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**GENERAL COVARIANCE VIOLATION AND  
THE GRAVITATIONAL DARK MATTER.  
II. VECTOR GRAVITON**

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**Abstract**

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The (four component) vector graviton contained in metric, the scalar component incorporated, is attributed to the violation of the general covariance to the residual isoharmonic one. In addition to the previously studied (singlet) scalar graviton, the vector graviton may constitute one more fraction of the gravitational dark matter. The gravity interactions of the vector graviton, as well as its impact on the continuous medium are studied.

**Аннотация**

Пирогов Ю.Ф. Нарушение общей ковариантности и гравитационная темная материя. II. Векторный гравитон: Препринт ИФВЭ 2005-34. – Протвино, 2005. – 10 с., библиогр.: 4.

Содержащийся в метрике (четырёхкомпонентный) векторный гравитон, включающий скалярную компоненту, приписывается нарушению общей ковариантности до остаточной изогармонической ковариантности. В дополнение к ранее изученному (синглетному) скалярному гравитону, векторный гравитон может составлять еще одну часть гравитационной темной материи. Изучены гравитационные взаимодействия векторного гравитона, а также его влияние на непрерывную среду.

## 1. Introduction

In the preceding paper [1], the author put forward the concept that the violation of the general covariance (GC) may serve as a reason d'être for the existence of the dark matter (DM) of the gravitational origin (or v.v.). The case study for the (singlet) scalar graviton, as a simplest representative of such a matter, was worked out. The consistency of the theory of such a graviton considered as a part of the metric (in addition to the massless tensor graviton) was assured there by the existence the residual unimodular covariance (UC). We refer the reader to ref. [1] for the general discussion of the GC violation in the context of the gravitational DM and for the details of the UC case. The GC violation gets its natural description in the framework of the affine Goldstone approach to gravity developed in ref. [2], to which we refer the reader as well <sup>1</sup>.

In the present paper, we continue studying the GC violation for the next in complexity case of the (four component) vector graviton, the respective scalar component incorporated. The consistency of the theory is assured in this case by the residual isoharmonic covariance (see later on). Section 2 of the paper is devoted to the gravity interactions of the vector graviton. First, the theory is constructed in the distinguished background coordinates. Then, it is developed in the arbitrary observer's coordinates. In Section 3, the impact of the vector graviton on the continuous medium is studied.

## 2. Gravity

### 2.1. Background coordinates

**Lagrangian** Remind shortly the framework of the affine Goldstone approach to gravity [2] to be used in what follows in describing the GC violation. It is postulated in the approach that there exists the physical, not just auxiliary, gravitational background (continuum). Let  $x^\mu$ ,  $\mu, \nu, \dots = 0, \dots, 3$  be the observer's coordinates of a point in the space-time and let  $\bar{\xi}^\alpha = \bar{\xi}^\alpha(x^\mu)$ ,  $\alpha, \beta, \dots = 0, \dots, 3$  be the background-attached coordinates of the point. The indices  $\alpha$ , etc., undergo the (global) affine transformations. Relative to the observer's coordinate transformations, these

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<sup>1</sup>For a short exposition of the approach, see refs. [3], [4].

indices are blind and can be considered as those just numerating the scalars. The theory starts in the background coordinates, wherein all the generic physical properties are already predestined. The observer's coordinates only replicate these properties in the coordinate dependent fashion. The background coordinates may be considered as an analogue of the comoving coordinates for the continuous medium. The dynamical variable for gravity in the background coordinates is the metric  $\gamma_{\alpha\beta}(\bar{\xi})$ . The latter is, roughly speaking, the square of the affine Goldstone boson which proves to be the original field variable for the gravity <sup>2</sup>.

Let us study the classical theory of the metric and the matter with the action

$$I = \int (L_g + \Delta L_g + L_m) \sqrt{-\gamma} d^4 \bar{\xi}, \quad (1)$$

where  $L_g$  and  $\Delta L_g$  are the gravity Lagrangians,  $L_m$  is the matter one and  $\gamma = \det \gamma_{\alpha\beta}$ . By the very construction, the action is to be invariant under the (global) affine symmetry (AS). Nevertheless, some parts of the Lagrangian may formally admit more wide sets of the coordinates. As for the gravity,  $L_g$  is chosen so as to allow the arbitrary coordinates, possessing thus the GC.  $\Delta L_g$  is supposed to be restricted to a subset of the general coordinates, being GC violating with some residual covariance. As for the matter,  $L_m$  may in general violate the GC, too.

