$\mathcal{CP}$ Violation in K-Meson Decays

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Here we present a pedagogical review of $\mathcal{CP}$ and $\mathcal{T}$ violation in the decays of K mesons. Diagonalization of the quark mass matrix and the emergence of the complex phase in both the standard and the left–right symmetric models is considered in great detail. A special emphasis is given to a correct definition of $\mathcal{CP}$-violating quantities: ϵ, ϵ' *etc.* (with due regard for the Wu–Yang phase convention) and to formulation of the time-reversal invariance criterion in the elementary particle physics. A particular attention has been concentrated on theoretical evaluation of the parameters ϵ and ϵ' and the $\mathcal{CP}$- and $\mathcal{T}$-violating asymmetries in the decays $K \to \mu\nu\gamma$ and $K \to 3\pi$.

1 The CKM matrix

1.1 Diagonalization of the Mass Matrix

The initial Lagrangian of the Standard Model (SM) involves 12 massless chiral quarks

$$
\begin{array}{ll}
D_1^R \quad U_1^R & \qquad d^R \quad u^R \\[2em]
\underline{D_1^L \quad U_1^L} & \qquad \underline{d^L \quad u^L} \\[2em]
D_2^R \quad U_2^R & \qquad s^R \quad c^R \\[1em]
\qquad\qquad\qquad \sim & \\[1em]
\underline{D_2^L \quad U_2^L} & \qquad \underline{s^L \quad c^L} \\[2em]
D_3^R \quad U_3^R & \qquad b^R \quad t^R \\[2em]
\underline{D_3^L \quad U_3^L} & \qquad \underline{b^L \quad t^L}
\end{array}
$$

which acquire mass due to spontaneous breaking of the $SU(2)_L$ symmetry. The most general quark mass matrix induced by the Higgs fields has the form

$$
\sum_{i,j=1}^{3} \left(\bar{D}_i^L M'_{ij} D_j^R + \bar{U}_i^L N'_{ij} U_j^R \right) + \text{H.c.}, \tag{1}
$$

where M'_{ij} and N'_{ij} are arbitrary 3×3 matrices. This mass term can be considered as a perturbation of the initial SM Hamiltonian, the above quark states form the basis associated with the 12-fold

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degenerate zero-mass level of the unperturbed Hamiltonian. According to the quantum mechanics, one should find the basis in which the perturbation operator takes the diagonal form

$$M' = V_d M Y_d^\dagger, \qquad N' = V_u M Y_u^\dagger, \tag{2}$$

where $M = \mathrm{diag}(m_d, m_s, m_b)$ and $N = \mathrm{diag}(m_u, m_c, m_t)$; Y_d, Y_u, V_d, and V_u are unitary 3×3 matrices; and

$$d_i^R, \ u_i^R, \ d_i^L, \ u_i^L, \ (i = 1, 2, 3) \tag{3}$$

are the so called mass eigenstates, which form the sought-for basis. The interaction eigenstates (the eigenstates of the interaction Hamiltonian) are expressed in terms of the mass eigenstates as follows:

$$D^R = Y_d \, d^R, \qquad U^R = Y_u \, u^R,$$
$$D^L = V_d \, d^L, \qquad U^L = V_u \, u^L.$$

This being so, the interaction between left charged currents and gauge bosons

$$\sum_{i=1}^{3} \left(\bar{D}_i^L \hat{W}^- U_i^L + \bar{U}_i^L \hat{W}^+ D_i^L \right) \tag{4}$$

takes the form

$$\sum_{i,j=1}^{3} \bar{d}_i^L (V_d^\dagger V_u)_{ik} \hat{W}^- u_j^L + \bar{u}_i^L (V_u^\dagger V_d)_{ik} \hat{W}^+ d_j^L \ = \ \sum_{i=1}^{3} \bar{u}_i \gamma^\mu \frac{1 - \gamma^5}{2} W_\mu^+ d_i' + \mathrm{H.c.}, \tag{5}$$

where

$$d_i' = V_{ij} d_j, \quad \text{and} \quad V = V_u^\dagger V_d \tag{6}$$

is the Cabibbo–Kobayashi–Maskawa (CKM) matrix [1]. **Note that the matrices Y_d and Y_u play no role when considering interaction.** Now the left doublets (that appear in the interaction Lagrangian) have the form

$$\begin{pmatrix} u \\ V_{ud}d + V_{us}s + V_{ub}b \end{pmatrix}, \quad \begin{pmatrix} c \\ V_{cd}d + V_{cs}s + V_{cb}b \end{pmatrix}, \quad \begin{pmatrix} t \\ V_{td}d + V_{ts}s + V_{tb}b \end{pmatrix}.$$

It should be noticed that mass eigenstates are invariant under the transformations $V_u \to V_u \Phi_u$, $V_d \to V_d \Phi_d$, where

$$\Phi_u = \begin{pmatrix} e^{i\phi_u} & 0 & 0 \\ 0 & e^{i\phi_c} & 0 \\ 0 & 0 & e^{i\phi_t} \end{pmatrix}, \qquad \Phi_d = \begin{pmatrix} e^{i\phi_d} & 0 & 0 \\ 0 & e^{i\phi_s} & 0 \\ 0 & 0 & e^{i\phi_b} \end{pmatrix};$$

thus any matrix of the family $\Phi_u^\dagger V \Phi_d$ may be chosen as the quark-mixing matrix. To exclude this arbitrariness, one should fix the phases of the quark states:

	u		V_{ud}	
	c		V_{cd}	
fixing the phase of the	t	quark allows to make	V_{td}	real.
	s		V_{us}	
	b		V_{ub}	

The phase of the d quark is fixed by the requirement that the above matrix elements are real.

Thus 5 independent conditions can be imposed on the elements of the mixing matrix and so it has 4 independent parameters, 1 of which has to be complex (all real-valued $U(3)$ matrices belong to the $SO(3)$ group). This means that there is one $\mathcal{CP}$- and $\mathcal{T}$- violating parameter in the SM interaction Lagrangian, whose value cannot be determined from the general principles. In the case of N flavors, a similar reasoning implies the existence of $(N-1)(N-2)/2$ complex phases; in the case of two generations, all the elements of the mixing matrix can be made real and the theory is $\mathcal{CP}$ invariant.

It is instuctive to show in detail how the imaginary part of the fermion–fermon–vector-boson coupling constant implies $\mathcal{CP}$ and $\mathcal{T}$-violation. We define the action of the $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ transformations on the fermion fields as follows:

$$\mathcal{P}\psi(x)\mathcal{P}^\dagger = \gamma^0\psi(\tilde{x}), \qquad \mathcal{C}\psi(x)\mathcal{C}^\dagger = -i\gamma^2\psi^*(\tilde{x}), \qquad \mathcal{T}\psi(x)\mathcal{T}^\dagger = i\gamma^1\gamma^3\psi(-\tilde{x}), \qquad (7)$$

where $\tilde{x}^\mu = x_\mu$. Then the action of these operators on the fermion densities

$$
\begin{aligned}
S(x) &= \; :\bar{q}(x)q(x): \\
V^\mu(x) &= \; :\bar{q}(x)\gamma^\mu q(x): \\
T^{\mu\nu}(x) &= \; :\bar{q}(x)\sigma^{\mu\nu}q(x): \\
A^\mu(x) &= \; :\bar{q}(x)\gamma^\mu\gamma^5 q(x): \\
P(x) &= \; :i\bar{q}(x)\gamma^5 q(x):
\end{aligned}
\qquad (8)
$$

takes the form [2] (note that $\mathcal{T}$ conjugates all complex constants):

Table 1. $\mathcal{P}, \mathcal{C}$, and $\mathcal{T}$ transformations of the fermion densities.

Transformation	$S(x)$	$V^\mu(x)$ and vector field	$T^{\mu\nu}(x)$	$A^\mu(x)$	$P(x)$
$\mathcal{P}$	$S(\tilde{x})$	$V_\mu(\tilde{x})$	$T_{\mu\nu}(\tilde{x})$	$A_\mu(\tilde{x})$	$-P(\tilde{x})$
$\mathcal{C}$	$S(x)$	$-V^\mu(x)$	$-T^{\mu\nu}(x)$	$-A^\mu(x)$	$P(x)$
$\mathcal{T}$	$S(-\tilde{x})$	$V_\mu(-\tilde{x})$	$-T_{\mu\nu}(-\tilde{x})$	$A_\mu(-\tilde{x})$	$-P(-\tilde{x})$
$\Theta = \mathcal{CPT}$	$S(-x)$	$-V^\mu(-x)$	$T^{-\mu\nu}(x)$	$-A^\mu(-x)$	$P(x)$

These definitions agree with the transformations of various physical quantities under $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$ (see Table 1). It is seen now that the term[1], say,

$$\int dx \; \left(V_{td}\, \bar{t}\gamma^\mu(1-\gamma^5)d\; W_\mu^+ \;+\; V_{td}^*\, \bar{d}\gamma^\mu(1-\gamma^5)t\; W_\mu^- \right) \qquad (9)$$

is both $\mathcal{CP}$ and $\mathcal{T}$ violating due to imaginary part of V_{td}: $\mathcal{CP}$ transformation changes the fermion densities: $\bar{t}\gamma^\mu(1-\gamma^5)d\; W_\mu^+ \;\leftrightarrow\; \bar{d}\gamma^\mu(1-\gamma^5)t\; W_\mu^-$, whereas $\mathcal{T}$ has no effect on the operators, however, conjugates the coefficients: $V_{td} \leftrightarrow V_{td}^*$. Note that the $\mathcal{CPT}$ transformation leaves the term (9) invariant.

