discovered in the year 1958 by Harry Lehmann [4] using Jost-Lehman-Dyson representation especially Dyson's theorem for a representation of causal commutators in local Quantum Field Theory [5,6,7]. Lehmann proved that imaginary part of two-body interaction amplitude is analytic function of $\cos\theta$, regular inside an ellipse in $\cos\theta$-plane with center at the origin and with semi-major axis

$$z_0(s) = 1 + \epsilon_L(s), \quad \epsilon_L(s) = \frac{2(m_1^2 - m_a^2)(m_2^2 - m_b^2)}{\mathbf{q}^2[s - (m_1 - m_2)^2]}, \quad (4)$$

5

where m_1 and m_2 define the support of spectral function in the JLD representation by the requirements of spectral condition or spectrality. Actually, m_1 and m_2 are the lowest mass values of the physical states for which the following matrix elements are not equal to zero

$$< 0|J_a(0)|m_1 >\neq 0, \quad < 0|J_b(0)|m_2 >\neq 0,$$

where $J_a(x)$ and $J_b(x)$ are local Heisenberg's currents of particles a and b. He also shown that two-body interaction amplitude, as itself, is analytic function of $\cos\theta$, regular inside an ellipse in $\cos\theta$-plane with center at the origin and with semi-major axis $x_0(s)$ which is related to $z_0(s)$ by the Equation

$$x_0(s) = \sqrt{\frac{z_0(s) + 1}{2}}. \tag{5}$$

Afterwards the fundamental results of Harry Leman were improved by Martin [8] and Sommer [9]: it was shown that imaginary part of two-body interaction amplitude is analytic function of $\cos\theta$, regular inside an ellipse in $\cos\theta$-plane with semi-major axis

$$z_0(s) = 1 + \epsilon_M(s), \quad \epsilon_M(s) = \frac{t_0}{2\mathbf{q}^2}, \quad t_0 = 4m_\pi{}^2, \tag{6}$$

$$\mathbf{q}^2 = \frac{\lambda(s, m_a^2, m_b^2)}{4s} = \frac{[s - (m_a + m_b)^2][s - (m_a - m_b)^2]}{4s}, \tag{7}$$

where m_π is pion mass. Correspondingly two-body interaction amplitude, as itself, appears as analytic function of $\cos\theta$, regular inside an ellipse in $\cos\theta$-plane with semi-major axis $x_0(s)$ which is related to $z_0(s)$ by Eq. (5).

The fundamental results derived by Lehmann and improved by his followers are of great importance because it has been shown that the partial wave expansions (2) which define physical scattering amplitudes continue to converge for complex values of the scattering angle, and define uniquely the amplitudes appearing in the unphysical region of non-forward dispersion relations. In fact, expansions converge for all values of momentum transfer for which dispersion relations have been proved. The proved analyticity of two-body interaction amplitudes as functions of two complex Mandelstam variables s and t in a topological product of cut s-plane with the cuts $(s_{thr} \leq s \leq \infty, u_{thr} \leq u \leq \infty)$ except for possible fixed poles and circle $|t| \leq t_0$ in t-plane allowed in a more general case to save the fundamental Froissart results previously obtained at a more restricted Mandelstam analyticity. Really, let us wright Cauchy representation for imaginary part of two-body interaction amplitude

$$Im\mathcal{F}_2(s; \cos\theta) = \frac{1}{2\pi i} \oint_C dz \frac{Im\mathcal{F}_2(s; z)}{z - \cos\theta},$$

where contour C is a boundary of an ellipse in $\cos\theta$-plane with semi-major axis given by Eq. (6). Using Heine formula

$$\frac{1}{z - \cos\theta} = \sum_{l=0}^{\infty}(2l + 1)Q_l(z)P_l(\cos\theta),$$

we obtain

$$Imf_l(s) = \frac{\Gamma_2(s)}{2i} \oint_C dz Im\mathcal{F}_2(s; z)Q_l(z). \tag{8}$$

6

From Eq. (8) it follows

$$Im f_l(s) \leq \frac{1}{2}\Gamma_2(s) \cdot \max_{z \in C} |Im\mathcal{F}_2(s; z)| \cdot \max_{z \in C} |Q_l(z)| \cdot \mathcal{L}(C), \tag{9}$$

where $\mathcal{L}(C)$ is a length of contour C. Representation (8) where estimate (9) followed from is a good tool to study an asymptotic behaviour of partial waves at large orbital momentum. Using asymptotic properties of the Legendre functions Q_l [10]

$$Q_l(z) \simeq \sqrt{\frac{\pi}{2l}} (z^2 - 1)^{-\frac{1}{4}} (z + \sqrt{z^2 - 1})^{-l-\frac{1}{2}}, \quad |l| \to \infty, \quad |\arg l| < \pi, \quad z \in C$$

and polynomial boundedness

$$\max_{z \in C} |Im\mathcal{F}_2(s; z)| \leq P_2(s),$$

$P_2(s)$ is some polynomial in s, we find

$$Im f_l(s) \leq \sqrt{\frac{2\pi}{l}} \Gamma_2(s) P_2(s) \left(\frac{z_0(s) + \sqrt{z_0^2(s) - 1}}{\sqrt{z_0^2(s) - 1}} \right)^{\frac{1}{2}} [z_0(s) + \sqrt{z_0^2(s) - 1}]^{-l}, \quad |l| \to \infty. \tag{10}$$

If we put $z_0(s) = 1 + \epsilon(s)$, $\epsilon(s) << 1$, $s \to \infty$ then estimate (10) at large values of s may be rewritten in the form

$$Im f_l(s) \leq \frac{\tilde{P}_2(s)}{\sqrt{l}} \exp\left(-l\sqrt{2\epsilon(s)}\right), \quad s \to \infty, \tag{11}$$

where

$$\tilde{P}_2(s) = \left(\frac{2\pi}{\sqrt{2\epsilon(s)}} \right)^{\frac{1}{2}} \Gamma_2(s) P_2(s). \tag{12}$$

Thus we have obtained a very important result: analyticity of two-body interaction amplitudes as functions of $\cos\theta$, regular inside an ellipse in $\cos\theta$-plane, results in exponential decrease of partial waves as functions of orbital momentum l at large values of l. This means that the significant contribution to the partial wave expansion (2) is determined by partial waves for which the orbital momentum does not exceed the quantity

$$L = \left[\frac{\ln \tilde{P}_2(s)}{\sqrt{2\epsilon(s)}} \right]. \tag{13}$$

The contribution of partial waves with $l > L$ to the partial wave expansion will be exponentially small. Let us decompose the partial wave expansion in two terms

$$Im\mathcal{F}_2(s; \cos\theta = 1) = \frac{1}{\pi\Gamma_2(s)} \sum_{l=0}^{L-1} (2l + 1)Im f_l(s) + Im\mathcal{F}_2^L(s), \tag{14}$$

where the second term in Eq. (14) contains the contribution of partial waves with $l \geq L$. Now we would like to take advantage of unitarity condition which can be written for the partial waves as the following sequence of inequalities

$$0 \leq |f_l(s)|^2 \leq Im f_l(s) \leq |f_l(s)| \leq 1. \tag{15}$$

7

Taking into account the unitarity condition we get for the first term in Eq. (14) an estimate in the form

$$\frac{1}{\pi\Gamma_2(s)}\sum_{l=0}^{L-1}(2l+1)Imf_l(s) \le \frac{1}{\pi\Gamma_2(s)}\sum_{l=0}^{L-1}(2l+1) = \frac{L^2}{\pi\Gamma_2(s)} = \frac{\ln^2\tilde{P}_2(s)}{2\pi\epsilon(s)\Gamma_2(s)}, \qquad (16)$$

where expression (13) for the quantity L has been used.

