Symplectic VS Pseudo-Orthogonal Structure of Space-Time

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The advantages to consider the ordinary space-time as the symplectic rather than pseudo-orthogonal one are indicated, and the consequences of extending this view to extra space/time dimensions are discussed.

1. Symplectic vs pseudo-orthogonal space-time

The space-time, or the world we live in is generally adopted to be (locally) the Minkowski one. Its structure group is the pseudo-orthogonal group $SO(1,3)$. To a space-time point there corresponds a real four-vector.

On the other hand, the spinor calculus in our space-time heavily relies on the isomorphy of the noncompact groups $SO(1,3) \simeq SL(2,C)/Z_2$, as well as that $SO(3) \simeq SU(2)/Z_2$ for their maximal compact subgroups. In fact, the whole relativistic field theory in four space-time dimensions can equivalently be formulated in the framework of the complex unimodular group $SL(2,C)$ alone. In a sense, it is even preferable. In this approach, to a space-time point there corresponds a Hermitian spin-tensor of the second-rank.

There is a choice for the ordinary space-time structure group: either the pseudo-orthogonal symmetry $SO(1,3)$, with vectors as defining representation and spinors as a kind of artifact, or the complex symplectic group $Sp(2,C) \simeq SL(2,C)$ with spinors as defining representations and vectors as a secondary object. The two approaches are mathematically equivalent. Nevertheless, the symplectic approach seems physically more appropriate.

Then, in searching for the space-times with extra dimensions it is natural to look for the extensions in the symplectic framework with the structure group $Sp(2l,C)$, $l>1$ instead of $SO(1,d-1)$, $d>4$. The symplectic series of the groups (contrary to $SL(l+1,C)$) is peculiar quantum-mechanically because it retains the invariant bilinear spinor product at any $l>1$.

Two alternative directions of the space-time extension can schematically be pictured as follows:

$$\begin{array}{llcll} d=4 & SO(1,3) & \simeq & Sp(2,C) & l=1 \\ & \downarrow & & \downarrow & \\ d>4 & SO(1,d-1) & \not\simeq & Sp(2l,C) & l>1\,. \end{array}$$

In the pseudo-orthogonal direction of extension, the local metric properties of the space-times, i.e., their dimensionalities and signatures, are to be put in from the very beginning. In the symplectic direction, these properties are not to be considered as the primary ones but, instead, they should emerge as a manifestation of the adopted symplectic structure[1].

2. General symplectic framework

Arbitrary symplectic space-time: l = 1, 2, ... Let ψ_A and $\bar\psi^{\bar A} \equiv (\psi_A)^*$, as well as their respective duals ψ^A and $\bar\psi_{\bar A} \equiv (\psi^A)^*$, with indices A, $\bar A = 1,\ldots,2l$, are the spinor representations of $Sp(2l,C)$. There exist the invariant second-rank spin-tensors $\epsilon_{AB} = -\epsilon_{BA}$ and $\epsilon^{AB} = -\epsilon^{BA}$ such that $\epsilon_{AC}\epsilon^{CB} = \delta_A{}^B$, with $\delta_A{}^B$ being the Kroneker symbol (and similarly for $\epsilon_{\bar A\bar B} \equiv (\epsilon^{BA})^*$ and $\epsilon^{\bar A\bar B} \equiv (\epsilon_{BA})^*$). Owing to these tensors the spinor indices of the upper and lower positions are

[1] For more detail we refer to the literature: Yu.F. Pirogov, IHEP 2001-19 (2001), hep-ph/0104119.

pairwise equivalent ($\psi_A \equiv \epsilon_{AB}\psi^B$ and $\bar{\psi}_{\bar{A}} \equiv \epsilon_{\bar{A}\bar{B}}\bar{\psi}^{\bar{B}}$), so that there are left just two inequivalent spinor representations (generically, ψ and $\bar{\psi}$). These are spinors of the first and the second kind, respectively.

Let us put in correspondence to an event point P a second-rank $2l \times 2l$ spin-tensor $X_A{}^{\bar{B}}(P)$, which is Hermitian, i.e., fulfil the restriction

$$X_A{}^{\bar{B}} = (X_B{}^{\bar{A}})^* \equiv \bar{X}^{\bar{B}}{}_A \,,$$

or in other terms

$$X^{A\bar{B}} = (X_{B\bar{A}})^* \equiv \bar{X}^{\bar{B}A}.$$

The quadratic scalar product is defined as follows:

$$(X,X) \equiv \mathrm{tr}\, X\bar{X} = X_A{}^{\bar{B}}\bar{X}_{\bar{B}}{}^A = -X_{A\bar{B}}(X_{B\bar{A}})^* \,.$$

Here (X,X) is real though not sign definite. Under arbitrary transformation $S \in Sp(2l,C)$ one has in short notations:

$$\begin{array}{lcl} X & \to & SXS^\dagger \,, \\ \bar{X} & \to & S^{\dagger -1}\bar{X}S^{-1} \end{array}$$

and hence (X,X) is invariant. At $l > 1$, the quadratic invariant above is just the lowest order one in a series of independent invariants $\mathrm{tr}\,(X\bar{X})^k$, $k = 1,\ldots,l$. The highest order one with $k = l$ is equivalent to $\det X$.