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[1]Operator $\bar{t}...d$ creates t and anti-d quarks and annihilates d and anti-t quarks.

1.2 The strong CP problem

Strictly speaking, one could fix the phases of left-handed and right-handed states separately, excluding arbitrariness in the chiral phases α:

$$q^R \to e^{i\alpha}q^R, \qquad q^L \to e^{-i\alpha}q^L. \tag{10}$$

This would give rise to the terms

$$ih_1^u \bar{u}\gamma^5 u + ih_1^d \bar{d}\gamma^5 d + ... \tag{11}$$

in the interaction Lagrangian. A strong limit on the CP-odd parameters h_i comes from the mechanism of the spontaneous breaking of the chiral $SU_L(N_f) \times SU_3(N_f)$ symmetry in strong interactions. Let us assume that the vacuum is CP-even. In this case, the v.e.v. of the mass term

$$H'_m = \sum_{i=1}^{N_F} \bar{q}_i(m_i - ih_i\gamma^5)q_i \tag{12}$$

can approach its minimum only provided that $h_u = h_d = h_s = ... = \lambda$ (a consequence of the Dashen theorem [3]). If we fix the chiral phases of quarks so that the mass term is γ^5-free, we have to consider the spontaneous breaking of the CP symmetry:

$$\langle 0|\bar{U}_i^L U_j^R|0\rangle = \langle 0|\bar{D}_i^L D_j^R|0\rangle = Ce^{i\phi_i}\delta_{ij}, \tag{13}$$

where the angles ϕ_i are CP-violating parameters. Thus we got rid of the $SU(N_f)$ chiral phases. The $U_A(1)$ chiral phase can be absorbed in the so called θ term giving rise to the strong CP violation: the measurable quantities are independent of $\theta - 2N_F\alpha$, where α is the chiral phase and θ is the coefficient of the CP-odd term [4]

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} G_{\mu\nu}G^{\mu\nu}, \tag{14}$$

which is equivalent to (that is, can be replaced with) the "pseudoscalar mass term"

$$\sum_{i=1}^{N_F} \frac{i}{\left(\dfrac{1}{m_1} + ... + \dfrac{1}{m_{N_F}}\right)} \; \bar{q}_i \gamma^5 q_i. \tag{15}$$

1.3 The CKM Matrix

Performing the above procedure, we arrive at the expression for the quark mixing matrix in the form proposed by Kobayashi and Maskawa:

$$\begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix},$$

where $c_i = \cos\theta_i, s_i = \sin\theta_i$. This matrix differs from the mixing matrix advocated by the Particle Data Group [5],

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix},$$

in the phases of the c, b, t quarks. A popular approximation that emphasizes the hierarchy in the size of the angles

$$s_{12} \gg s_{23} \gg s_{13},\tag{16}$$

where $s_{12} \equiv \lambda$ is the sine of the Cabibbo angle, is that one expands the other elements in terms of the parameter λ. Up to and including terms of order λ^3, the mixing matrix is given by

$$\begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix},$$

where A, ρ, η are assumed to be of order unity. The unitarity of the CKM matrix implies

$$V_{ub}^* V_{ud} + V_{cb}^* V_{cd} + V_{tb}^* V_{td} = 0.\tag{17}$$

In the approximation $V_{ud} \simeq V_{tb} \simeq 1$ one obtains

$$\frac{V_{ub}^*}{A\lambda^3} + \frac{V_{td}}{A\lambda^3} - 1 = 0.\tag{18}$$

interacts with quarks as follows:

$$\bar{q}_i^L(h_{ij}\phi + f_{ij}\tilde{\phi})q_j^R + \text{H.c.}, \tag{20}$$

where $\tilde{\phi} = -\sigma_2 \, \phi^* \, \sigma_2$ and the indices i, j denote generation. Vacuum expectation values (v.e.v.) of the Higgs fields can be taken to be

$$\langle\phi\rangle = \begin{pmatrix} \kappa & 0 \\ 0 & \kappa' \end{pmatrix}$$

giving rise to the mass term of the form

$$\sum_{i,j=1}^{3} \left(\bar{U}_i^L, \bar{D}_i^L\right) \begin{pmatrix} h_{ij}\kappa + f_{ij}\kappa' & 0 \\ 0 & h_{ij}\kappa' + f_{ij}\kappa \end{pmatrix} \begin{pmatrix} U_j^R \\ D_j^R \end{pmatrix} = \bar{D}_i^L M'_{ij} D_j^R + \bar{U}_i^L N'_{ij} U_j^R,$$

where $M'_{ij} = f_{ij}\kappa + h_{ij}\kappa'$ and $N'_{ij} = f_{ij}\kappa' + h_{ij}\kappa$. Diagonalization of the mass matrices M' and N' can be made by the same token as in the case of the SM:

$$D^R = Y_d \, d^R, \qquad U^R = Y_u \, u^R,$$
$$D^L = V_d \, d^L, \qquad U^L = V_u \, u^L,$$

where

$$d_i^R, \ u_i^R, \ d_i^L, \ u_i^L, \ (i = 1, 2, 3) \tag{21}$$

are the mass eigenstates. Mixing in the left sector and the elimination of arbitrariness in the choice of the matrices V_d and V_u can be considered in the same way as in the case of the SM. The additional interaction Lagrangian of the right currents is expressed in terms of the mass eigenstates as follows:

$$\sum_{i=1}^{3} \left(\bar{U}_i^R \hat{W}^+ D_i^R + \text{H.c.}\right) = \sum_{i=1}^{3} \bar{u}_i \gamma^\mu \hat{W}_R^+ \frac{1 + \gamma^5}{2} d_i'' + \text{H.c.}, \tag{22}$$

where

$$d_i'' = Y_{ij} d_j, \qquad Y_{ij} = (Y_u^\dagger)_{ik}(Y_d)_{kj}. \tag{23}$$

In the case of two flavors,

$$Y = \begin{pmatrix} \cos\theta_R & e^{i\delta_R}\sin\theta_R \\ -e^{i\delta_R}\sin\theta_R & \cos\theta_R \end{pmatrix}.$$

In the case of N flavors the number of independent $\mathcal{CP}$-violating phases is equal to $\frac{N(N-1)}{2}$.

2　The Low-Energy Effective Lagrangian

It is well to recollect that any effective Lagrangian derives from an expansion of the exact amplitudes at small external momenta. Keeping only a few terms of such expansion (denote them by T), we find the Lagrangian $\mathcal{L}_{eff}$ such that $T = \langle out|\mathcal{L}_{eff}|in\rangle$.

Now we turn to the consideration of the consequences of the $\mathcal{CP}$-violating phase for the hadronic physics. This can be made in the two stages:

- we construct the low-energy effective Lagrangian in terms of the quark fields, where the CKM phase gives rise to the imaginary parts of the effective coupling constants, represented by the Wilson coefficients;

- using some model assumptions, we derive the expression for the Lagrangian in terms of the meson fields.

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The remaining operators

$$Q_{3,5} = (\bar{s}d)_{\mathrm{V-A}} \sum_q (\bar{q}q)_{\mathrm{V \mp A}} \;, \tag{26}$$

$$Q_{4,6} = (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}} \sum_q (\bar{q}_\beta q_\alpha)_{\mathrm{V \mp A}} \;,$$

$$Q_{7,9} = \frac{3}{2} (\bar{s}d)_{\mathrm{V-A}} \sum_q e_q \, (\bar{q}q)_{\mathrm{V \pm A}} \;,$$

$$Q_{8,10} = \frac{3}{2} (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}} \sum_q e_q \, (\bar{q}_\beta q_\alpha)_{\mathrm{V \pm A}} \;,$$

come from the celebrated 'penguin' diagram:

The most general effective bosonic Lagrangian of the second order in derivatives, with the same $SU(3)_L \otimes SU(3)_R$ transformation properties and quantum numbers as the short–distance Lagrangian, contains three terms [12]:

$$
\mathcal{L}_2^{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \, f^4 \left\{ g_8 \left[\langle \lambda L_\mu L^\mu \rangle + e^2 f^2 g_{ew} \, \langle \lambda U^\dagger \mathcal{Q} U \rangle \right] \right.
$$

$$
\left. + g_{27} \left(L_{\mu 23} L_{11}^\mu + \frac{2}{3} L_{\mu 21} L_{13}^\mu \right) \right\}, \tag{29}
$$

where the matrix

$$
L_\mu = -i U^\dagger D_\mu U \quad \left(U = \exp\left(\frac{i\sqrt{2}\Phi}{F_\pi} \right) \right)
$$

represents the octet of $V - A$ currents at lowest order in derivatives, $\mathcal{Q} = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ is the quark charge matrix, $\lambda \equiv (\lambda^6 - i\lambda^7)/2$ projects onto the $\bar{s} \to \bar{d}$ transition $[\lambda_{ij} = \delta_{i3}\delta_{j2}]$ and $\langle A \rangle$ denotes the flavor trace of A.

The chiral couplings g_8 and g_{27} measure the strength of the two parts of the effective Lagrangian transforming as $(8_L, 1_R)$ and $(27_L, 1_R)$, respectively, under chiral rotations.