Froissart has shown that the second term in Eq. (11) is asymptotically small compared to the first one at large values of s, so that we finally get

$$Im\mathcal{F}_2(s;\cos\theta=1) < \frac{\ln^2\tilde{P}_2(s)}{2\pi\epsilon(s)\Gamma_2(s)}. \qquad (17)$$

The optical theorem relates a total cross section of two-body interaction with imaginary part of two-body forward elastic scattering amplitude

$$\sigma_{ab}^{tot}(s) = \frac{(2\pi)^3}{\lambda^{1/2}(s,m_a^2,m_b^2)}Im\mathcal{F}_2(s;\cos\theta=1),$$

λ-function is defined by Eq. (7). Hence from estimate (17) it follows an upper bound for the total cross section of two-body interaction

$$\sigma_{ab}^{tot}(s) < \frac{S_2\ln^2\tilde{P}_2(s)}{32s\epsilon(s)A_2^2(s)}. \qquad (18)$$

where, as it was mentioned above,

$$A_2(s) = \Gamma_2(s)/S_2 = \frac{\lambda^{1/2}(s,m_a^2,m_b^2)}{8s}.$$

Here is just the place to introduce the physical notion of the effective radius of two-body forces [12,13]. Let us define the effective radius $R_2(s)$ of two-body forces by the following equation

$$R_2(s) \stackrel{def}{=} \frac{L}{|\mathbf{q}|} = \frac{2\sqrt{s}\ln\tilde{P}_2(s)}{\sqrt{2\epsilon(s)\lambda(s,m_a^2,m_b^2)}}, \qquad (19)$$

where the definition (13) of the quantity L and expression (7) for $\mathbf{q}$ have been used. Now upper bound (18) in terms of such defined quantity $R_2(s)$ takes the form

$$\sigma_{ab}^{tot}(s) < 4\pi R_2^2(s). \qquad (20)$$

This form of the upper bound for experimentally measured quantity $\sigma_{ab}^{tot}(s)$ has a quite transparent physical and clear geometrical meanings: it means that the total cross section of two-body interaction is bounded by the area of a surface of two-dimensional sphere whose radius is defined by the effective radius of two-body forces. A remarkable property of upper bound (20) consist in the fact that here all information about analytic properties of two-body interaction amplitudes is hidden in the physically tangible quantity (19) which is the effective radius of two-body forces. If we put $\epsilon(s)$ equal to $\epsilon_M(s)$ given by Eq. (6) then from Eqs. (16) and (12) it follows that

$$\tilde{P}_2(s) \sim \tilde{c}_2\, s^{9/4}, \quad s \to \infty.$$

In that case for the the effective radius of two-body forces we find from Eq. (19)

$$R_2(s) = \frac{\ln \tilde{P}_2(s)}{\sqrt{t_0}} \sim \frac{9}{4\sqrt{t_0}} \ln(s/s_0) = \frac{9}{8m_\pi} \ln(s/s_0), \quad s \to \infty. \tag{21}$$

In the article Froissart gave an excellent semiclassical explanation corroborating his theorem. We would like to present here a remarkable fragment from section II of the Froissart paper [3]. He wrote: "*To get intuitive idea why the amplitude is bounded in the physical region, let us consider a classical problem: Two particles interact by means of absorptive Yukawa potential $g\,e^{-\kappa r}/r$. If a is the impact parameter, the total interaction seen by a particle for large a is likely to be approximately $g\,e^{-\kappa a}$. If this is small compared to one, there will be practically no scattering. If $|g\,e^{-\kappa a}|$ is large compared to one, there will be practically complete scattering, so that the cross section will be essentially determined by the value $a = (1/\kappa)\ln|g|$ where $|g\,e^{-\kappa a}| = 1$. It is $\sigma \cong (\pi/\kappa^2)\ln^2 |g|$. If we now assume that g is a function of the energy, and increases like a power of the energy, then σ will vary at most like the squared logarithm of the energy.*" In fact, Froissart anticipated here a running coupling and quasi-potential character of strong forces. Later on it was shown [14] that the hypothesis about validity of the dispersion relations in the momentum transfer leads, for any value of the energy s, to a potential which is a superposition of Yukawa potentials with energy dependent intensities. This fact together with a theorem on single-time reduction in Quantum Field Theory [15] provides a strong basis for semiclassical consideration given by Froissart. However, it should be stressed that upper bound (20) has a quite different geometrical sense compared to semiclassical consideration given by Froissart: Eq. (20) shows that the total cross section of two-body interaction is bounded by the area of a surface of the sphere with the radius equal to the effective radius of two-body forces but not by the area of a disk with the same radius.

Unitarity bound (20) states that the total probability (per unit volume per unit time in fraction of particles density flux) of all possible (elastic and inelastic) two-particle interactions, which take place in a limited volume V during a limited interval of time T, is limited by the area of a surface of the sphere which is, actually, a boundary of the volume V. This means that widely discussed in the recent literature concerning some physical problems at Planck scale *the holographic principle* [16] has been incorporated in the general scheme of axiomatic Quantum Field Theory and resulted from the general principles of local Quantum Field Theory.

2. Generalized Froissart theorem

In our works [17,18] it was shown that there is a quite natural geometrical generalization of the Froissart theorem to the case of multiparticle interaction. In this respect it should be noted that the problem of finding such generalization is non-trivial because at least the known singularities of multiparticle scattering amplitudes related to disconnected parts by cluster structure of the amplitudes point to the fact that for the total amplitude of n-particle scattering ($n \geq 3$) there is no such generalization. Connected part of n-particle ($n \geq 3$) scattering amplitudes contains singular rescattering terms as well. Therefore, the first problem which arises in this case is to define a suitable object connected with the $n \to n$ reaction amplitude which would permit a correct formulation of the problem. It turns out there is a wide class of many-particle reaction amplitudes for which such a problem would be quite meaningful. We have shown that these amplitudes should be understood as amplitudes of true n-particle interaction or n-body

forces amplitudes; see details in [17,18]. Here we reproduce our results taking a line stated in previous section.