Conventionally, take as  $L_g$  the modified Einstein-Hilbert Lagrangian of the General Relativity (GR):

$$L_g = -M_P^2 \left( \frac{1}{2} R(\gamma_{\alpha\beta}) - \Lambda \right), \quad (2)$$

with  $M_P$  being the Plank mass,  $R$  being the Ricci scalar and  $\Lambda$  being the cosmological constant. In principle, there is conceivable any generally covariant modification of the Lagrangian (2). As for the extra gravity Lagrangian  $\Delta L_g$ , decompose it generically in two terms:

$$\Delta L_g = \Delta K_g - \Delta V_g, \quad (3)$$

with  $\Delta K_g$  being the derivative kinetic term and  $\Delta V_g$  being the derivativeless potential. Consider these terms in turn.

**Kinetic term** Take  $\Delta K_g$  in the lowest approximation as follows:

$$\Delta K_g(\omega^\alpha) = \frac{1}{2} \kappa^2 \omega \cdot \omega, \quad (4)$$

with  $\kappa$  being a constant with the dimension of mass,  $|\kappa| < M_P$ . A priori, both  $\kappa^2 \geq 0$  and  $\kappa^2 < 0$  can be envisaged, corresponding to physical particles or ghosts, respectively. Here and in what follows in this subsection the notation for the dot product  $\omega \cdot \omega = \omega_\alpha \omega^\alpha$  is understood with the metric  $\gamma_{\alpha\beta}$ , until stated otherwise (and similarly for any two vectors). In the above, the Lagrangian variable for the vector graviton is defined as follows

$$\omega^\alpha = \gamma^{\beta\gamma} \Gamma^\alpha_{\beta\gamma} + k \gamma^{\alpha\gamma} \Gamma^\beta_{\beta\gamma}, \quad (5)$$

with

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \gamma^{\alpha\delta} \left( \partial_\beta \gamma_{\delta\gamma} + \partial_\gamma \gamma_{\delta\beta} - \partial_\delta \gamma_{\beta\gamma} \right) \quad (6)$$

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<sup>2</sup>For the spin-half particles, the affine Goldstone boson itself should be used instead of its square [2].

being the Christoffel connection and  $k$  any real. Eq. (5) is the most general expression for the gravity vector variable allowed by the AS [2]. Remind that

$$\begin{aligned}\gamma^{\beta\gamma}\Gamma^\alpha_{\beta\gamma} &= -\frac{1}{\sqrt{-\gamma}}\partial_\delta(\sqrt{-\gamma}\gamma^{\alpha\delta}), \\ \Gamma^\beta_{\beta\gamma} &= \partial_\gamma \ln \sqrt{-\gamma}\end{aligned}\quad (7)$$

and thus

$$\omega^\alpha = -\partial_\beta\gamma^{\alpha\beta} + (k-1)\partial^\alpha \ln \sqrt{-\gamma}, \quad (8)$$

where  $\partial^\alpha = \gamma^{\alpha\beta}\partial_\beta$ .

**Isosharmonic covariance** Consider the arbitrary change of the background coordinates:  $\bar{\xi}^\alpha \rightarrow \bar{\xi}'^\alpha = \bar{\xi}^\alpha + \bar{\varepsilon}^\alpha$ . Under this coordinate transformation, one has

$$\begin{aligned}\Gamma'^\alpha_{\beta\gamma}(\bar{\xi}') &= \partial'_\beta\bar{\xi}^\delta\partial'_\gamma\bar{\xi}^\varepsilon\left(\partial_\varphi\bar{\xi}'^\alpha\Gamma^\varphi_{\delta\varepsilon}(\bar{\xi}) - \partial_\delta\partial_\varepsilon\bar{\xi}'^\alpha\right), \\ \gamma'_{\beta\gamma}(\bar{\xi}') &= \partial_\beta\bar{\xi}'^\delta\partial_\gamma\bar{\xi}'^\varepsilon\gamma'_{\delta\varepsilon}(\bar{\xi}'),\end{aligned}\quad (9)$$

where  $\partial'_\alpha = \partial/\partial\bar{\xi}'^\alpha$ , etc. This gives for the small  $\bar{\varepsilon}^\alpha$ :

$$\omega'^\alpha(\bar{\xi}') = \omega^\alpha(\bar{\xi}) + \omega \cdot \partial\bar{\varepsilon}^\alpha - (\partial \cdot \partial\bar{\varepsilon}^\alpha + k\partial^\alpha\partial \cdot \bar{\varepsilon}), \quad (10)$$

where  $\omega \cdot \partial = \omega^\beta\partial_\beta$ ,  $\partial \cdot \bar{\varepsilon} = \partial_\beta\bar{\varepsilon}^\beta$  and  $\partial \cdot \partial = \partial^\beta\partial_\beta = \gamma^{\alpha\beta}\partial_\alpha\partial_\beta$ . The similar notations for the dot product containing the (covariant) derivatives, mutatis mutandis, will be used in what follows. For the theory to remain meaningful, only those background coordinates are allowed under the substitution of which  $\omega^\alpha$  transforms homogeneously as a vector. This gives

$$\partial \cdot \partial\bar{\varepsilon}^\alpha + k\partial^\alpha\partial \cdot \bar{\varepsilon} = 0. \quad (11)$$