In the presence of electroweak interactions, the explicit breaking of chiral symmetry generated by the quark charge matrix $\mathcal{Q}$ induces the $\mathcal{O}(p^0)$ operator $\langle \lambda U^\dagger \mathcal{Q} U \rangle$, transforming as $(8_L, 8_R)$ under the chiral group.

$$
|g_8| \simeq 5.1, \qquad |g_{27}| \simeq 0.29. \tag{30}
$$

The huge difference between these two couplings shows the well–known enhancement of the octet $|\Delta I| = 1/2$ transitions. In the $N_C \to \infty$ limit, the real parts of these constants are expressed in terms of the Wilson coefficients as follows [13]:

$$
g_8^\infty = -\frac{2}{5} C_1(\mu) + \frac{3}{5} C_2(\mu) + C_4(\mu) - 16 L_5 \left(\frac{\langle \bar{q}q \rangle^{(2)}(\mu)}{f^3} \right)^2 C_6(\mu),
$$

$$
g_{27}^\infty = \frac{3}{5} \left[C_1(\mu) + C_2(\mu) \right], \tag{31}
$$

$$
(g_8 e^2 g_{ew})^\infty = -3 \left(\frac{\langle \bar{q}q \rangle^{(2)}(\mu)}{f^3} \right)^2 C_8(\mu).
$$

The imaginary parts of them are responsible for $\mathcal{CP}$-violating effects and will be considered further. With the effective weak Lagrangian at hand, it is very helpful to consider the evolution of a meson system to the second order in the weak interaction.

3 $\mathcal{CP}$-Violation in the System of Neutral Kaons

It has become a tradition[2] to begin a description of the $K^0 - \bar{K}^0$ system with writing down the most general expression for $\mathcal{CPT}$-invariant Hamiltonian

$$
i \frac{d}{dt} \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{11} \end{pmatrix} \begin{pmatrix} K^0 \\ \bar{K}^0 \end{pmatrix},
$$

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[2]In this Section, we follow [1, 14, 15]

where K^0 and $\bar{K}^0$ are the eigenstates of the strong-interaction Hamiltonian and the matrix H_{ij} is non-Hermitean.

However, it is well to recollect a derivation of this formula and formulate the assumptions made in its derivation.

- We consider weak interactions as a perturbation to the strong interactions.

- We consider evolution of the eigenstates of the strong-interaction Hamiltonian to the second order in the weak interaction and then search for the effective Hamiltonian that would give the same evolution in the leading order of perturbation theory [16].

The perturbation expansion of the S matrix in the $K^0 - \bar{K}^0$ system has the form

$$S_{ab} = \langle b | T \exp\left(-i \int H_W^{int}(t) dt \right) |a\rangle = \delta_{ab} - 2\pi i T_{ab}. \tag{32}$$

where $a, b = K^0, \bar{K}^0$, $H_W^{int}(t) = e^{iHt} H_W e^{-iHt}$, and H_W is the weak-interaction Hamiltonian. To the second order in H_W, we obtain

$$
\begin{aligned}
T_{ab} &= \langle b | H_W | a \rangle - \frac{i}{2} \int dt \langle b | T \left(H_W^{int}(t) H_W^{int}(0) \right) |a\rangle \\
&= \langle b | H_W | a \rangle + \frac{1}{2} \sum_\lambda \left[\frac{\langle b | H_W | \lambda \rangle \langle \lambda | H_W | a \rangle}{E_b - E_\lambda + i\epsilon} + \frac{\langle b | H_W | \lambda \rangle \langle \lambda | H_W | a \rangle}{E_a - E_\lambda + i\epsilon} \right].
\end{aligned} \tag{33}
$$

Making use of the Sokhotsky relations one can represent the transition amplitudes $T_{ab} + \langle b | H_{strong} | a \rangle$ as the matrix elements of the effective Hamiltonian

$$H_{ab} = M_{ab} - i\frac{\Gamma_{ab}}{2}, \tag{34}$$

where

$$
\begin{aligned}
M_{ab} &= m_K \delta_{ab} + \langle b | H_W | a \rangle + \mathcal{P} \int d\lambda \frac{\langle b | H_W | \lambda \rangle \langle \lambda | H_W | a \rangle}{m_K - E_\lambda}, \\
\Gamma_{ab} &= 2\pi \sum_\lambda \langle b | H_W | \lambda \rangle \langle \lambda | H_W | a \rangle \delta(m_K - E_\lambda).
\end{aligned} \tag{35}
$$

Note that $M = M^\dagger$ and $\Gamma = \Gamma^\dagger$. Thus the Hamiltonian

$$
H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{11} \end{pmatrix} = \begin{pmatrix} (M'_{11} + iM''_{11}) - i\dfrac{(\Gamma'_{11} + i\,\Gamma''_{11})}{2}; & (M'_{12} - iM''_{12}) - i\dfrac{(\Gamma'_{12} - i\,\Gamma''_{12})}{2} \\ (M'_{12} + iM''_{12}) - i\dfrac{(\Gamma'_{12} + i\,\Gamma''_{12})}{2}; & (M'_{11} + iM''_{11}) - i\dfrac{(\Gamma'_{11} + i\,\Gamma''_{11})}{2} \end{pmatrix},
$$

is related to the Hamiltonian of the weak interactions. Eigenstates of this Hamiltonian are identified with the physical states K_L^0 and K_S^0, which are expressed through K^0 and $\bar{K}^0$ in terms of the parameter

$$\bar{\epsilon} = \frac{\sqrt{H_{12}} - \sqrt{H_{21}}}{\sqrt{H_{12}} + \sqrt{H_{21}}} : \tag{36}$$

$$K_L^0 = \frac{1}{\sqrt{2(1 + |\bar{\epsilon}|^2)}} \begin{pmatrix} 1 + \bar{\epsilon} \\ -1 + \bar{\epsilon} \end{pmatrix}, \qquad\qquad K_S^0 = \frac{1}{\sqrt{2(1 + |\bar{\epsilon}|^2)}} \begin{pmatrix} 1 + \bar{\epsilon} \\ 1 - \bar{\epsilon} \end{pmatrix};$$

the respective eigenvalues give the masses and widths of the K_L^0 and K_S^0 mesons:

$$
\begin{aligned}
\lambda_S &= H_{11} - \sqrt{H_{12}H_{21}} = M_S - i\,\frac{\Gamma_S}{2}, \\
\lambda_L &= H_{11} + \sqrt{H_{12}H_{21}} = M_L - i\,\frac{\Gamma_L}{2}.
\end{aligned}
\tag{37}
$$

$\mathcal{CP}$ transformation exchanges $|K^0\rangle$ and $|\bar{K}^0\rangle$ states:

$$
\mathcal{CP}|K^0\rangle = e^{i\theta}|\bar{K}^0\rangle, \quad \mathcal{CP}|\bar{K}^0\rangle = e^{-i\theta}|K^0\rangle.
\tag{38}
$$

The phase factor can be chosen arbitrarily, because any quantum-mechanical state is defined up to a phase. However **an interpretation** of the matrix elements H_{ab} and parameter $\bar{\epsilon}$ depends on a particular choice of the phase.

In the case $\theta = 0$, the parameters M'' and Γ'' are $\mathcal{CP}$-odd, whereas M' and Γ' are $\mathcal{CP}$-even. Assuming that $\mathcal{CP}$-odd parameters are small, $\Gamma_{12}'' \ll \Gamma_{12}'$ and $M_{12}'' \ll M_{12}'$, we obtain

$$
\bar{\epsilon} = \frac{H_{12} - H_{21}}{4\sqrt{H_{12}H_{21}} + (\sqrt{H_{12}} - \sqrt{H_{21}})^2} \approx \frac{i\,M_{12}''}{\lambda_S - \lambda_L}\,\mathrm{sign}(M_{12}').
\tag{39}
$$

The formulas (37) agree with the experimental fact $M_{K_L} > M_{K_S}$ if the sign of the square root $\sqrt{H_{12}H_{21}}$ is chosen so that

$$
\begin{aligned}
\sqrt{H_{12}H_{21}} &= M_{12}' - i\,\frac{\Gamma'}{2} \quad \text{for} \quad M_{12}' > 0, \quad \Delta M \equiv M_{K_S} - M_{K_L} = -2\,M_{12}'; \\
\sqrt{H_{12}H_{21}} &= -M_{12}' + i\,\frac{\Gamma'}{2} \quad \text{for} \quad M_{12}' < 0, \quad \Delta M \equiv M_{K_S} - M_{K_L} = 2\,M_{12}'.
\end{aligned}
$$

Experimental data indicate that $\Delta M = -\dfrac{\Delta\Gamma}{2}$, hence $\lambda_S - \lambda_L \approx \Delta M\,(1+i)$. Combining the above formulas, we arrive at

$$
\bar{\epsilon} = -\frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\,\frac{M_{12}''}{2M_{12}'}
\tag{40}
$$

for both $M_{12}' > 0$ and $M_{12}' < 0$. (The assumption that $M_{K_L} < M_{K_S}$ would give the phase factor $e^{\frac{-i\pi}{4}}$ instead of $e^{\frac{i\pi}{4}}$.)