The scattering amplitude of the n-body reaction may be considered as a function of the invariant variable $s = (p_1 + p_2 + \cdots + p_n)^2$ and two unit vectors $\mathbf{e}$ and $\mathbf{e}'$ on $(D-1)$-dimensional sphere S_{D-1}, which characterise the initial and final states of n-particle system:

$$\mathcal{F}_n(s; p'_1 p'_2 \cdots p'_n, p_1 p_2 \cdots p_n) = \mathcal{F}_n(s; \mathbf{e}', \mathbf{e}).$$

Dimensionality D of multidimensional space is related to the number of particles n by the equation $D = 3n - 3$. There are many ways to introduce the spherical coordinates in multidimensional space. Moreover, there are some peculiarities related to a parametrization of relativistic n-particle system. However, we will not concern this subject here because it does not play any role for our main goal. For the details we refer to [18] and references therein.

As above we may wright the partial wave expansion

$$\mathcal{F}_n(s; \mathbf{e}', \mathbf{e}) = \mathcal{F}_n(s; \cos\omega) =$$

$$= \frac{1}{\pi A_n(s)} \sum_{lm} Y_{lm}(\mathbf{e}') f_l(s) \overset{*}{Y}_{lm}(\mathbf{e}) = \frac{1}{\pi \Gamma_n(s)} \sum_l (\frac{l}{\nu} + 1) f_l(s) C_l^\nu(\cos\omega), \qquad (22)$$

where $A_n(s) = \Gamma_n(s)/S_{D-1}$, $\Gamma_n(s)$ is n-particle phase space volume, $S_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is a surface of $(D-1)$-dimensional unit sphere, $\cos\omega = \mathbf{e}' \cdot \mathbf{e}$, and we have used in second line of Eq. (22) an addition theorem for the (hyper)spherical harmonics in multidimensional space

$$\sum_{m=1}^{M(l,\nu)} Y_{lm}(\mathbf{e}') \overset{*}{Y}_{lm}(\mathbf{e}) = \left(\frac{l}{\nu} + 1\right) S_{D-1}^{-1} C_l^\nu(\mathbf{e}' \cdot \mathbf{e}),$$

$$\nu = \frac{D}{2} - 1, \quad M(l, \nu) = \frac{(2l + 2\nu)\Gamma(l + 2\nu)}{\Gamma(l + 1)\Gamma(2\nu + 1)},$$

where $C_l^\nu(z)$ is Gegenbauer polynomial. Here we contented ourself with a special class of n-body forces scattering amplitudes which are invariant under rotation in multidimensional space (so called $O(D)$-invariant amplitudes).

We will assume that for physical values of the variable s imaginary part of n-body forces scattering amplitude is analytic function of $\cos\omega$, regular inside an ellipse $E_n(s)$ in $\cos\omega$-plane with center at the origin and with semi-major axis

$$z_n(s) = 1 + \epsilon_n(s), \quad \epsilon_n(s) = \frac{M_n^2}{2\mathbf{Q}^2}, \qquad (23)$$

and for any $\cos\omega \in E_n(s)$ is polynomially bounded in the variable s, M_n is some constant of mass dimensionality independent of s, $\mathbf{Q}$ is global momentum (dependent of s) of n-particle system which will be defined later on. Such analyticity of n-body forces scattering amplitudes was called *global* [18]. If it is so, one can wright Cauchy representation for imaginary part of n-body interaction amplitude

$$Im\mathcal{F}_n(s; \cos\omega) = \frac{1}{2\pi i} \oint_{C_n} dz \frac{Im\mathcal{F}_n(s; z)}{z - \cos\omega},$$

where contour C_n is a boundary of an ellipse $E_n(s)$ in $\cos\omega$-plane with semi-major axis given by Eq. (23). There is a standard generalization of Heine's expansion of the Cauchy denominator [10]

$$\frac{1}{z-t} = \exp(-i\pi\nu)2^{2\nu}[\Gamma(\nu)]^2(z^2-1)^{\nu-1/2}\sum_{l=0}^{\infty}(l+\nu)\frac{\Gamma(l+1)}{\Gamma(l+2\nu)}D_l^\nu(z)C_l^\nu(t), \qquad (24)$$

which converges absolutely for

$$|[t+(t^2-1)^{1/2}]/[z+(z^2-1)^{1/2}]| < 1. \qquad (25)$$

In Eq. (24) $D_l^\nu(z)$ is a second solution to Gegenbauer's equation. The restriction (25) requires that the point t lie within that ellipse in the complex t-plane with foci at ± 1 which passes through the point $t = z$. In particular from Eq. (24) it follows

$$D_l^\nu(z) = \exp(i\pi\nu)(z^2-1)^{-\nu+1/2}\frac{1}{2\pi}\int_{-1}^{1}dt\frac{(1-t^2)^{\nu-1/2}C_n^\nu(t)}{z-t}.$$

As a result we obtain

$$Imf_l(s) = \frac{\exp(-i\pi\nu)\nu 2^{2\nu}[\Gamma(\nu)]^2\Gamma(l+1)}{\Gamma(l+2\nu)} \cdot \frac{\Gamma_n(s)}{2i}\oint_{C_n}dz(z^2-1)^{\nu-1/2}D_l^\nu(z)Im\mathcal{F}_n(s;z). \qquad (26)$$

Representation (26) is very useful to study an asymptotic behaviour of partial waves at large global orbital momentum. Taking into account asymptotic properties of the Gegenbauer functions D_l^ν [10]

$$D_l^\nu(z) \simeq \frac{\exp(i\pi\nu)l^{\nu-1}}{2^\nu\Gamma(\nu)}(z^2-1)^{-\nu/2}(z+\sqrt{z^2-1})^{-l-\nu}, \quad |l|\to\infty, \, |\arg l| < \pi, \, z\in C_n,$$

and polynomial boundedness

$$\max_{z\in C_n}|Im\mathcal{F}_n(s;z)| \le P_n(s),$$

$P_n(s)$ is some polynomial in s, we find

$$Imf_l(s) \le \Gamma_n(s)P_n(s)\frac{l^{\nu-1}\nu 2^{\nu+1}\Gamma(\nu)\Gamma(l+1)}{\Gamma(l+2\nu)}\times$$

$$\left(\frac{z_n(s)+\sqrt{z_n^2(s)-1}}{\sqrt{z_n^2(s)-1}}\right)^{1-\nu}\left(z_n(s)+\sqrt{z_n^2(s)-1}\right)^{-l}, \quad |l|\to\infty. \qquad (27)$$

Finally if we put $z_n(s) = 1 + \epsilon_n(s)$, $\epsilon_n(s) << 1$, $s\to\infty$ then we get at large values of s

$$Imf_l(s) \le \frac{P_n(s,\nu)}{l^\nu}\exp\left(-l\sqrt{2\epsilon_n(s)}\right), \quad s\to\infty, \qquad (28)$$

where

$$P_n(s,\nu) = \nu 2^{\nu+1}\Gamma(\nu)[2\epsilon_n(s)]^{(\nu-1)/2}\Gamma_n(s)P_n(s). \qquad (29)$$