Call this equation the isosharmonic one<sup>3</sup>. The respective transformation group will be called the isosharmonic one, resulting in the isosharmonic covariance (IC). At  $k=0$ , eq. (11) remains to be valid for the finite  $\bar{\varepsilon}^\alpha$ . In the limit  $|k| \gg 1$ , it reduces (up to a constant) to the unimodularity condition,  $\partial \cdot \bar{\varepsilon} = 0$  (so that  $\delta\gamma = 0$ ), corresponding to the (singlet) scalar graviton [1].

**Weak-field limit** For the physics interpretation, consider the weak-field limit of the theory corresponding to the decomposition

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad (12)$$

with  $|h_{\alpha\beta}| \ll 1$ . Accounting for eq. (8) and  $\gamma = -(1+h)$ ,  $h = \eta^{\alpha\beta}h_{\alpha\beta}$ , one gets

$$\omega^\alpha = \partial_\beta\left(h^{\alpha\beta} + \frac{1}{2}(k-1)\eta^{\alpha\beta}h\right). \quad (13)$$

All the indices are manipulated in the weak-field limit by means of  $\eta_{\alpha\beta}$  and  $\eta^{\alpha\beta}$ . In the limit  $|k| \gg 1$ , one gets  $\omega_\alpha \sim \partial_\alpha h$  and thus this limit corresponds to the (singlet) scalar graviton, indeed.

<sup>3</sup>It is not to be mixed with the conventional harmonicity condition,  $\gamma^{\beta\gamma}\Gamma^\alpha_{\beta\gamma} = 0$ .

Consider the “gauge” transformations for the “potentials”  $h_{\alpha\beta}$ , initiated by the small transformations of the background coordinates:

$$h'_{\alpha\beta}(\bar{\xi}) = h_{\alpha\beta}(\bar{\xi}) - (\partial_\alpha \bar{\varepsilon}_\beta + \partial_\beta \bar{\varepsilon}_\alpha) \quad (14)$$

(and thus  $h' = h - 2\partial \cdot \bar{\varepsilon}$ ). Under these gauge transformations, the field strength changes as

$$\omega'^\alpha(\bar{\xi}) = \omega^\alpha(\bar{\xi}) - (\partial \cdot \partial \bar{\varepsilon}^\alpha + k \partial^\alpha \partial \cdot \bar{\varepsilon}). \quad (15)$$

Clearly,  $\omega^\alpha$  is gauge invariant only under the isoharmonic transformations eq. (11), with the dot product defined now by  $\eta_{\alpha\beta}$ .

In the linearized GR in the flat background, the requirement for the r.h.s. of eq. (13) to be zero is nothing but the Hilbert-Lorentz gauge condition eliminating from  $h_{\alpha\beta}$  the three vector and one scalar degrees of freedom. On the contrary, abandoning this requirement in the present paper puts the respective degrees of freedom in action. It is for this reason, that we interpret the extra gravity components contained in  $\omega^\alpha$  as those corresponding to the vector graviton, the scalar component incorporated. In this,  $\omega^\alpha$  is nothing but the field strength for such a graviton. The existence of the residual IC is crucial for the theory. It allows one to preserve the well-established properties of the tensor gravity. Namely, the IC serves as the gauge symmetry to remove from  $h_{\alpha\beta}$  the remaining (singlet) scalar and three tensor components, leaving thus six physical components: four for the vector graviton, the scalar component incorporated, and two for the massless tensor graviton.

Note for completeness, that including in  $\Delta K_g$  the term quadratic in  $\Gamma^\alpha_{\alpha\beta}$  with the independent coefficient  $\kappa_0^2$  produces additionally the physical (singlet) scalar graviton, followed by the unimodularity condition,  $\partial \cdot \bar{\varepsilon} = 0$ , and the residual unimodular isoharmonic covariance (UIC). This realizes the most general case for the scalar and vector gravitons, described by the three independent constants.