3.1 Phase Convention

It is important and helpful to keep track of the phase arbitrariness stemming from the fact that

$$
\text{both} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e^{i\phi} \\ 0 \end{pmatrix} \quad \text{describe the same physical state.}
$$

The transformation rules induced by the rotation of the phase of the s quark

$$
s \to e^{i\phi}s, \quad \bar{s} \to e^{-i\phi}\bar{s},
\tag{41}
$$

are as follows:

$$
K^0 \to e^{-i\phi}K^0, \quad \bar{K}^0 \to e^{i\phi}\bar{K}^0, \quad H_{12} \to e^{-2i\phi}H_{12}, \quad H_{21} \to e^{2i\phi}H_{21},
$$

$$
V_{us} \to e^{i\phi}V_{us} \quad \bar{\epsilon} \to \frac{\bar{\epsilon} - i\,\mathrm{tg}\phi}{1 - i\,\bar{\epsilon}\mathrm{tg}\phi}, \quad A_I \to e^{-i\phi}A_I,
$$

$$\mathcal{CP} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & e^{-2i\phi} \\ e^{2i\phi} & 0 \end{pmatrix}.$$

Let the phases of the K^0 and $\bar{K}^0$ be chosen so that the matrix of $\mathcal{CP}$ transformation has the form

$$\mathcal{CP}_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(we call it the "default" phase convention). It should be compared with the widely used Wu–Yang phase convention, in which the phase of A_0 is set equal to zero. It should be noticed that in the Wu–Yang phase convention the operator of $\mathcal{CP}$ transformation has the form

$$\mathcal{CP}_{ab} = \begin{pmatrix} 0 & e^{-2i\alpha} \\ e^{2i\alpha} & 0 \end{pmatrix}.$$

It should also be emphasized that the value $\bar{\epsilon}$ as was defined above is phase-dependent and so **does not measure $\mathcal{CP}$ violation**, and the imaginary part of the effective weak Hamiltonian H_W is not associated with $\mathcal{CP}$ violation (and so it may be larger than real).

We adopt the "default" phase convention. In this case

- the quantities M'' and Γ'' are $\mathcal{CP}$-odd;

- M' and Γ' are $\mathcal{CP}$-even;

- V_{us} is real.

Upon fixing a phase convention, the parameter $\bar{\epsilon}$ makes physical sense and can be related to measurable quantities; the $\mathcal{CP}$-violating parameters in the effective weak Hamiltonian are M'' and Γ''.

3.2 Mixing of K^0 and $\bar{K}^0$ in the Standard Model

As has been demonstrated, such mixing is accounted for by the $\Delta S = 2$ effective weak Lagrangian.

Let us consider the computation of the diagrams in Fig. 5 (giving the transition amplitude $\bar{s}d \rightarrow s\bar{d}$).
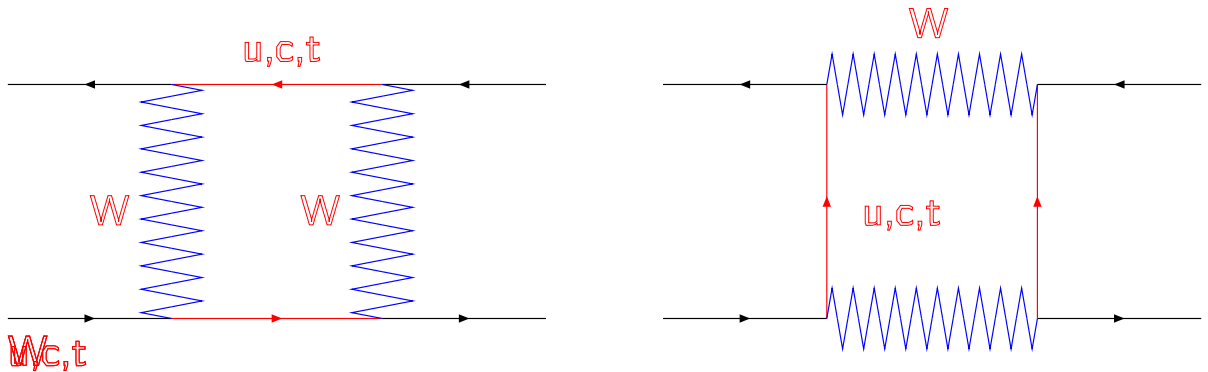
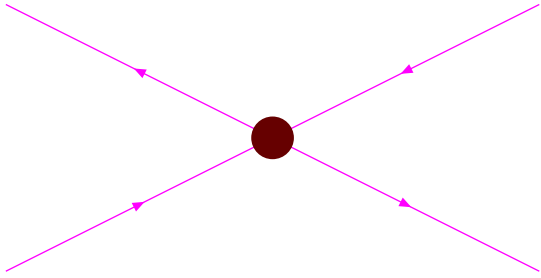Gaillard and Lee in the pioneer work [17] obtained

$$T(\bar{s}d \to s\bar{d}) = -\frac{G_{\mathrm{F}}}{\sqrt{2}} \frac{\alpha}{\pi \sin^2 \theta_{\mathrm{W}}} (\bar{d}_L \gamma^\mu s_L)(\bar{d}_L \gamma_\mu s_L) \sum_{i,j=u,c,t} \xi_i \xi_j A(x_i, x_j), \tag{42}$$

where

$$A(x_i, x_j) = \frac{J(x_i) - J(x_j)}{x_i - x_j}, \quad J(x) \equiv \frac{1}{1-x} + \frac{x^2 \ln x}{(1-x)^2}, \tag{43}$$

$$\xi_i = V_{is} V_{id}^*, \quad \text{and} \quad x_i = \frac{m_i^2}{M_{\mathrm{W}}^2}.$$

Thus we obtain the effective $\Delta S = 2$ Lagrangian

$$\mathcal{L}_{eff}^{\Delta s=2} =$$

is named "the bag constant". The computation of the bag factors presents the major challenge in the calculations of $\mathcal{CP}$-violating quantities in nonleptonic reactions.

The short-distance contribution to ΔM comprises $70 \div 80\%$ of the total SM contribution:

$$\Delta M = -2\, M_{12}'' = -\frac{2}{3} \frac{G_{\mathrm{F}}}{\sqrt{2}}\, \eta B_K\, \frac{\alpha}{4\pi} \left(\frac{m_c}{37 \mathrm{GeV}}\right)^2 \mathrm{Re}\lambda, \tag{49}$$

where (in the approximation $x_u = 0, x_c \ll 1, x_t \sim 1)^4$

$$\mathrm{Re}\lambda = \mathrm{Re}\left(\xi_c^2 + \xi_t^2\, \frac{x_t}{x_c}\, \frac{(1 - x_t^2 + 2x_t \ln x_t)}{(1 - x_t)^3} + 2\xi_c \xi_t \left(\frac{x_t}{1 - x_t} + \frac{\ln x_t}{(1 - x_t)^2} - \ln x_c\right)\right) \tag{50}$$

and the factor η accounts for the corrections due to strong interactions, evaluated in perturbative QCD. The fact that $\mathrm{Re}\lambda \approx 1$ indicates that the c quark gives the dominant contribution to the mass difference. The remaining $20 \div 30\%$ are attributed to the long-distance contribution (that is the contribution of the $\pi\pi$, $\pi\pi\pi$ *etc* intermediate states in formula (35)), which is extremely difficult to compute exactly.
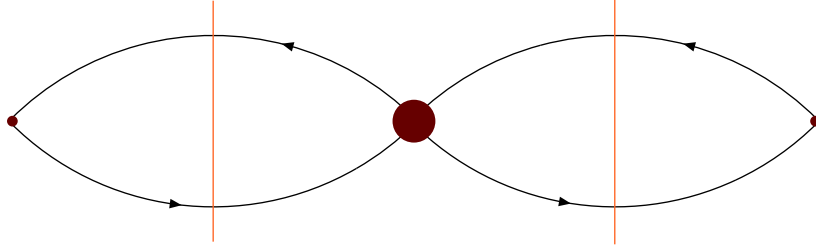
where

$$C_\epsilon = \frac{G_F^2 F_K^2 \, M_K \, M_W^2}{6\sqrt{2}\pi^2 \, \Delta M_K} = 3.837 \times 10^4 \tag{54}$$

(here we have used the experimental value of M'_{12}, determined from ΔM_K, in spite of the theoretical value (49)),

$$
\begin{aligned}
S_0(x_c) &\approx x_c, \\
S_0(x_t) &\approx 2.46 \cdot \left(\frac{m_t}{170\text{GeV}}\right)^{1.52}, \\
S_0(x_c, x_t) &\approx x_c \left[\ln\left(\frac{x_t}{x_c}\right) - \frac{3x_t}{4(1-x_t)} - \frac{3x_t^2 \ln x_t}{4(1-x_t)^2}\right],
\end{aligned}
\tag{55}
$$

$$m_c \simeq 1.3 \text{ GeV}, \quad m_t = (174.3 \pm 5.1) \text{ GeV};$$

the short-distance corrections due to the strong interactions are absorbed in the coefficients [21]

$$\eta_1 = 1.38 \pm 0.20, \quad \eta_2 = 0.57 \pm 0.01, \quad \eta_3 = 0.45 \pm 0.04.$$

Formula (53) allows to set a limitation on the $\mathcal{CP}$-violating parameter η of the SM from the experimental limitations on ϵ (see Fig.7).
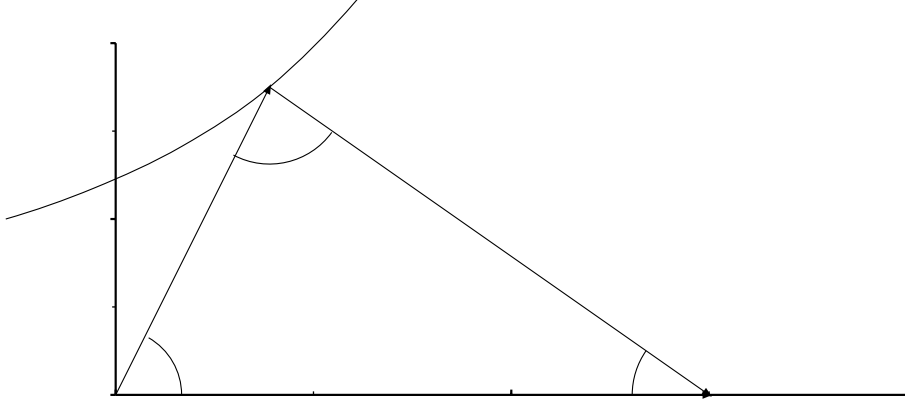
where δ_I is the S-wave phase shift of the $\pi\pi$ scattering;

$$|\pi\pi,\ I\ \mathrm{out}\rangle = e^{2i\delta_I}|\pi\pi,\ I\ \mathrm{in}\rangle \tag{57}$$

and

$$A_I = A_I' + iA_I'', \qquad B_I = B_I' + iB_I''. \tag{58}$$

The properties of these amplitudes under the $\mathcal{CP}$-transformation and time reversal $\mathcal{T}$ are seen from Table 2.