Estimate (29) shows that partial waves as functions of global orbital momentum l exponentially decrease at large values of l, i.e. the significant contribution to the partial wave expansion (22) is resulted from partial waves for which the global orbital momentum does not exceed the quantity

$$\Lambda = \left[\frac{\ln P_n(s,\nu)}{\sqrt{2\epsilon_n(s)}}\right]. \qquad (30)$$

11

The contribution of partial waves with $l > \Lambda$ to the partial wave expansion will be exponentially small. So, we decompose the partial wave expansion in two terms

$$Im\mathcal{F}_n(s; \cos\omega = 1) = \frac{1}{\pi\Gamma_n(s)} \sum_{l=0}^{\Lambda} (\frac{l}{\nu} + 1)Imf_l(s)C_l^{\nu}(1) + Im\mathcal{F}_n^{\Lambda}(s), \tag{31}$$

where the second term in Eq. (31) contains the contribution of partial waves with $l > \Lambda$. Taking into account the unitarity condition (15) for the partial waves we get for the first term in Eq. (31) an estimate

$$\frac{1}{\pi\Gamma_n(s)} \sum_{l=0}^{\Lambda} (\frac{l}{\nu} + 1)Imf_l(s)C_l^{\nu}(1) \leq \frac{1}{\pi\Gamma_n(s)} \sum_{l=0}^{\Lambda} (\frac{l}{\nu} + 1)C_l^{\nu}(1) =$$

$$\frac{(2\Lambda + 2\nu + 1)\Gamma(\Lambda + 2\nu + 1)}{\pi\Gamma_n(s)\Gamma(2\nu + 2)\Gamma(\Lambda + 1)} = \frac{2\Lambda^{2\nu+1}}{\pi\Gamma_n(s)\Gamma(2\nu + 2)} \left(1 + O(\frac{1}{\Lambda})\right), \tag{32}$$

where we inserted $C_l^{\nu}(1) = \Gamma(l + 2\nu)/[\Gamma(2\nu)\Gamma(l+1)]$. It can easily be seen that the second term in Eq. (27) is asymptotically small compared to the first one at large values of s, so that we finally get

$$Im\mathcal{F}_n(s; \cos\omega = 1) < \frac{2\left[\ln P_n(s, \nu)\right]^{D-1}}{\pi\Gamma(D)\Gamma_n(s)[2\epsilon_n(s)]^{(D-1)/2}}. \tag{33}$$

where we have used expression (30) for Λ and relation $2\nu = D - 2$. By analogy with Eq. (19) let us introduce the effective radius $R_n(s)$ of n-body forces

$$R_n(s) \stackrel{def}{=} \frac{\Lambda}{|\mathbf{Q}|} = \frac{1}{M_n} \ln P_n(s, \nu), \tag{34}$$

where the definition (30) of the quantity Λ and expression (23) for $\epsilon_n(s)$ have been used. Now upper bound (33) in terms of such defined quantity $R_n(s)$ takes the form

$$Im\mathcal{F}_n(s; \cos\omega = 1) < \frac{2\left[R_n(s)\right]^{D-1}}{\pi\Gamma(D)\Gamma_n(s)[2\epsilon_n(s)/M_n^2]^{(D-1)/2}} = J_n(s)S_{D-1}[R_n(s)]^{D-1}, \tag{35}$$

where

$$J_n(s) = \frac{2}{\pi\Gamma(D)S_{D-1}^2 A_n(s)[2\epsilon_n(s)/M_n^2]^{(D-1)/2}} = \frac{2|\mathbf{Q}|^{D-1}}{\pi\Gamma(D)S_{D-1}^2 A_n(s)}. \tag{36}$$

With account of the generalized optical theorem relating a total cross section of n-body interaction with imaginary part of n-body forces forward scattering amplitude [18]

$$\sigma_n^{tot}(s) = \frac{1}{J_n(s)}Im\mathcal{F}_n(s; \cos\omega = 1),$$

from estimate (35) we obtain an upper bound for the total cross section of n-body interaction

$$\sigma_n^{tot}(s) < S_{D-1}[R_n(s)]^{D-1}. \tag{37}$$

Here again, as it should be, upper bound (37) has a quite clear geometrical meaning: the total cross section of n-body interaction is bounded by the area of a surface of $(D-1)$-dimensional sphere whose radius is defined by the effective radius of n-body forces. Again all information

about global analyticity of n-body interaction amplitudes is hidden in the physical quantity (34) which is the effective radius of n-body forces. From Eqs. (29) and (33) it follows that

$$P_n(s, \nu) \sim c_n \, s^{(3n+3)/4}, \quad s \to \infty.$$

For the the effective radius of n-body forces we find from Eq. (34) in that case

$$R_n(s) \sim \frac{r_n}{M_n} \ln(s/s_0), \quad r_n = \frac{3n+3}{4}, \quad s \to \infty. \tag{38}$$

Upper bounds (35,37) are a direct consequence of global analyticity of n-body forces scattering amplitudes which, in one's turn, is a direct geometrical generalization of analytic properties of two-body scattering amplitude strictly proved in axiomatic Quantum Field Theory. At present we do not know to what extend global analyticity of n-particle scattering amplitudes ($n \geq 3$) is a consequence of general principles of local Quantum Field Theory. The validity of such an assumption is obvious to us if we rely on the physical nature of n-body forces: our intuition tells us that true n-body interactions should manifest themselves only in the case when all the n particles are in a sufficiently limited volume. On the other hand, from the beginning one may, by definition, consider the n-body forces scattering amplitude to be a globally analytic part of the total S-matrix which may always be singled out from it [18].

At last, we have to give the definition of global momentum $|\mathbf{Q}|$ for the relativistic n-particle system. In this respect, first of all, note that momentum $\mathbf{q}$ for two-particle system has been defined in a relativistic covariant way. Under any Lorentz transformation Λ from the restricted Lorentz group $\Lambda \in \mathcal{L}_+^\uparrow$ momentum $\mathbf{q}$ is transforming by Wigner rotation:

$$\mathbf{q} \to \mathbf{q}' = R_W \mathbf{q}, \quad R_W = L^{-1}(\Lambda P_{ab}) \Lambda L(P_{ab}),$$

$L(P_{ab})$ is Lorentz boost. This means that $|\mathbf{q}|$ defined by Eq. (7) is a Lorentz invariant quantity. Moreover, we would like to emphasize the following asymptotic properties

$$\mathbf{q}^2 \simeq \frac{1}{4}s, \, s \to \infty; \quad \mathbf{q}^2 \simeq 2\mu_2(\sqrt{s} - M_2), \, \sqrt{s} \to M_2, \, M_2 = m_a + m_b, \, \mu_2 = \frac{m_a m_b}{M_2}. \tag{39}$$