**Mass term** To attribute the mass to the extra graviton one should account additionally for the potential  $\Delta V_g$ . The latter is a scalar function  $\Delta V_g((\gamma\eta)_{\alpha\beta})$  depending on the determinant  $\gamma$  and  $\text{tr}((\gamma\eta)_{\alpha\beta})^n$ , with  $n$  being an arbitrary integer, positive or negative, and  $\eta^{\alpha\beta}$  being the Minkowski symbol. Clearly, using the latter violates explicitly the AS to the Lorentz one, which is supposed to be exact. In this, the general covariance is violated completely. The degree of this violation is characterized by a mass parameter  $\mu$ . Thus in the framework of the affine Goldstone approach to gravity, one can structure the gravity Lagrangians, with  $L_g(R)$  being both affine invariant and generally covariant,  $\Delta L_g(\omega^\alpha)$  being also affine invariant though GC violating, whereas  $\Delta V_g((\gamma\eta)_{\alpha\beta})$  violating both the AS and the GC. The AS being the basic one in the given framework, one expects  $M_P > |\kappa| \gg \mu$  with the natural hierarchy of the residual covariance groups

$$GC \xrightarrow{\kappa} IC \xrightarrow{\mu} TC \quad (16)$$

for  $L_g$ ,  $\Delta K_g$  and  $\Delta V_g$ , respectively (TC meaning the trivial covariance). Because the potential supplies the mass to the tensor graviton, too, we postpone the mass issue to the cumulative study of the graviton mass mixing in the future.

Varying the action (1) with respect to  $\gamma^{\alpha\beta}$  one would get the equation of motion for the gravity in the basic form. Then one could transform the results into the observer’s coordinates. For the physics generality, we rewrite the Lagrangian directly in the observer’s coordinates and proceed therein. The results in the background coordinates will be recovered as a marginal case.

## 2.2. Observer's coordinates

**Lagrangian** The action now looks like

$$I = \int (L_g + \Delta L_g + L_m) \sqrt{-g} d^4x, \quad (17)$$

with the metric

$$g_{\mu\nu} = \partial_\mu \bar{\xi}^\alpha \partial_\nu \bar{\xi}^\beta \gamma_{\alpha\beta}, \quad (18)$$

(the inverse one  $g^{\mu\nu} = \partial_\alpha x^\mu \partial_\beta x^\nu \gamma^{\alpha\beta}$ ) and  $g = \det g_{\mu\nu}$ . Throughout this subsection, all the indices are manipulated by means of  $g_{\mu\nu}$  and  $g^{\mu\nu}$ , unless stated otherwise. The generally covariant Lagrangian  $L_g$  gets unchanged:

$$L_g = -M_P^2 \left( \frac{1}{2} R(g_{\mu\nu}) - \Lambda \right). \quad (19)$$

To proceed, introduce the auxiliary fields

$$\begin{aligned} \bar{g}_{\mu\nu} &= \partial_\mu \bar{\xi}^\alpha \partial_\nu \bar{\xi}^\beta \eta_{\alpha\beta}, \\ \bar{g}^{-1\mu\nu} &= \partial_\alpha x^\mu \partial_\beta x^\nu \eta^{\alpha\beta}. \end{aligned} \quad (20)$$

By definition, call  $\bar{g}_{\mu\nu}$  the background metric ( $\bar{g}^{-1\mu\nu}$  being the inverse one)<sup>4</sup>. The mass term is modified straightforwardly to the scalar function  $\Delta V_g((g\bar{g}^{-1})_\mu{}^\nu)$  which depends on the ratio of the determinants,  $g/\bar{g}$ , and  $\text{tr}((g\bar{g}^{-1})_\mu{}^\nu)^n$ . The matter Lagrangian  $L_m$  will be discussed in the next Section.

**Kinetic term** As for  $\Delta K_g$ , it gets modified as follows:

$$\omega^\lambda = \Omega^\lambda - \bar{\Omega}^\lambda, \quad (21)$$

where

$$\begin{aligned} \Omega^\lambda &= g^{\mu\nu} \Gamma^\lambda_{\mu\nu} + k g^{\lambda\nu} \Gamma^\mu_{\mu\nu}, \\ \bar{\Omega}^\lambda &= g^{\mu\nu} \bar{\Gamma}^\lambda_{\mu\nu} + k g^{\lambda\nu} \bar{\Gamma}^\mu_{\mu\nu}. \end{aligned} \quad (22)$$

In the above,  $\Gamma^\lambda_{\mu\nu}$  is the dynamical Christoffel connection defined, mutatis mutandis, by eq. (6) through  $g_{\mu\nu}$ . Accounting for the reduced connections  $g^{\mu\nu} \Gamma^\lambda_{\mu\nu}$  and  $\Gamma^\mu_{\mu\nu}$ , given by eq. (7) with the proper substitutions, one gets similarly to eq. (8):

$$\Omega^\lambda = -\partial_\nu g^{\lambda\nu} + (k-1) \partial^\lambda \ln \sqrt{-g}. \quad (23)$$