Definition: the Hermitian spin-tensor set $\{X\}$ equipped with the structure group $Sp(2l,C)$ and the interval between points X_1 and X_2 equal to $(X_1 - X_2, X_1 - X_2)$ constitutes the flat symplectic space-time.

The noncompact transformations out of $Sp(2l,C)$ are counterparts of the Lorentz boosts in the ordinary space-time $l = 1$, while transformations out of the compact subgroup $Sp(2l) = Sp(2l,C) \cap SU(2l)$ correspond to rotations. With account for translations $X_A{}^{\bar{B}} \to X_A{}^{\bar{B}} + \Xi_A{}^{\bar{B}}$, where $\Xi_A{}^{\bar{B}}$ is an arbitrary constant Hermitian spin-tensor, the whole theory in the flat symplectic space-time should be covariant relative to the inhomogeneous symplectic group $ISp(2l,C)$.

Restricting by the maximal compact subgroup $Sp(2l)$, the indices of the first and the second kinds in the same position are indistinguishable relative to their transformation properties. Hence, under $Sp(2l)$ one can reduce the event tensor $X_{A\bar{B}}$ into two irreducible parts, symmetric and antisymmetric ones: $X = X_+ + X_-$, where $X_\pm = \pm(X_\pm)^T$ have dimensionalities $d_\pm = l(2l \pm 1)$, respectively. The scalar product decomposes as follows:

$$(X,X) = \sum_\pm (\mp 1)(X_\pm)_{A\bar{B}}[(X_\pm)_{A\bar{B}}]^* \,.$$

Thus one of two pieces $X_\pm$ is the spatial part of coordinate while the rest is the time part.

At $l > 1$, one can further reduce the antisymmetric part X_- of the event spin-tensor into the trace relative to ϵ and a traceless part: $X_- = X_-^{(0)} + X_-^{(1)}$. The trace $X_-^{(0)}$ is rotationally invariant and hence represents the true time. In short:

$$\text{1-time} = \text{trace}\,.$$

The traceless part $X_-^{(1)}$ is uniquely associated with extra times.

Relative to the rotational subgroup, the whole extended space-time can be decomposed into three irreducible subspaces of the dimensionalities 1, $(l-1)(2l+1)$ and $l(2l+1)$, respectively.

The first two subspaces correspond to time directions, while the third subspace corresponds to the spatial ones.

Though any particular decomposition into $X_\pm$ is noncovariant and depends on the boosts, the number of the positive and negative components in (X, X) is invariant under the whole $Sp(2l, C)$. In other words, there emerges the invariant metric tensor of the d-dimensional flat space-time:

$$\eta_d = (\underbrace{+1, \dots}_{d-}; \underbrace{-1, \dots}_{d_+}) \,.$$

Thus, at $l > 1$ the structure group $Sp(2l, C)$, acting on the Hermitian second-rank spin-tensors with $d = 4l^2$ components, is a subgroup of the embedding pseudo-orthogonal group $SO(d_-, d_+)$, acting on the pseudo-Euclidean space of the same dimensionality. What distinguishes $Sp(2l, C)$ from $SO(d_-, d_+)$, is the total set of independent invariants $\mathrm{tr}(X\bar{X})^k$, $k = 1, \dots, l$. The isomorphy between the groups is valid only at $l = 1$, i.e., for the ordinary space-time $d = 4$ where there is just one invariant.

In the symplectic approach, neither the discrete set of dimensionalities, $d = 4l^2$, of the extended space-time, nor its signature, nor the existence of the rotationally invariant one-dimensional time subspace are postulated from the beginning. These properties are the attributes of the very symplectic structure.

In particular, the symplectic structure provides the simple rationale for the four-dimensionality of the ordinary space-time, as well as for its signature $(+---)$. Namely, these properties just reflect the existence of one antisymmetric and three symmetric 2×2 Hermitian spin-tensors. In short:

$$(2 \times 2)_{\mathrm{H}} = 1_{\mathrm{A}} \oplus 3_{\mathrm{S}} \,.$$

The set of the second-rank Hermitian tensors, in its turn, is the lowest admissible real space to accommodate the symplectic structure.

On the other hand, right the one-dimensionality of time allows one to put the events in order at any fixed boosts and hence to insure the causal description. Therefore, the causality might ultimately be attributed to the underlying symplectic structure, too. At $l > 1$, the one-dimensional time and the extra times mix with each other via boosts. Because of this influence of extra times, the causality is expected not to fulfil for relativistic events.