The expression of $\mathbf{q}^2$ given by Eq. 7 can be rewritten in the form

$$\mathbf{q}^2 = 16s \left(\Gamma_2(s)/S_2\right)^2 = 16s A_2^2(s). \tag{40}$$

The definition of global momentum for the relativistic n-particle system should be given such as to save the asymptotic properties shown by Eqs. (39). Such generalization for any number of particles looks like

$$\mathbf{Q}^2 = \gamma_n s^{(n-1)/(3n-5)} A_n^{2/(3n-5)}, \tag{41}$$

where γ_n is dimensionless constant

$$\gamma_n = 2^{2n/(3n-5)} \left(\frac{\mu_n}{M_n}\right)^{(2n-4)/(3n-5)}, \quad \mu_n = \left(\frac{\prod_{i=1}^n m_i}{M_n}\right)^{1/(n-1)}, \quad M_n = \sum_{i=1}^n m_i. \tag{42}$$

From the definition (41) we have the following asymptotic properties:

$$\mathbf{Q}^2 \simeq a_n^2 s, \quad s \to \infty, \tag{43}$$

13

where a_n is dimensionless constant

$$a_n^2 = \left(\frac{\Gamma(3n/2 - 3/2)}{\pi^{(n-1)/2}(n-1)!(n-2)!} \right)^{2/(3n-5)} \left(\frac{\mu_n}{M_n} \right)^{(2n-4)/(3n-5)}, \tag{44}$$

for example

$$a_2^2 = \frac{1}{4}, \quad a_3^2 = \left(\frac{\mu_3}{\pi M_3} \right)^{1/2}, \quad a_3^{-1}(m_1 = m_2 = m_3) = 2.0100..., $$

and

$$\mathbf{Q}^2 \simeq 2\mu_n(\sqrt{s} - M_n), \quad \sqrt{s} \to M_n. \tag{45}$$

3. Physical applications and discussion

Let us come back to Eq. (16) and remind an ancient Pythagoras theorem stated that the sum of first N odd numbers beginning from unity is equal exactly to the square of N i.e.

$$\underbrace{1 + 3 + 5 + 7 + \cdots}_{N} = N^2. \tag{46}$$

This Pythagoras theorem can easily be proved with the help of the formula for an arithmetical progression. However, Pythagoras theorem can be proved without a knowledge of the formula for an arithmetical progression but using only some remarkable observations in a game with the numbers. We will not touch here the simplest proof, we would only like to stress a deep link between the Froissart bound and this Pythagoras theorem. Of course, to take advantage of this link we have to learn apart from differential calculus and integral calculus that:

- Symmetry properties of the space-time continuum are described by inhomogeneous Lorentz group or Poincaré group. We had also to know how to construct the unitary representations of this group as well, as it was made in the fundamental paper of Wigner [11].
- There is a very deep connexion between general physical principles such as causality, spectrality, unitarity and analytic properties of physical scattering amplitudes. The very essence of this connexion is expressed by brilliant Jost-Lehmann-Dyson representation which provided the fundamental results of Lehmann.
- It takes many other attainments and the knowledge acquisitions as well.

There is a generalization of Pythagoras theorem (46). Really, let us consider any polynomial $P_n(x)$ degree of n

$$P_n(x) = c_0(P) + c_1(P)x + c_2(P)x^2 + \cdots + c_n(P)x^n,$$

let $S(N)$ be a sum of the polynomial values when the argument x takes an integer value

$$S(N) \stackrel{def}{=} \sum_{k=0}^{N} P_n(k),$$

then it can be proved that $S(N)$ is also a polynomial $Q_{n+1}(N)$ in N degree of $(n+1)$

$$S(N) = Q_{n+1}(N), \quad Q_{n+1}(x) = c_0(Q) + c_1(Q)x + c_2(Q)x^2 + \cdots + c_{n+1}(Q)x^{n+1}, \tag{47}$$

14

and there is correspondence between $c_n(P)$ and $c_n(Q)$: $c_{n+1}(Q) = c_n(P)/(n+1), \cdots$. For example, if $P_4(x)$ is polynomial of fourth degree then we have

$$c_5(Q) = c_4(P)/5,$$
$$c_4(Q) = c_3(P)/4 + c_4(P)/2,$$
$$c_3(Q) = c_2(P)/3 + c_3(P)/2 + c_4(P)/3,$$
$$c_2(Q) = c_1(P)/2 + c_2(P)/2 + c_3(P)/4,$$
$$c_1(Q) = c_0(P) + c_1(P)/2 + c_2(P)/6 - c_4(P)/30.$$
$$c_0(Q) = c_0(P). \tag{48}$$

We will call that statement as a generalized Pythagoras theorem. It can easily be seen that usual Pythagoras theorem (46) corresponds to $P_1(x) = 2x + 1$. From Eq. (32) it's clear that the generalized Froissart theorem is related to the generalized Pythagoras theorem where $P_{D-2}(x)$ is being used.

In according with the theory held by Pythagoras the unitarity bounds (20) and (37) give form to the Unlimited and therefore they are Limit; see Introduction.

Recently [19,20,21] a simple theoretical formula describing the global structure of pp and $p\bar{p}$ total cross-sections in the whole range of energies available up today has been derived by an application of single-time formalism in QFT and general theorems a là Froissart. The fit to the experimental data with the formula was made, and it was shown that there is a very good correspondence of the theoretical formula to the existing experimental data obtained at the accelerators. Moreover, it turned out there is a very good correspondence of the theory to all existing cosmic ray experimental data as well [21]. The predicted values for σ_{pp}^{tot} obtained from theoretical description of all existing accelerators data are completely compatible with the values obtained from cosmic ray experiments. The global structure of (anti)proton-proton total cross section is shown in Figs. 1-2 extracted from papers [20,21].

The theoretical formula describing the global structure of (anti)proton-proton total cross section has the following structure

$$\sigma_{(\bar{p})pp}^{tot}(s) = \sigma_{asmpt}^{tot}(s)\left[1 + \chi_{(\bar{p})pp}(s)\right], \tag{49}$$

where

$$\sigma_{asmpt}^{tot}(s) = 2\pi\left[B_{el}(s) + (1-\beta)R_3^2(s)\right] = \left[42.0479 + 1.7548\ln^2(\sqrt{s}/20.74)\right](mb), \tag{50}$$

$$B_{el}(s) = R_2^2(s)/2 = \left[11.92 + 0.3036\ln^2(\sqrt{s}/20.74\right](GeV^{-2}),$$

$$R_3^2(s)|_{\beta<<1} = \left[0.40874044\sigma_{asmpt}^{tot}(s)(mb) - B_{el}(s)\right](GeV^{-2}) =$$

$$= \left[5.267 + 0.4137\ln^2\sqrt{s}/20.74\right](GeV^{-2}), \tag{51}$$

$$\beta = \frac{x_{inel}^2}{4(1+x_{inel}^2)}, \quad x_{inel}^2 = \frac{R_3^2(s)}{R_d^2} = \frac{2B_{sd}}{R_d^2},$$

$B_{el}(s)$ is the slope of nucleon-nucleon differential elastic scattering cross section, $R_2(s)$ is the effective radius of two-nucleon forces, $R_3(s)$ is the effective radius of three-nucleon forces, R_d characterizes the internucleon distance in a deuteron, the functions $\chi_{(\bar{p})pp}(s)$ describe low-energy