By the construction, the symbol  $\bar{\Gamma}^\lambda_{\mu\nu}$  is as follows

$$\bar{\Gamma}^\lambda_{\mu\nu} = \partial_\alpha x^\lambda \partial_\mu \partial_\nu \bar{\xi}^\alpha. \quad (24)$$

It can be expressed in terms of  $\bar{g}_{\mu\nu}$  as the respective Christoffel connection

$$\bar{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} \bar{g}^{-1\lambda\rho} \left( \partial_\mu \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\mu} - \partial_\rho \bar{g}_{\mu\nu} \right), \quad (25)$$

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<sup>4</sup>Note that  $\bar{g}^{-1\mu\nu}$  is to be distinguished from  $\bar{g}^{\mu\nu} = g^{\mu\lambda} g^{\nu\rho} \bar{g}_{\lambda\rho}$ .

so that one has, in particular,

$$\bar{\Gamma}^\mu{}_{\mu\nu} = \partial_\nu \ln \sqrt{-\bar{g}}. \quad (26)$$

Thus ultimately, the gravity is described by the fourteen independent fields: ten for the tensor  $g_{\mu\nu}$  and four for the scalars  $\bar{\xi}^\alpha$ . The latter ones are to be found through observations, with the former ones being reproduced by the dynamics in the self-consistent fashion. More particularly, only the quadratic combination of  $\partial_\mu \bar{\xi}^\alpha$  in the form of  $\bar{g}_{\mu\nu}$ , eq. (20), enters<sup>5</sup>. Due to the compensating terms with the background connection, the theory can be studied in the arbitrary observer's coordinates, in contrast to the background coordinates. It should be stressed that  $\bar{g}_{\mu\nu}$  (and thus  $\bar{\Gamma}^\lambda{}_{\mu\nu}$ ) is not the most general one, but depends on the four scalar parameter-fields  $\bar{\xi}^\alpha$ . It is for this reason, that there can be chosen the coordinates where  $\bar{\Gamma}^\lambda{}_{\mu\nu} = 0$ . From the observer viewpoint, the last property is precisely what distinguishes the background coordinates from the remaining ones, all of the coordinates being for the observer a priori equivalent.

**Isosharmonic covariance** It follows from eqs. (21), (22) that  $\omega^\lambda$  is the vector transforming homogeneously under the arbitrary change of the coordinates  $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu$ . This allows  $\omega^\lambda$  to serve as the Lagrangian variable. The infinitesimal changes of the background and observer's coordinates being related as  $\bar{\epsilon}^\alpha = \partial_\mu \bar{\xi}^\alpha \epsilon^\mu$ , the residual covariance group corresponds now to those  $\epsilon^\mu$  which are related with  $\bar{\epsilon}^\alpha$ , satisfying eq. (11). This results straightforwardly in the modified isosharmonic equation:

$$\bar{\nabla} \cdot \bar{\nabla} \epsilon^\lambda + k \bar{\nabla}^\lambda \bar{\nabla} \cdot \epsilon = 0. \quad (27)$$

Here  $\bar{\nabla}_\mu$  is the background covariant derivative and

$$\begin{aligned} \bar{\nabla} \cdot \bar{\nabla} &= g^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu, \\ \bar{\nabla}^\lambda \bar{\nabla} \cdot \epsilon &= g^{\lambda\mu} \partial_\mu \left( \frac{1}{\sqrt{-\bar{g}}} \partial_\nu (\sqrt{-\bar{g}} \epsilon^\nu) \right). \end{aligned} \quad (28)$$

At  $\bar{\Gamma}^\lambda{}_{\mu\nu} = 0$ , eq. (27) reduces, mutatis mutandis, to eq. (11). In the limit  $|k| \gg 1$ , it reduces to the modified unimodularity condition  $\bar{\nabla} \cdot \epsilon = 0$  (up to a constant), or otherwise  $\partial \cdot (\sqrt{-\bar{g}} \epsilon) = 0$ .