Table 2. Transformation properties of the isospin amplitudes.

Transformation	A'	A''	B'	B''
$\mathcal{CP}$	$+$	$-$	$-$	$+$
$\mathcal{T}$	$+$	$-$	$+$	$-$

It is seen that the amplitudes A_I are $\mathcal{CPT}$-even, whereas the B_I amplitudes are $\mathcal{CPT}$-odd.

Using the relations

$$\frac{1}{\sqrt{2}}|\pi^+\pi^- + \pi^-\pi^+\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|I=0\rangle + |I=2\rangle),$$

$$|\pi^0\pi^0\rangle = \frac{1}{\sqrt{3}}(|I=0\rangle - \sqrt{2}|I=2\rangle),$$

we obtain the expressions for the amplitudes $\mathcal{A}(K^0 \to \pi^+\pi^-)$ and $\mathcal{A}(K^0 \to \pi^0\pi^0)$:

$$\mathcal{A}(K^0 \to \pi^+\pi^-) = A_0 e^{i\delta_0} + \frac{1}{\sqrt{2}}A_2 e^{i\delta_2}, \tag{59}$$

$$\mathcal{A}(K^0 \to \pi^0\pi^0) = \frac{1}{\sqrt{2}}A_0 e^{i\delta_0} - A_2 e^{i\delta_2}.$$

A_0 describes the transitions with $\Delta I = 1/2$, whereas A_2 describes the transitions with $\Delta I = 3/2$. Assuming that the Hamiltonian for the transitions $K \to 2\pi$ contains only terms with quantum numbers $I = 1/2, I_3 = 1/2$ and $I = 3/2, I_3 = 1/2$ (that is, it does not contain the terms with $I = 5/2, I_3 = 1/2$ *etc.*), we can use the Clebsch–Gordan coefficients

$$\langle J, M|\quad j_1, m_1;\quad |j_2, m_2\rangle,$$

$$\langle \pi^+\pi^0|\quad H_W\quad |K^+\rangle \quad \sim \quad \langle 2, 1|\quad \tfrac{3}{2}, \tfrac{1}{2};\quad \tfrac{1}{2}, \tfrac{1}{2}\rangle \quad = \quad \sqrt{3}/2,$$

$$\langle \pi\pi|\quad H_W;\quad |K\rangle,$$

$$\langle \pi\pi, I=2|\quad H_W\quad |K^+\rangle \quad \sim \quad \langle 2, 0|\quad \tfrac{3}{2}, \tfrac{1}{2};\quad \tfrac{1}{2}, -\tfrac{1}{2}\rangle \quad = \quad 1/\sqrt{2}$$

to obtain the $\mathcal{A}(K^+ \to \pi^+\pi^0)$ amplitude

$$\mathcal{A}(K^+ \to \pi^+\pi^0) = \frac{3}{2}A_2 e^{i\delta_2}. \tag{60}$$

A comparison of the lifetimes of K_S^0 and K^+ leads one to a conclusion that the ratio

$$|\omega| \;=\; \frac{|A_2|}{|A_0|} \;=\; \frac{1}{22} \tag{61}$$

is very small, which is referred to as the long-standing "$\Delta I = 1/2$ problem". The experimental value $\delta_2 - \delta_0 \simeq (45 \pm 6)^o$.

The tree–level $K \to \pi\pi$ amplitudes generated by the $\mathcal{O}(p^2)$ χPT Lagrangian are:

$$
\begin{aligned}
A_0 &= -\frac{G_F}{\sqrt{2}} \, V_{ud} V_{us}^* \, \sqrt{2} \, f \, \left\{ \left(g_8 + \frac{1}{9} \, g_{27} \right) (M_K^2 - M_\pi^2) - \frac{2}{3} \, f^2 \, e^2 \, g_8 \, g_{ew} \right\}, \\
A_2 &= -\frac{G_F}{\sqrt{2}} \, V_{ud} V_{us}^* \, \frac{2}{9} \, f \, \left\{ 5 \, g_{27} \, (M_K^2 - M_\pi^2) - 3 \, f^2 \, e^2 \, g_8 \, g_{ew} \right\}.
\end{aligned} \tag{62}
$$

Let us introduce the notation

$$\omega \;=\; \frac{A_2}{A_0} \;=\; |\omega| e^{i\chi}, \quad A_0 = |A_0| e^{i\alpha}, \quad \delta = \delta_2 - \delta_0, \quad \tilde{\omega} = \sqrt{2} |\omega| e^{i\delta}. \tag{63}$$

We can now express the amplitudes $\mathcal{A}(K_L^0 \to \pi^+\pi^-)$, $\mathcal{A}(K_L^0 \to \pi^0\pi^0)$ $\mathcal{A}(K_S^0 \to \pi^+\pi^-)$ and $\mathcal{A}(K_S^0 \to \pi^0\pi^0)$ in terms of A_1, A_2 and the parameter $\bar{\epsilon}$. For example,

$$
\begin{aligned}
\mathcal{A}(K_L^0 \to \pi^0\pi^0) &= \frac{1}{\sqrt{2(1 + |\bar{\epsilon}|^2)}} \{ [\mathcal{A}(K^0 \to \pi^0\pi^0) - \mathcal{A}(\bar{K}^0 \to \pi^0\pi^0)] \\
&\qquad\qquad + \bar{\epsilon} [\mathcal{A}(K^0 \to \pi^0\pi^0) + \mathcal{A}(\bar{K}^0 \to \pi^0\pi^0)] \} \\
&= \frac{|A_0| e^{i\delta_0}}{\sqrt{2(1 + |\bar{\epsilon}|^2)}} \left(i \, \sin\alpha - i\tilde{\omega} \sin(\alpha + \chi) + \bar{\epsilon}[\cos\alpha - \tilde{\omega} \cos(\alpha + \chi)] \right).
\end{aligned} \tag{64}
$$

Let us consider the experimentally measurable values

$$\eta_{00} = \frac{\langle \pi^0\pi^0 | K_L^0 \rangle}{\langle \pi^0\pi^0 | K_S^0 \rangle}, \quad \eta_{+-} = \frac{\langle \pi^+\pi^- | K_L^0 \rangle}{\langle \pi^+\pi^- | K_S^0 \rangle}. \tag{65}$$

Proceeding as indicated above, we obtain

$$\eta_{00} = \frac{i \, [\sin\alpha - \tilde{\omega} \sin(\alpha + \chi)] + \bar{\epsilon}[\cos\alpha - \tilde{\omega} \cos(\alpha + \chi)]}{[\cos\alpha - \tilde{\omega} \cos(\alpha + \chi)] + i\bar{\epsilon}[\sin\alpha - \tilde{\omega} \sin(\alpha + \chi)]} \tag{66}$$

and a similar expression for η_{+-}.

Assuming that $\mathcal{CP}$-violating parameters are small, that is $\bar{\epsilon}, \alpha, \chi \ll 1$, we arrive at

$$\eta_{00} = \bar{\epsilon} + i\alpha - i \frac{\chi\tilde{\omega}}{1 - \tilde{\omega}}, \tag{67}$$

where

$$\chi \approx \frac{A_2''}{A_2'} - \frac{A_0''}{A_0'}.$$

In a similar way, one obtains

$$\eta_{+-} \equiv \frac{\langle \pi^+\pi^- | K_L^0 \rangle}{\langle \pi^+\pi^- | K_S^0 \rangle} = \bar{\epsilon} + i\alpha - i \frac{\chi\tilde{\omega}}{1 - \tilde{\omega}}. \tag{68}$$

Let us introduce the parameters, which are conventionally used for a description of the effects of $\mathcal{CP}$-violation:

$$\epsilon = \frac{\langle \pi\pi, I = 0|K_L^0\rangle}{\langle \pi\pi, I = 0|K_S^0\rangle} = \bar{\epsilon} + i\alpha, \tag{69}$$

$$\epsilon' = \frac{i}{\sqrt{2}}e^{i\delta}Im\frac{A_2}{A_0} = \frac{i\,\tilde{\omega}}{2}\left(\frac{A_2''}{A_2'} - \frac{A_0''}{A_0'}\right).$$