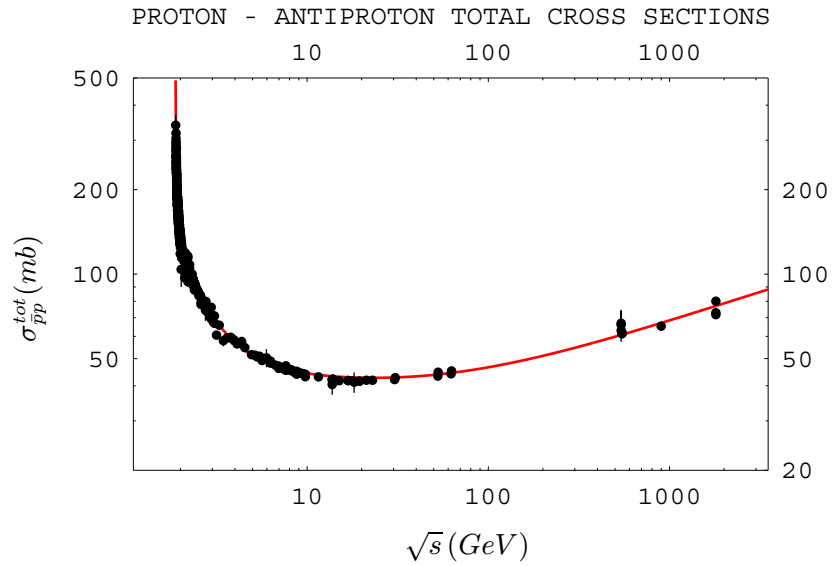PROTON - ANTIPROTON TOTAL CROSS SECTIONS

Fig. 1.   The proton-antiproton total cross sections versus $\sqrt{s}$ compared with the theory. Solid line represents our fit to the data [19,20]. Statistical and systematic errors added in quadrature.

parts of (anti)proton-proton total cross sections and asymptotically tend to zero at $s \to \infty$ (see details in the original paper [20]). The mathematical structure of the formula (49) is very simple and physically transparent: the total cross section is represented in a factorized form. One factor describes high energy asymptotics of total cross section and it has the universal energy dependence predicted by the general Froissart theorem in local Quantum Field Theory.
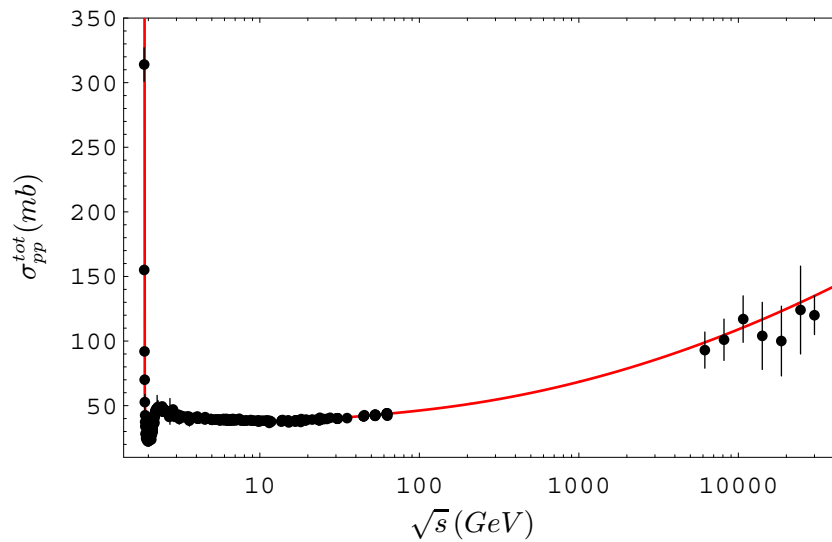
Fig. 2.   The proton-proton total cross-section versus $\sqrt{s}$ with the cosmic rays data points from Akeno Observatory and Fly's Eye Collaboration. Solid line corresponds to our theory predictions [21].

16

The other factor is responsible for the behaviour of total cross section at low energies and it has a complicated resonance structure. However this factor has also the universal asymptotics at elastic threshold. It is a remarkable fact that the low energy part of total cross section has been derived by application of the generalized Froissart theorem for a three-body forces scattering amplitude.

Eq. (50) shows that geometrical scaling in a naive form $\sigma_{asmpt}^{tot}(s) = \text{Const} B_{el}(s)$ is not valid. However, from Eq. (50) it follows the generalized geometrical scaling which looks like

$$\sigma_{asmpt}^{tot}(s) = 2\pi B_{el}(s)[1 + 2\gamma(1 - \beta)], \tag{52}$$

where β is defined above and

$$\gamma = \frac{R_3^2(s)}{2B_{el}(s)} = \frac{R_3^2(s)}{R_2^2(s)}.$$

Here, we would like to point out some remarkable features of the global structure in the (anti)proton-proton total cross sections. First of all, the (anti)proton-proton total cross sections have a minimum at $s = s_0$, and the question is what this minimum corresponds to. It turns out that the effective radius of three-nucleon forces at the point $s = s_0$ satisfys the following harmonic ratio

$$\boxed{R_3(s_0) : r_p^{ch} = 1 : 2}, \tag{53}$$

where $r_p^{ch} = 0.88\,fm$ is the proton charge radius. In other words, at the minimum $s = s_0$ it takes place the *"octave consonance"* of the three-nucleon forces with the proton charge distribution.

Going further on, we have applied our approach to study a shadow dynamics in scattering from deuteron in some details. In this way a new simple formula for the shadow corrections to the total cross-section in scattering from deuteron has been derived and new scaling characteristics with a clear physical interpretation have been established. We shall briefly sketch the basic results of our analysis of high-energy particle scattering from deuteron. As has been shown in [22], the total cross-section in the scattering from deuteron can be expressed by the formula

$$\sigma_{hd}^{tot}(s) = \sigma_{hp}^{tot}(\hat{s}) + \sigma_{hn}^{tot}(\hat{s}) - \delta\sigma(s),$$

where $\sigma_{hd}, \sigma_{hp}, \sigma_{hn}$ are the total cross-sections in scattering from deuteron, proton and neutron,

$$\delta\sigma(s) = \delta\sigma^{el}(s) + \delta\sigma^{inel}(s) = 2\sigma^{el}(s)a^{el}(x_{el}) + 2\sigma_{sd}^{ex}(s)a^{inel}(x_{inel}), \tag{54}$$

$$\sigma^{el}(s) \equiv \frac{\sigma_{hN}^{tot\,2}(s)}{16\pi B_{el}(s)}, \quad a^{el}(x_{el}) = \frac{x_{el}^2}{1 + x_{el}^2}, \quad x_{el}^2 \equiv \frac{2B_{el}(s)}{R_d^2} = \frac{R_2^2(s)}{R_d^2},$$

$$a^{inel}(x_{inel}) = \frac{x_{inel}^2}{(1 + x_{inel}^2)^{3/2}}, \quad x_{inel}^2 \equiv \frac{R_3^2(s)}{R_d^2} = \frac{2B_{sd}(s)}{R_d^2},$$