**Weak-field limit** In the observer's coordinates, the weak-field decomposition becomes

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (29)$$

with  $h_{\mu\nu} = \partial_\mu \bar{\xi}^\alpha \partial_\nu \bar{\xi}^\beta h_{\alpha\beta}$ ,  $|h_{\mu\nu}| \ll 1$ . From eqs. (22) and (23) with account for  $\delta \bar{g}_{\mu\nu} = 0$ , one gets

$$\begin{aligned} \delta \Omega^\lambda &= -\partial_\nu \delta g^{\lambda\nu} + (k-1) g^{\lambda\nu} \partial_\nu \delta \ln \sqrt{-g} \\ &\quad + (k-1) \partial_\nu \ln \sqrt{-g} \delta g^{\lambda\nu}, \\ \delta \bar{\Omega}^\lambda &= \bar{\Gamma}^\lambda{}_{\mu\nu} \delta g^{\mu\nu} + k \bar{\Gamma}^\mu{}_{\mu\nu} \delta g^{\lambda\nu}. \end{aligned} \quad (30)$$

Accounting for the the relations  $\delta \sqrt{-g} = -1/2 \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$ ,  $\delta g^{\lambda\rho} = -g^{\lambda\mu} g^{\rho\nu} \delta g_{\mu\nu}$  and substituting  $\delta g_{\mu\nu} \rightarrow h_{\mu\nu}$ ,  $g_{\mu\nu} \rightarrow \bar{g}_{\mu\nu}$ , one gets from eqs. (21) – (23)

$$\omega^\lambda = \bar{\nabla}_\mu \left( h^{\lambda\mu} + \frac{1}{2} (k-1) \bar{g}^{\lambda\mu} h \right), \quad (31)$$

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<sup>5</sup>For the spin-half particles,  $\partial_\mu \bar{\xi}^\alpha$  themselves are operative.



which clearly generalizes eq. (13). Here one puts  $h^{\mu\nu} = \bar{g}^{\lambda\mu}\bar{g}^{\rho\nu}h_{\mu\nu}$ ,  $h = \bar{g}^{\mu\nu}h_{\mu\nu}$ . In the weak-field limit, the indices are manipulated by means of  $\bar{g}_{\mu\nu}$  and its inverse  $\bar{g}^{\mu\nu}$ <sup>6</sup>. Remind that one has, in particular,  $\bar{\nabla}_\lambda\bar{g}^{\mu\nu} = 0$ , etc.

The gauge transformations for the potentials  $h_{\mu\nu}$  initiated by the infinitesimal transformations of the observer's coordinates, the dependence of  $\bar{g}_{\mu\nu}$  on the coordinates including, now look like

$$h'_{\mu\nu}(x) = h_{\mu\nu}(x) - (\bar{\nabla}_\mu\epsilon_\nu + \bar{\nabla}_\nu\epsilon_\mu) \quad (32)$$

(and thus  $h' = h - 2\bar{\nabla} \cdot \epsilon$ ). Respectively, the field strength  $\omega^\lambda$  changes under the gauge transformations as

$$\omega'^\lambda(x) = \omega^\lambda(x) - (\bar{\nabla} \cdot \bar{\nabla}\epsilon^\lambda + k\bar{\nabla}^\lambda\bar{\nabla} \cdot \epsilon), \quad (33)$$

where

$$\begin{aligned} \bar{\nabla} \cdot \bar{\nabla} &= \frac{1}{\sqrt{-\bar{g}}}\partial_\mu(\sqrt{-\bar{g}}\bar{g}^{\mu\nu}\partial_\nu), \\ \bar{\nabla}^\lambda\bar{\nabla} \cdot \epsilon &= \partial^\lambda\left(\frac{1}{\sqrt{-\bar{g}}}\partial \cdot (\sqrt{-\bar{g}}\epsilon)\right). \end{aligned} \quad (34)$$

Clearly, the requirement for  $\omega^\lambda$  to be gauge invariant,  $\omega'^\lambda(x) = \omega^\lambda(x)$ , results in the isoharmonic condition, eqs. (27), (28), with the metric  $g_{\mu\nu}$  substituted by  $\bar{g}_{\mu\nu}$ .

**Equations of motion** Varying the action (17) with respect to  $g^{\mu\nu}$  ( $\bar{g}_{\mu\nu}$  being unchanged) one arrives at the gravity equation:

$$G_{\mu\nu} + \Delta G_{\mu\nu} = M_P^{-2} T_{\mu\nu}^{(m)}. \quad (35)$$

Here  $G_{\mu\nu}$  is the gravity tensor defined as usually:

$$-M_P^2 G_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_g}{\delta g^{\mu\nu}}, \quad (36)$$

with  $\mathcal{L}_g = \sqrt{-g} L_g$  being the gravity Lagrangian density (and similarly for  $\Delta G_{\mu\nu}$  corresponding to  $\Delta L_g$ ).  $T_{\mu\nu}^{(m)}$  is the conventional energy-momentum tensor of the matter defined through  $\mathcal{L}_m = \sqrt{-g} L_m$  by the r.h.s. of eq. (36). Introduce the notation  $(\nabla \cdot G)^\nu = \nabla_\mu G^{\mu\nu}$ , etc, with  $\nabla_\mu$  being the generally covariant derivative. Due to the GC of  $L_g$  and thus the relation  $(\nabla \cdot G)^\mu = 0$ , one gets from eq. (35) the modified conservation law for the matter:

$$\left(\nabla \cdot (T_m + \Delta T)\right)^\mu = 0. \quad (37)$$

The extra term  $\Delta T^{\mu\nu} \equiv -M_P^2 \Delta G^{\mu\nu}$  is to be interpreted as the contribution of the vector graviton to the DM. In other terms, the equation above can be written as

$$(\nabla \cdot T_m)^\mu = Q^\mu, \quad (38)$$

where

$$Q^\mu = M_P^2 (\nabla \cdot \Delta G)^\mu \quad (39)$$

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<sup>6</sup>In this limit  $\bar{g}^{\mu\nu} = \bar{g}^{-1\mu\nu}$ .

is the external force acting on the matter from the site of the vector gravitons.

Varying the Einstein-Hilbert Lagrangian density  $\mathcal{L}_g$  with respect to  $g^{\mu\nu}$  one gets the gravity tensor as usually

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu}. \quad (40)$$

With account for eq. (30), one gets from  $\Delta\mathcal{L}_g$

$$\begin{aligned} -M_P^2 \Delta G_{\mu\nu} &= \kappa^2 \left( k(\omega_\mu \partial_\nu \ln \sqrt{g/\bar{g}} + \omega_\nu \partial_\mu \ln \sqrt{g/\bar{g}}) - \omega_\mu \omega_\nu - \frac{1}{2} \omega \cdot \omega g_{\mu\nu} \right) \\ &+ \kappa^2 \left( (k-1) \nabla \cdot \omega g_{\mu\nu} + (\bar{\nabla}_\mu \omega_\nu + \bar{\nabla}_\nu \omega_\mu) \right) \\ &+ \Delta V_g g_{\mu\nu} - 2\partial \Delta V_g / \partial g^{\mu\nu}, \end{aligned} \quad (41)$$

where

$$\nabla \cdot \omega = \frac{1}{\sqrt{-g}} \partial \cdot (\sqrt{-g} \omega). \quad (42)$$

In the weak-field limit,  $\bar{\nabla}_\mu$  coincides with  $\nabla_\mu$ . In the limit  $|k| \gg 1$  with the redefinition  $\kappa \rightarrow \kappa/k$ , one recovers the results for the (singlet) scalar graviton [1]. At  $\bar{\Gamma}^\lambda{}_{\mu\nu} = 0$ , mutatis mutandis, the results in the background coordinates follow.

### 3. Matter

**Energy-momentum tensor** For the applications to cosmology, it suffices to treat the matter as the continuous medium. Describe it directly in the observer's coordinates with the metric  $g_{\mu\nu}$ . To apply the Lagrangian framework to the medium [1], characterize the latter by the proper (i.e., measured in the comoving coordinates) concentration  $n$  of the medium particles and by the specific entropy  $\sigma$  (the entropy per particle), in addition to the medium 4-velocity  $U^\mu$ ,  $U \cdot U = 1$ . Besides, the medium is characterized by the nondynamical parameters such as the particle mass  $m$ , etc. The particle number current  $N^\mu = nU^\mu$  satisfies the generic continuity condition  $\nabla \cdot N = 0$ . This constraint has to be valid identically, independently of the equations of motion.

Take the Lagrangian for the medium generically as

$$L_m(n, \sigma, U^\mu) = -E(|N|, \sigma), \quad (43)$$

with the scalar  $E(|N|, \sigma)$  being the energy function. Here one puts  $|N| = (N \cdot N)^{1/2}$ . One can also add the interactions of the vector graviton with the medium as follows:

$$L'_m(n, \sigma, U^\mu) = -F(|N|, \sigma) \frac{1}{|N|} N \cdot \omega, \quad (44)$$

with the scalar  $F(|N|, \sigma)$  being the formfactor. Introducing the vector density  $\mathcal{N}^\mu = \sqrt{-g} N^\mu$  as the independent variable and the respective scalar density  $|\mathcal{N}| = (\mathcal{N} \cdot \mathcal{N})^{1/2} = \sqrt{-g} |N|$ , write the total Lagrangian density as follows:

$$\mathcal{L}_m^{(tot)} = -\sqrt{-g} \left( E\left(\frac{|\mathcal{N}|}{\sqrt{-g}}, \sigma\right) + F\left(\frac{|\mathcal{N}|}{\sqrt{-g}}, \sigma\right) \frac{1}{|\mathcal{N}|} \mathcal{N} \cdot \omega \right) + \lambda \partial \cdot \mathcal{N}, \quad (45)$$

with  $\lambda$  being the Lagrange's multiplier.