Note that, in contrast to $\bar{\epsilon}$, the parameter ϵ is independent of the phase convention. These values coincide **only in the Wu–Yang phase convention.** In terms of the introduced parameters, we have

$$\eta_{00} = \frac{\langle \pi^0\pi^0|K_L^0\rangle}{\langle \pi^0\pi^0|K_S^0\rangle} = \epsilon - \frac{2\epsilon'}{1 - |\omega|e^{i\delta}\sqrt{2}} \approx \epsilon - 2\epsilon', \tag{70}$$

$$\eta_{+-} = \frac{\langle \pi^+\pi^-|K_L^0\rangle}{\langle \pi^+\pi^-|K_S^0\rangle} = \epsilon + \frac{\epsilon'}{1 + \frac{|\omega|e^{i\delta}}{\sqrt{2}}} \approx \epsilon + \epsilon',$$

where

$$|\omega| = Re\frac{A_2}{A_0} \simeq 0.045, \quad \delta \simeq 45^o.$$

From the above it follows that

$$Re\frac{\epsilon'}{\epsilon} \simeq \frac{1 - |\omega|}{6}\left(1 - \left|\frac{\eta_{00}}{\eta_\pm}\right|^2\right). \tag{71}$$

Note that ϵ'/ϵ is approximately real. Using the short–distance Lagrangian, the $\mathcal{CP}$–violating ratio ϵ'/ϵ can be written as follows [22]:

$$\frac{\epsilon'}{\epsilon} = \text{Im}\,(V_{ts}^*V_{td})\, e^{i\Phi}\, \frac{G_F}{2|\varepsilon|}\, \frac{\omega}{|\text{Re}(A_0)|}\, \left[P^{(0)}\,(1 - \Omega_{IB}) - \frac{1}{\omega}\,P^{(2)}\right], \tag{72}$$

where the quantities

$$P^{(I)} = \sum_i y_i(\mu)\,\langle(\pi\pi)_I|Q_i|K\rangle \tag{73}$$

contain the contributions from hadronic matrix elements with isospin I and

$$\Omega_{IB} = \frac{1}{\omega}\,\frac{\text{Im}(A_2)_{\text{IB}}}{\text{Im}(A_0)} \tag{74}$$

parameterizes isospin breaking corrections. The factor $1/\omega$ enhances the relative weight of the $I = 2$ contributions.

The hadronic matrix elements $\langle(\pi\pi)_I|Q_i|K\rangle$ are usually parameterized in terms of the bag parameters B_i, which measure them in units of their vacuum insertion approximation values. In the SM, $P^{(0)}$ and $P^{(2)}$ turn out to be dominated by the contributions from the QCD penguin operator Q_6 and the electroweak penguin operator Q_8, respectively [23]. Thus, to a very good approximation, ϵ'/ϵ can be written (up to global factors) as [24, 25]

$$\frac{\epsilon'}{\epsilon} \sim \left[B_6^{(1/2)}(1 - \Omega_{IB}) - 0.4\,B_8^{(3/2)}\right]. \tag{75}$$

34

The isospin–breaking correction coming from π^0-η mixing was originally estimated to be $\Omega_{IB}^{\pi^0\eta} = 0.25$ [26]. Together with the usual ansatz $B_i \sim 1$, this produces a large numerical cancellation in (72) [27] leading to low values of ϵ'/ϵ around $7 \cdot 10^{-4}$. A recent improved calculation of π^0-η mixing at $\mathcal{O}(p^4)$ in χPT has found the result [28]

$$\Omega_{IB}^{\pi^0\eta} = 0.16 \pm 0.03. \tag{76}$$

This smaller number, slightly increases the naive estimate of ϵ'/ϵ.

Table 3. Theoretical and experimental values of ϵ'/ϵ.

Year	Theory (models: $1/N_c$, unitarization)	Theory (lattice)	Experiment
~ 1988	$0.01 \div 0.03$ [1]		$(3.2 \pm 1.0) \times 10^{-3}$ [31]
1995	$(6.7 \pm 2.6) \times 10^{-4}$ [25]	$(3.1 \pm 2.5) \times 10^{-4}$ [29]	$(1.5 \pm 0.8) \times 10^{-3}$ [32]
1999 –2000	$(-1 \div 35) \times 10^{-4}$ [22]	$(0.44 \div 2.1) \times 10^{-3}$ [30]*	$(2.1 \pm 1.5) \times 10^{-3}$ [5]
2001	$(1.7 \pm 0.9) \times 10^{-3}$ [27]		$(1.53 \pm 0.24) \times 10^{-3}$ [33]**

* This value depends crucially on the mass of the s quark: 0.44×10^{-3} for $m_s(m_c) = 150$ MeV and 2.1×10^{-3} for $m_s(m_c) = 80$ MeV.

** The most recent data: $\mathrm{Re}\frac{\epsilon'}{\epsilon} = (1.73 \pm 0.18) \times 10^{-3}$.
(http://na48.web.cern.ch/NA48/Welcome/images/talks/win02/win02.pdf)

4 Time-Reversal Invariance

Throughout this Section it is assumed that all the processes under consideration are adequately described by some local quantum field theory, that is, $\mathcal{CPT}$ is an exact symmetry of the theory. In this case $\mathcal{CP}$-violation is equivalent to $\mathcal{T}$-violation and so we turn to the consideration of the time-reversal invariance.

<u>Time-reversal invariance in the classical mechanics</u> (Fig. 8):

motion from A to B $\quad \{x(t): \ x(T_1) = A, \ x(T_2) = B\}$

$$\text{and}$$

motion from B to A $\quad \{x(t): \ x(T_1) = B, \ x(T_2) = A\}$

are described with the same Hamiltonian.
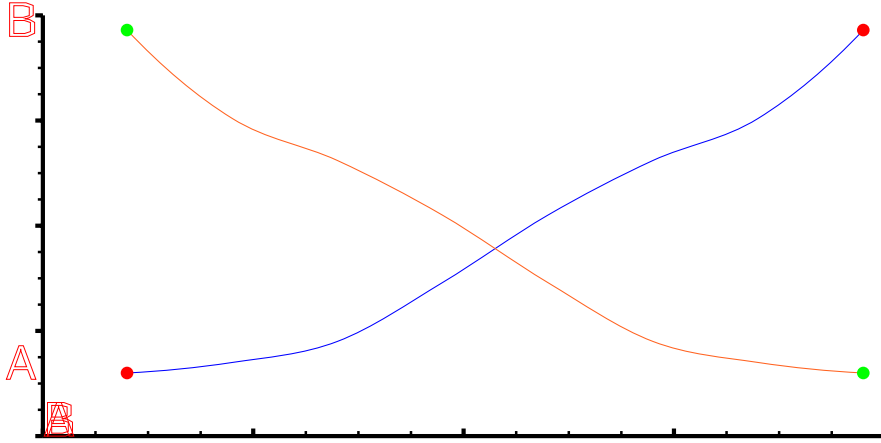
This is the case provided that $H(p, x) = H(-p, x)$.
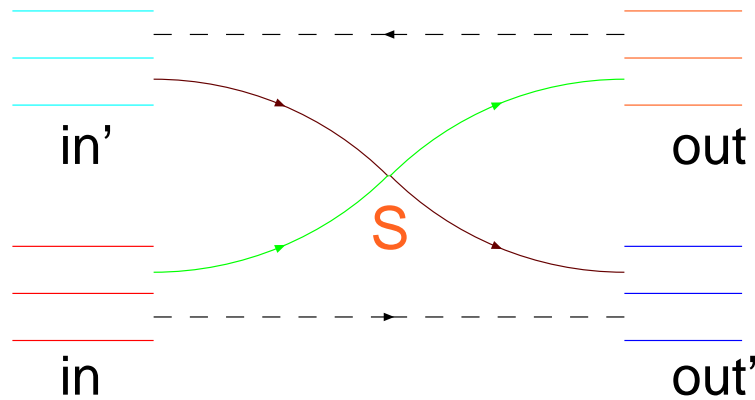
- there exists (anti-unitary) operator of time reversal

$$\mathcal{T} : in \to out, \quad \mathcal{T} : out \to in, \quad \text{such that} \quad \langle \mathcal{T}I | \mathcal{T}F \rangle = \langle F|I \rangle = \langle I|F \rangle^*,$$

which changes the signs of spins and momenta of all particles; this condition can be cast in the form $(\phi, \mathcal{T}\psi) = (\mathcal{T}\phi, \psi)^*$ for all vectors ϕ and $\psi \in \mathcal{H}$;

- the transition from the state $|I\rangle$ in the state $\langle F|$ and the transition from the state $|\mathcal{T}F\rangle$ to the state $\langle \mathcal{T}I|$ are described with the same S-matrix (see Fig. 9):

$$\mathcal{M}_{I \to F} = \langle F|S|I \rangle = \langle \mathcal{T}I|S|\mathcal{T}F \rangle = \mathcal{M}_{\mathcal{T}F \to \mathcal{T}I}. \tag{77}$$

Table 4. $\mathcal{P}$ and $\mathcal{T}$ transformations for various quantities.

Action of these operators on the quantum-mechanical states is determined by the condition that the $\mathcal{T}$- (or $\mathcal{P}$-)transformed states are characterized by the $\mathcal{T}$- (or $\mathcal{P}$-)transformed (eigen)values of the respective operators.