the total single diffractive dissociation cross-section $\sigma_{sd}^{ex}(s)$ is defined by the following equation [22]

$$\sigma_{sd}^{\varepsilon}(s) = \pi \int_{M_{min}^2}^{\varepsilon s} \frac{dM_X^2}{s} \int_{t_-(M_X^2)}^{t_+(M_X^2)} dt \frac{d\sigma}{dt dM_X^2}, \tag{55}$$

where

$$\varepsilon = \varepsilon^{ex} = \sqrt{2\pi}/2M_N R_d, \tag{56}$$

17

and we supposed that $\sigma_{hp}^{tot} = \sigma_{hn}^{tot} = \sigma_{hN}^{tot}$ and $B_{el}^{hp} = B_{el}^{hn} = B_{el}$ at high energies. The first term in the R.H.S. of Eq. (54) generalizes the known Glauber correction

$$\delta\sigma^{el}(s) = \delta\sigma_G(s) = \frac{\sigma_{hN}^{tot\,2}(s)}{4\pi R_d^2}, \quad x_{el}^2 << 1,$$

but the second term in the R.H.S. of Eq. (54) is totally new and comes from the contribution of the three-body forces to the hadron-deuteron total cross section.

The expressions for the shadow corrections have quite a transparent physical meaning, both the elastic a^{el} and inelastic a^{inel} scaling functions have a clear physical interpretation [23]. The function a^{el} measures out a portion of elastic rescattering events among of all the events during the interaction of an incident particle with a deuteron as a whole, and this function attached to the total probability of elastic interaction of an incident particle with a separate nucleon in a deuteron. Correspondingly, the function a^{inel} measures out a portion of inelastic events of inclusive type among of all the events during the interaction of an incident particle with a deuteron as a whole, and this function attached to the total probability of single diffraction dissociation of an incident particle on a separate nucleon in a deuteron. The scaling variables x_{el} and x_{inel} have quite a clear physical meaning too. The dimensionless quantity x_{el} characterizes the effective distances measured in the units of "fundamental length", which the deuteron size is, in elastic interactions, but the similar quantity x_{inel} characterizes the effective distances measured in the units of the same "fundamental length" during inelastic interactions.

The functions a^{el} and a^{inel} have a different behaviour: a^{el} is a monotonic function while a^{inel} has the maximum at the point $x_{inel}^{max} = \sqrt{2}$ where $a^{inel}(x_{inel}^{max}) = 2/3\sqrt{3}$. The existence of the maximum in the function a^{inel} results an interesting physical effect of weakening the inelastic eclipsing (screening) at superhigh energies. The energy s_m at the maximum of a^{inel} can easily be calculated from the equation $R_3^2(s_m) = 2R_d^2$ and here we faced with the harmonic ratio (in square)

$$\boxed{R_3^2(s_m) : R_d^2 = 2 : 1}. \tag{57}$$

Using the above mentioned global structure for the (anti)proton-proton total cross sections, we have made a preliminary comparison of the new structure for the shadow corrections in elastic scattering from deuteron with the existing experimental data on proton-deuteron and antiproton-deuteron total cross sections. The results of this comparison are shown in Figs. 3-4.

We would like to emphasize that in the fit to the data on antiproton-deuteron total cross sections R_d^2 was considered as a single free fit parameter. After that a comparison with the data on proton-deuteron total cross sections has been made without any free parameters: R_d^2 was fixed by the previous fit to the data on antiproton-deuteron total cross sections, and our fit yielded $R_d^2 = 66.61 \pm 1.16\,GeV^{-2}$. If we take into account the latest experimental value for the deuteron matter radius $r_{d,m} = 1.963(4)\,fm$ [24] then we can find that the fitted value for the R_d^2 satisfies with a good accuracy the equality

$$R_d^2 = \frac{2}{3}r_{d,m}^2, \quad (r_{d,m}^2 = 3.853\,fm^2 = 98.96\,GeV^{-2}). \tag{58}$$

So, we have established a harmonic *"consonance"* between the internucleon distance in a deuteron and the deuteron matter distribution

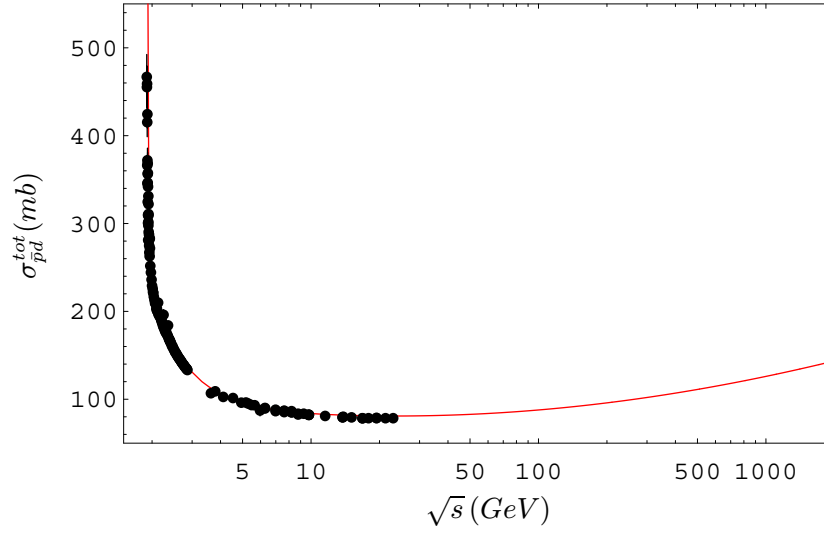$$\boxed{R_d^2 : r_{d,m}^2 = 2 : 3}. \tag{59}$$

18

Fig. 3. The total antiproton-deuteron cross-section compared with the theory. Statistical and systematic errors added in quadrature.

Now, let us come back to Eq. (50). Taking into account that $0 \leq \beta \leq 1/4$, from the Froissart bound (20) and Eq. (50) we have the following bound
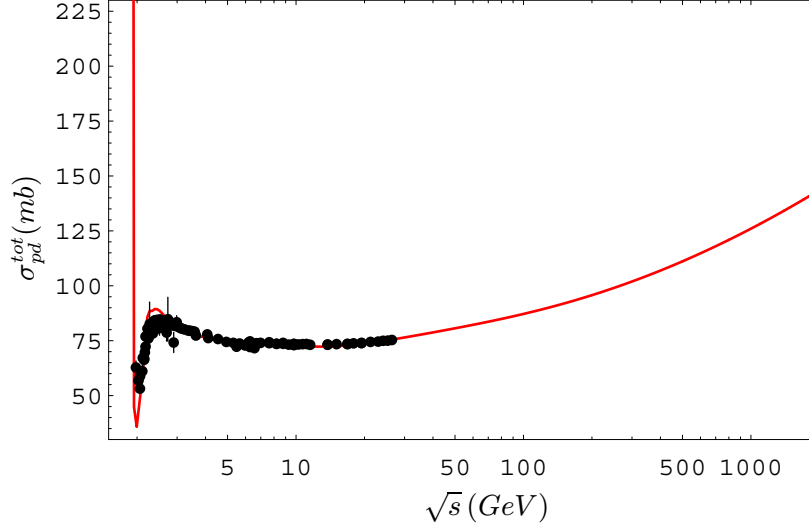
$$\boxed{R_3^2(s) < 2R_2^2(s)}.$$
(60)

Fig. 4. The total proton-deuteron cross-section compared with the theory without any free parameters. Statistical and systematic errors added in quadrature.