Varying eq. (45) relative to  $\lambda$  one reproduces the continuity condition,  $\partial \cdot \mathcal{N} = 0$ . Varying  $\mathcal{L}_m^{(tot)}$  with respect to  $g^{\mu\nu}$  and accounting for  $\delta \mathcal{N}^\mu = 0$ ,  $\delta(|\mathcal{N}|/\sqrt{-g}) = n/2 (g_{\mu\nu} - U_\mu U_\nu) \delta g^{\mu\nu}$  one gets the energy-momentum tensor for the matter

$$T_{\mu\nu}^{(m)} = \rho U_\mu U_\nu + p(U_\mu U_\nu - g_{\mu\nu}) + \tau_{\mu\nu} , \quad (46)$$

with

$$\begin{aligned} \rho &= e - (k-1) \nabla \cdot (fU), \\ p &= ne' - e + (nf' - f)U \cdot \omega + (k-1) \nabla \cdot (fU), \\ \tau_{\mu\nu} &= f(U_\mu \omega_\nu + U_\nu \omega_\mu) - kf \left( U_\mu \partial_\nu \ln \sqrt{g/\bar{g}} + U_\nu \partial_\mu \ln \sqrt{g/\bar{g}} \right) \\ &\quad - \left( \bar{\nabla}_\mu (fU_\nu) + \bar{\nabla}_\nu (fU_\mu) \right). \end{aligned} \quad (47)$$

Here  $e(n, \sigma) \equiv E(|N|, \sigma)|_{|N|=n}$ ,  $e' = \partial e(n, \sigma)/\partial n$  and likewise for  $f$ . In the equations above,  $\rho$  is the scalar coinciding with the energy per unit proper volume,  $p$  is the scalar coinciding with the (isotropic) pressure, while  $f$  is the new scalar state function. The terms proportional to  $f$  distort the medium. Being of the odd degree in the medium velocity these terms reflect the energy dissipation/pumping for the medium in the vector graviton environment. As a result, there appears in  $T_{\mu\nu}^{(m)}$  one more independent tensor structure  $\tau_{\mu\nu}$  accounting, in particular, for the anisotropy of the medium. In the limit  $|k| \gg 1$  with the redefinition  $f \rightarrow f/k$ , eq. (47) reproduces the results for the (singlet) scalar graviton [1], in particular,  $\tau_{\mu\nu} = 0$ . The trace of the energy-momentum tensor gets modified as

$$T_{m\mu}^\mu = \rho - 3p + \tau_\mu^\mu . \quad (48)$$

**Equations of motion** As the equation of motion for the continuous medium, there serves the conservation condition in the external field, eq. (38), for the matter energy-momentum tensor. This equation can be divided into two parts. First, projecting it on the streamlines by multiplying on  $U_\mu$  and accounting for  $U \cdot \nabla_\nu U = 0$ , one gets the energy equation for the medium in the vector graviton field:

$$\nabla \cdot \left( (\rho + p)U \right) - U \cdot \partial p + U \cdot (\nabla \cdot \tau) = q. \quad (49)$$

The scalar  $q = U \cdot Q$  on the r.h.s. of the equation above coincides with the power  $Q_0$  depositing in (dissipating from) the medium per unit proper volume due to interactions with the vector gravitons. Second, restricting eq. (38) by the projector  $\Pi_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu$ ,  $(\Pi \cdot U)_\mu = (g_{\mu\nu} - U_\mu U_\nu)U^\nu = 0$ , on the hypersurface orthogonal to the streamlines one gets the modified Euler equation:

$$(\rho + p)U \cdot \nabla U_\mu + U \cdot \partial p U_\mu - \partial_\mu p + (\Pi \cdot (\nabla \cdot \tau))_\mu = Q_\mu - qU_\mu. \quad (50)$$

When all the terms above, but the first one proportional to  $\rho$ , are missing eq. (50) is nothing but the geodesic condition:  $U \cdot \nabla U_\mu = 0$ . Otherwise, it describes the deviation of the flow from the geodesics due to the medium pressure and the influence of the vector graviton field.

## 4. Conclusion

The classical theory of gravity with the GC violation and the residual IC can consistently be constructed. The theory describes the vector graviton, the scalar component incorporated, as a part of the metric (in addition to the massless tensor graviton). Similarly to the previously studied (singlet) scalar graviton, the vector graviton may constitute one more fraction of the gravitational DM. The case study for the tensor graviton, as the remaining part of the gravitational DM, is to be given in the future.

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