Value	Notation	$\mathcal{P}$-transformed value	$\mathcal{T}$-transformed value	Comment
Coordinate	$\vec{\mathbf{x}}$	$-\vec{\mathbf{x}}$	$\vec{\mathbf{x}}$	
Momentum	$\vec{\mathbf{p}}$	$-\vec{\mathbf{p}}$	$-\vec{\mathbf{p}}$	$\vec{\mathbf{p}} = m\, \dfrac{d\vec{\mathbf{x}}}{dt}$
Angular momentum	$\vec{\mathbf{l}}$	$\vec{\mathbf{l}}$	$-\vec{\mathbf{l}}$	$(\vec{\mathbf{r}} \times \vec{\mathbf{p}})$
Spin	$\vec{\mathbf{s}}$	$\vec{\mathbf{s}}$	$-\vec{\mathbf{s}}$	Like $\vec{\mathbf{l}}$
Electric field	$\vec{\mathbf{E}}$	$-\vec{\mathbf{E}}$	$\vec{\mathbf{E}}$	$\vec{\mathbf{E}} = -\dfrac{dA_0}{d\vec{\mathbf{r}}}$
Magnetic field	$\vec{\mathbf{B}}$	$\vec{\mathbf{B}}$	$-\vec{\mathbf{B}}$	$\vec{\mathbf{B}} \sim \vec{r} \times \vec{j}$
Potential	A_0	A_0	A_0	
Vector potential	$\vec{A}$	$-\vec{A}$	$-\vec{A}$	
Helicity	λ	$-\lambda$	λ	$\lambda = \vec{\mathbf{s}}\vec{\mathbf{p}}$
Transverse polarization*	ξ	ξ	$-\xi$	$\xi = (\vec{\mathbf{s}}, [\vec{\mathbf{p}_1} \times \vec{\mathbf{p}_2}])$
Triple correlation**	η	$-\eta$	$-\eta$	$\eta = (\vec{\mathbf{p}_1}, [\vec{\mathbf{p}_2} \times \vec{\mathbf{p}_3}])$

* A characterisitic of a three-particle state, if at least one particle has a nonzero spin.
** A characterisitic of a multiparticle state (number of particles must be greater than 3).

Having in mind that

$$\langle \mathcal{T}I|S|\mathcal{T}F\rangle = (\mathcal{T}I,\ S\mathcal{T}F) = (S^\dagger \mathcal{T}I,\ \mathcal{T}F) = (\mathcal{T}F,\ S^\dagger \mathcal{T}I)^* = \langle \mathcal{T}F|S^\dagger|\mathcal{T}I\rangle^*, \qquad (78)$$

we obtain the condition of $\mathcal{T}$-invariance in terms of the decay and scattering amplitudes:

$$\langle\ F|S|I\rangle = \langle\ \mathcal{T}F|S^\dagger|\mathcal{T}I\rangle^*. \qquad (79)$$

In perturbation theory, the S matrix is expanded in the coupling constant g, which is assumed to be a small parameter:

$$S = 1 + igT_1 + g^2 T_2 + ... \qquad (80)$$

Here T_1 is nothing but the interaction Lagrangian, $T_1 = \mathcal{L}_{int}$; the unitarity of the S matrix implies the Hermiticity of the interaction lagrangian T_1. Thus the $\mathcal{T}$-invariance condition in the leading order of perturbation theory takes the form

$$\langle\ \mathcal{T}F|\mathcal{L}_{int}|\mathcal{T}I\rangle = \langle\ F|\mathcal{L}_{int}|I\rangle^*. \qquad (81)$$

To put it differently, if the complex-conjugated amplitude of the transition between the states I and F differs **in the leading order of perturbation theory** from the amplitude of the transition between the states $\mathcal{T}I$ and $\mathcal{T}F$ then the dynamics of such system is not invariant under the time reversal.

Let me illustrate this statement by considering the example of the decay $K \rightarrow \mu\nu\gamma$; for definiteness, we consider the reference frame comoving with the kaon. Let the average transverse[6] polarization of the muon $\xi \neq 0$. This is possible only if the probabilities of the decay into the states with positive and negative transverse polarizations of the muon differ from each other:

$$|\langle\mu(\vec{k},\vec{o})\nu(\vec{k}')\gamma(\vec{q},\vec{\epsilon})|S|K\rangle|^2 \neq |\langle\mu(\vec{k},-\vec{o})\nu(\vec{k}')\gamma(\vec{q},\vec{\epsilon})|S|K\rangle|^2. \qquad (82)$$

Note that the state $\langle\mu(\vec{k},-\vec{o})\nu(\vec{k}')\gamma(\vec{q},\vec{\epsilon})|$ can be obtained from the state

$$\langle\mu(-\vec{k},-\vec{o})\nu(-\vec{k}')\gamma(-\vec{q},-\vec{\epsilon})| = \langle \mathcal{T}\ (\mu(\vec{k},\vec{o})\nu(\vec{k}')\gamma(\vec{q},\vec{\epsilon}))|$$

as the result of the rotation by the angle of 180° in the reaction plane. For this reason,

$$\langle\mu(\vec{k},-\xi)\nu(\vec{k}')\gamma(\vec{q},\vec{\epsilon})|S|K\rangle = \langle \mathcal{T}\ \mu(\vec{k},\xi)\nu(\vec{k}')\gamma(\vec{q},\vec{\epsilon})|S|\mathcal{T}\ K\rangle. \qquad (83)$$

The equations (82) and (83) imply the conclusion as follows: if the transverse polarization of the muon emerges in the first order of perturbation theory, then

$$\langle \mathcal{T}\ \mu\nu\gamma|\mathcal{L}_{int}|\mathcal{T}\ K\rangle \neq \langle\mu\nu\gamma|\mathcal{L}_{int}|K\rangle, \qquad (84)$$

that is, the dynamics is not invariant under the time reversal.

However, we should take into account the following reasoning. Since the transverse polarization of the muon is determined by the imaginary part of the decay amplitude, which does not vanish in higher orders of perturbation theory due to unitarity condition, the transverse polarization of the muon emerges in higher orders even in the case of $\mathcal{T}$-even interactions.

Thus the transverse polarization of the muon in the decay $K \rightarrow \mu\nu\gamma$ can be caused by both electromagnetic and $\mathcal{T}$-odd interactions (beyond the SM)

$$\xi = \xi_{\text{EM}} + \xi_{\text{odd}}, \qquad (85)$$

---

[6]That is, transverse with respect to the momenta of the outgoing particles

where $\xi_{\rm EM}$ is the electromagnetic contribution to the transverse polarization of the muon and $\xi_{\rm odd}$ is the contribution of the $\mathcal{T}$-odd (and, therefore $\mathcal{CP}$-odd) interactions.

The $\mathcal{CP}$-violating interactions can be accounted for by the imaginary parts of the coupling constants in the effective quark–lepton Lagrangian

$$\begin{aligned}
\mathcal{L}_{eff} \;=\; & -\frac{G_F}{\sqrt{2}} sin\theta_c \bar{s}\gamma^\alpha(1-\gamma_5)u\bar{\nu}\gamma_\alpha(1-\gamma_5)\mu + \\
& +G_S\bar{s}u\bar{\nu}(1+\gamma_5)\mu + G_P\bar{s}\gamma_5 u\bar{\nu}(1+\gamma_5)\mu + \\
& +G_V\bar{s}\gamma^\alpha u\bar{\nu}\gamma_\alpha(1-\gamma_5)\mu + G_A\bar{s}\gamma^\alpha\gamma_5 u\bar{\nu}\gamma_\alpha(1-\gamma_5)\mu + \\
& +G_T\bar{s}\sigma^{\alpha\beta}(1-\gamma_5)u\bar{\nu}\sigma_{\alpha\beta}(1-\gamma_5)\mu \; + \; {\rm H.c.}
\end{aligned} \tag{86}$$

The interactions in (86) arise from new physics. Nonvanishing imaginary parts in the effective coupling constants G_P, G_V, G_A, G_T gives rise to the imaginary parts of the form factors F_{IB}, F_A, F_V, F_T parametrizing the matrix element of the decay $K^+(p) \to \mu^+(k)\nu(k')\gamma(q)$:

Current limitations on the $\mathcal{T}$-violation parameters in various extensions of the SM allow the transverse polarization of the muon in the decay $K \to \mu\nu\gamma$ to be rather large: the left-right symmetric models based on the symmetry group $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ with one doublet Φ and two triplets $\Delta_{L,R}$ of Higgs bosons can give $\xi_{\rm odd} \sim 7 \times 10^{-3}$, supersymmetric models — $\xi_{\rm odd} \sim 5 \times 10^{-3}$, leptoquark models — $\xi_{\rm odd} \sim 5 \times 10^{-3}$ [34].

The respective $\mathcal{T}$-even contribution to the transverse polarization of the muon is determined by the imaginary part of the decay amplitude (see Fig. 10) and emerges in the second order in α_{em}. Straightforward computations of $\xi_{\rm EM}$ were made by several authors; recent results [35] agree with each other and give the average value $\langle\xi_{\rm EM}\rangle \sim 0.5 \times 10^{-3}$ (with the photon cutoff energy ~ 25 MeV). The previous computations [36] are incomplete: either diagrams in Fig. 10g–10h or the diagrams in Fig. 10a–10h were not taken into account.

A similar $\mathcal{T}$-even contribution to the correlation $\eta = (\vec{\mathbf{p_1}}, [\vec{\mathbf{p_2}} \times \vec{\mathbf{p_3}}])$ in the decay $K^+ \to \pi^0\mu_+\nu\gamma$ is given by similar diagrams and has the same order of magnitude 1.1×10^{-4} [37].

4.1 $\mathcal{CP}$ and $\mathcal{T}$ violation in the decays $K \to 3\pi$

The imaginary part of the effective coupling constants g_8, g_{27} and g_{ew} in the effective weak Lagrangian (29) gives rise to the $\mathcal{CP}$-violating effects in the decays $K \to 3\pi$.