19

On the other hand for the effective radii of n-body forces we have obtained an asymptotic behaviour given by Eq. (38) where it follows from

$$\frac{R_3(s)}{R_2(s)} = \frac{4}{3} \cdot \frac{M_2}{M_3}, \quad s \to \infty. \tag{61}$$

Bound (60) with account of Eq. (61) gives

$$M_3 > \frac{4}{3\sqrt{2}} M_2 = \frac{8 m_\pi}{3\sqrt{2}}, \quad (M_2 = 2 m_\pi). \tag{62}$$

However, if we conjecture that $M_n = n m_\pi$ which is fulfilled for $n = 2$ then

$$\frac{R_3(s)}{R_2(s)} = \frac{8}{9}, \quad s \to \infty. \tag{63}$$

The ratio given by Eq. (63) corresponds to the harmonic ratio for the major second in the major scale in according to Pythagoras tune; see Introduction. We would like especially to emphasize that the ratio (63) is compatible with the global structure of (anti)proton-proton(deuteron) total cross sections described above.

4. Conclusion

In this minireview we have tried in the spirit of the Pythagorean school to show the mathematical, physical and geometrical beauty of the Froissart theorem. No doubt, we were enchanted with the aesthetic aspects of the Froissart theorem: there were heard *the new notes of the music of the spheres* produced by the Froissart theorem in the fundamental dynamics of particles and nuclei. Starting from abstract mathematical structures of axiomatic Quantum Field Theory by applying the general theorems, a physically transparent intuitively clear and visual picture of particles and nuclei interactions was arisen before our eyes. We found a very simple relations between physically tangible quantities which looked like Pythagoras harmonic ratios mentioned above and hence might be considered as a "hadronic symphony" in the fundamental dynamics. In fact, we came back to the great Pythagorean ideas reformulated in terms of the objects living in the microcosmos.

It appears that the study of fundamental processes in high energy elementary particle physics makes it possible to establish a missing link between cosmos and microcosmos, between the great ancient ideas and recent investigations in particle and nuclear physics and to confirm the unity of physical picture of the World. Anyway, we believe in it.

At last, in our previous papers we repeatedly criticized the so called supercritical pomeron phenomenology in hadronic physics. In our opinion this phenomenology might be compared with a "cacophony" in particle physics. Certainly, someone likes cacophony in the music. However, we prefer a symphony in the music and a harmony in the fundamental dynamics as well.

Acknowledgements

References

[1] V.V. Belokurov, O.A. Khrustalev, O.D. Timofeevskaya, Quantum Teleportation – usual Wonder, in russian, Izhevsk, 2000.

[2] A.V. Lebedev, Fragments of ancient Greek philosophers, Part I, in russian, Moscow, "Nauka", 1989.

[3] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[4] H. Lehmann, Nuovo Cim. **10**, 579 (1958).

[5] R. Jost, H. Lehmann, Nuovo Cim. **5**, 1598 (1957).

[6] F.J. Dyson, Phys. Rev. **110**, 1460 (1958).

[7] H. Lehmann, Nuovo Cimento Suppl. **14**, 153 (1959).

[8] A. Martin, Nuovo Cim. **42A**, 930 (1966); ibid **44A**, 1219 (1966).

[9] G. Sommer, Nuovo Cim. **48A**, 92 (1967); **52A**, 373 (1967); **52A**, 850 (1967); **52A**, 866 (1967).

[10] L. Durand, P.M. Fishbane, L.M. Simmons, Jr., Journ. of Math. Phys. **17**, 1933 (1976).

[11] E.P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. of Math. **40**, 149 (1939).

[12] O.A. Khrustalev, V.I. Savrin, Preprint IHEP 68-19K, Serpukhov, 1968.

[13] O.A. Khrustalev, A.A. Logunov, Nguen Van Hieu, in: *Problems of Theoretical Physics*, Essays dedicated to N.N. Bogolyubov on the occasion of his Sixtieth Birthday, Publishing House "Nauka", Moscow, 1969, p. 90.

[14] O.A. Khrustalev, A.A. Logunov, A.N. Tavkhelidze, I.T. Todorov, Nuovo Cim. **30**, 134 (1963).

[15] A.A. Arkhipov, Sov. J. Theor. Math. Phys. **74**, 69 (1988); ibid **83**, 247 (1990).

[16] G.'t Hooft, *The Holographic Principle*, e-print hep-th/0003004.

[17] A.A. Arkhipov, V.I. Savrin, Sov. J. Theor. Math. Phys. **49**, 3 (1981).

[18] A.A. Arkhipov, Rep. on Math. Phys. **20**, 303 (1984).

[19] A.A. Arkhipov, *What Can we Learn from the Study of Single Diffractive Dissociation at High Energies?* – in Proceedings of VIIIth Blois Workshop on Elastic and Diffractive Scattering, Protvino, Russia, June 28–July 2, 1999, World Scientific, Singapore, 2000, pp. 109-118; REPORT IHEP 99-43, Protvino, 1999; e-print hep-ph/9909531.

[20] A.A. Arkhipov, *On Global Structure of Hadronic Total Cross Sections*, preprint IHEP 99-45, Protvino, 1999; e-print hep-ph/9911533.

[21] A.A. Arkhipov, *Proton-Proton Total Cross Sections from the Window of Cosmic Ray Experiments*, preprint IHEP 2001-23, Protvino, 2001; e-print hep-ph/0108118; in Proceedings of IXth Blois Workshop on Elastic and Diffractive Scattering, Pruhonice near Prague, Czech Republic, June 9-15, 2001, eds. V. Kundrat, P. Zavada, Institute of Physics, Prague, Czech Republic, 2002, pp. 293-304.

[22] A.A. Arkhipov, *Three-Body Forces, Single Diffraction Dissociation and Shadow Corrections in Hadron-Deuteron Total Cross Sections*, preprint IHEP 2000-59, Protvino, 2000; e-print hep-ph/0012349; in Proceedings of XVth Workshop on High Energy Physics and Quantum Field Theory, Tver, Russia, September 7-13, 2000, eds. M. Dubinin, V. Savrin, Institute of Nuclear Physics, Moscow State University, Russia, 2001, pp. 241-257.

[23] A.A. Arkhipov, *DIFFRACTION 2000: New Scaling Laws in Shadow Dynamics*, Nucl. Phys. B (Proc. Suppl.) **99A**, 72 (2001).

[24] F. Schmidt-Kaler et al., Phys. Rev. Lett. **70**, 2261 (1993).