The kinematical variables used to describe the decay $K(p) \to \pi_1(p_1)\pi_2(p_2)\pi_3(p_3)$ are as follows:

$$s_i = (p-p_i)^2, \quad X = \frac{(s_1-s_2)^2}{m_\pi^2}, \quad Y = \frac{(s_3-s_0)}{m_\pi^2},$$

where "3" is the "odd" pion in either the $\tau(\pi^\pm\pi^\mp\pi^\mp)$ or $\tau'(\pi^\pm\pi^0\pi^0)$ decay mode. The slope parameters g and j are defined by the formula for the differential probability of the decay:

$$|A(K \to 3\pi)|^2 \sim 1 + gY + jX + hY^2 + kX^2. \tag{87}$$

The $\mathcal{CP}$-violating quantities are as follows:

$$\delta_\Gamma = \frac{\Gamma(K^+ \to 3\pi) - \Gamma(K^- \to 3\pi)}{\Gamma(K^+ \to 3\pi) + \Gamma(K^- \to 3\pi)} \tag{88}$$

and

$$\delta_g = \frac{g(K^+ \to 3\pi) - g(K^- \to 3\pi)}{g(K^+ \to 3\pi) + g(K^- \to 3\pi)}. \tag{89}$$
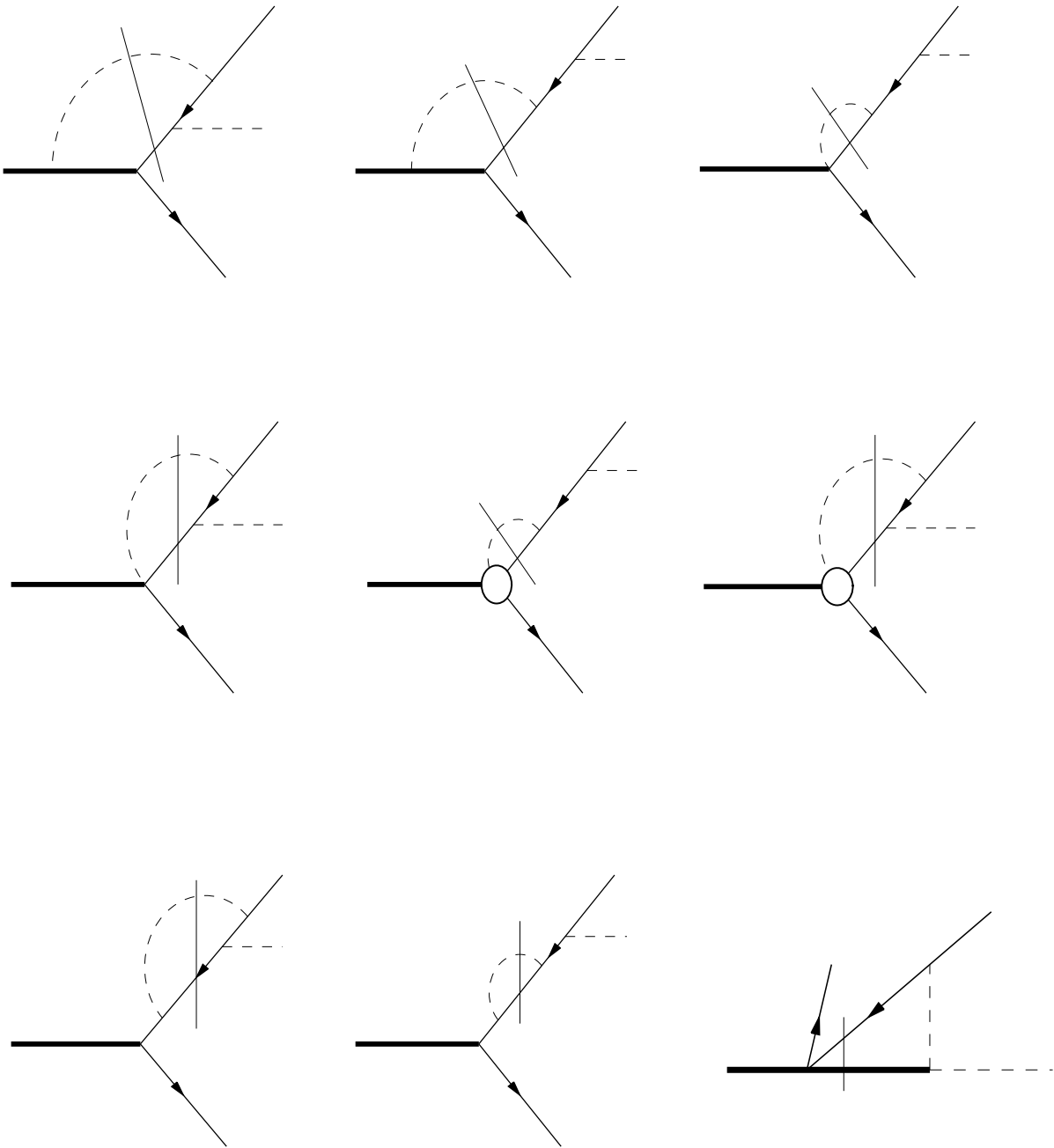
a

With the assumption that $\Delta I \le 1/2$, the relevant $K \to 3\pi$ amplitudes can be expanded as follows:

$$A(K^+ \to \pi^+\pi^+\pi^-) = 2a_c\left(1 + i\alpha^0 + i\frac{\alpha_0'}{2}Y\right) + \left[b_c(1 + i\beta_0) + b_2(1 + i\delta_0)\right]Y, \qquad (90)$$

$$A(K^+ \to \pi^0\pi^0\pi^+) = a_c\left(1 + i\alpha^0 - i\alpha_0'Y\right) - \left[b_c(1 + i\beta_0) + b_2(1 + i\delta_0)\right]Y.$$

From here on we restrict our attention to the slope asymmetry (89) in the τ decay mode. As it usually is, this asymmetry is determined by the interplay of (i) the imaginary parts of the parameters a_c and b_2, stemming from the CP-odd effective weak Lagrangian (29) and (ii) the imaginary part coming about the CP-even final-state interactions [38]:

$$(\delta g)_\tau = \frac{\alpha_0 - \beta_0}{a_c(b_c + b_2)}\left(a_c\mathrm{Im}b_c - b_c\mathrm{Im}a_c\right) + \frac{\alpha_0 - \delta_0}{a_c(b_c + b_2)}\left(a_c\mathrm{Im}b_2 - b_2\mathrm{Im}a_c\right). \qquad (91)$$

An evaluation of the strong rescattering phases in the one-loop approximation of the χPT gives

$$\alpha_0 = \frac{\sqrt{1 - (4m_\pi^2/s_0)}}{32\pi F_\pi^2}(2s_0 + m_\pi^2) \simeq 0.13, \qquad (92)$$

$$\beta_0 = -\delta_0 = \frac{\sqrt{1 - (4m_\pi^2/s_0)}}{32\pi F_\pi^2}(s_0 - m_\pi^2) \simeq 0.05.$$

The values a_c, b_c, b_2, α_0, β_0, and δ_0 can be expanded in powers of the χPT expansion parameter $\lambda \simeq p/(4\pi F)$, where p defines the momentum scale and $F = 93$ MeV. In the case of K-meson decays, $\lambda \simeq 0.4$. In each order of the chiral expansion it is helpful to isolate the $\Delta I = 1/2$ and $\Delta I = 3/2$ contributions to Δg, the latter contribution being suppressed by the factor $\omega = 0.045$. The point is that, in the order $O(p^2)$, the neglect of the $\Delta I = 3/2$ contribution gives rise to the relations $a_c = -\frac{1}{3}\frac{M_K^2}{M_\pi^2}b_c$ and $b_2 = 0$, which, in their turn imply that $\Delta g_{O(p^2),\,\Delta I=1/2} = 0$. However, the $\Delta I = 1/2$ contribution does not vanish in the order $O(p^4)$ and so it dominates the total $O(p^4)$ contribution. It is natural to assume that it is enhanced by the factor $\omega^{-1} = 22.5$ (see 61) as compared to the $\Delta I = 3/2$ contribution in the order $O(p^4)$ of the χPT. The $O(p^4)$ $\Delta I = 3/2$ contribution is suppressed by the ChPT expansion parameter λ^2 as compared to the $O(p^2)$ $\Delta I = 3/2$ contribution A. The above reasoning is summarized in Table 5.

Table 5. Various contributions to $\delta_g = \dfrac{\Delta g}{2g}$.

Order of χPT	$\Delta I = 1/2$	$\Delta I = 3/2$	Numerical estimate
$O(p^2)$	0	A	$(1 \div 3) \times 10^{-6}$
$O(p^4)$	$\sim \dfrac{\lambda^2 A}{\omega}$	$\sim \lambda^2 A$	$(0.4 \div 1) \times 10^{-5}$
$O(p^6)$	$\sim \dfrac{\lambda^4 A}{\omega}$	$\sim \lambda^4 A$	$(0.8 \div 2) \times 10^{-6}$

We see that the total contribution is dominated by the $O(p^4)$ contribution, which is $\omega^{-1}\lambda^2 \simeq 4$ times greater than the $O(p^2)$ contribution—due to vanishing of the $O(p^2)$ $\Delta I = 1/2$ contribution. Maiani and Paver [38] assume that the enhancement factor may run up to $10 \div 20$. Therewith, the conclusion by Bel'kov *et al.* [39] that the $O(p^6)$ corrections increase the enhancement factor by the order of magnitude appear, in view of the above reasoning, highly questionable. It should also be notices that the multi-Higgs models allow a two-fold increase of the parameter δ_g as compared with the SM prediction [40].

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