Gauge interactions Let $D_A{}^{\bar{B}} \equiv \partial_A{}^{\bar{B}} + igG_A{}^{\bar{B}}$ be the generic covariant derivative, where $\partial_A{}^{\bar{B}} \equiv \partial/\partial X^A{}_{\bar{B}}$ is the ordinary derivative, g is the gauge coupling and the Hermitian spin-tensor $G_A{}^{\bar{B}}$ is the gauge fields. One can introduce the gauge invariant strength tensor

$$F^{[\bar{B}_1 \bar{B}_2]}_{\{A_1 A_2\}} = \partial^{[\bar{B}_1}_{\{A_1} G^{\bar{B}_2]}_{A_2\}} + igG^{[\bar{B}_1}_{\{A_1} G^{\bar{B}_2]}_{A_2\}} \,.$$

The total number of the real components in tensor F precisely coincides with the number of components of the antisymmetric second-rank tensor $F_{[\mu\nu]}$, $\mu, \nu = 0, 1, \dots, 4l^2 - 1$ in the embedding pseudo-Euclidean space of the dimensions $d = 4l^2$. But in the symplectic case, tensor F is reducible and splits into a trace relative to ϵ and a traceless part, $F = F^{(0)} + F^{(1)}$.

For an unbroken gauge theory with fermions, the generic gauge, fermion and mass terms of the Lagrangian $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_M$ are, respectively,

$$\begin{aligned}
\mathcal{L}_G &= \sum_{s=0,1} (c_s + i\theta_s)\, F^{(s)} F^{(s)} + \mathrm{h.c.}\,, \\
\mathcal{L}_F &= \frac{i}{2} \sum_{\pm} (\psi^\pm)^\dagger \overleftrightarrow{D}\, \psi^\pm \,, \\
\mathcal{L}_M &= \psi^+ m_0\, \psi^- + \sum_{\pm} \psi^\pm m_\pm \psi^\pm + \mathrm{h.c.}\,,
\end{aligned}$$

where $\psi^\pm$ are pairs of the charged conjugate fermions.

The Lagrangian results in the following generalization of the Dirac equation

$$iD^C{}_{\bar{B}}\psi^{\pm}_C = m_0^\dagger \overline{\psi}^{\pm}_{\bar{B}} + \sum_{\pm} m^\dagger_{\pm}\, \overline{\psi}^{\mp}_{\bar{B}}$$

and the pair of Maxwell equations ($c_0 \equiv 1$ and $c_1 = \theta_1 = 0$ for simplicity)

$$\begin{aligned}(1 + i\theta_0)D^{C\bar{B}}F^{(0)}{}_{\{CA\}} - \text{h.c.} &= 0\,, \\ (1 + i\theta_0)D^{C\bar{B}}F^{(0)}{}_{\{CA\}} + \text{h.c.} &= 2gJ_A{}^{\bar{B}}\,,\end{aligned}$$

with $J_A{}^{\bar{B}} \equiv \sum_{\pm}(\pm 1)\psi^{\pm}_A(\psi^{\pm}_B)^\dagger$ being the fermion Hermitian current.

Tensors $F^{(s)}$, $s = 1, 2$ are non-Hermitian, but under restriction by the maximal compact subgroup $Sp(2l)$ they split into a pair of the Hermitian ones: $F^{(s)} = E^{(s)} + iB^{(s)}$. There is the duality transformation $F^{(s)} \to \tilde{F}^{(s)} \equiv iF^{(s)}$, so that $\tilde{E}^{(s)} = -B^{(s)}$ and $\tilde{B}^{(s)} = E^{(s)}$. Though the splitting into $E^{(s)}$ and $B^{(s)}$ is noncovariant with respect to the whole $Sp(2l, C)$, the duality transformation is covariant.

Tensors $E^{(s)}$ and $B^{(s)}$ are the counterparts of the ordinary electric and magnetic strengths, and θ_0, θ_1 are the T-violating θ-parameters. Due to electric-magnetic duality, the electric and magnetic strengths stay in the framework of symplectic extension on equal footing. This is to be contrasted with the pseudo-orthogonal extension where these strengths have unequal number of components at $d \neq 4$.

Gravity The considerations above refer to the flat extended space-time or, otherwise, to the inertial local frames. To go beyond, one can introduce the Hermitian local fielbeins $e_{MA}{}^{\bar{B}}(X)$, with $M = 0, 1, \ldots, 4l^2 - 1$ being the world vector index, and the real world coordinates $x_M \equiv \mathrm{tr}X\bar{e}_M$. Now, one can equip space-time with the pseudo-Riemannian structure, i.e., the real symmetric metrics $g_{MN}(x) = \mathrm{tr}\, e_M\bar{e}_N$. Introducing the generally covariant derivative $\nabla_M(e)$ one can adapt the field theory to the $d = 4l^2$ dimensional curved space-time.

One can also supplement gauge equations by the generalized Hilbert-Einstein gravity equations. But now the group of equivalence of the local fielbeins (structure group) is just the symplectic group $Sp(2l, C)$ rather than the whole pseudo-orthogonal group $SO(d_-, d_+)$. This permits more independent components in the local symplectic fielbeins compared to the metrics. The curvature tensor in the symplectic case splits additionally into irreducible parts which can enter the gravity Lagrangian with the independent coefficients. Hence, the symplectic gravity is in general not equivalent to the metric one.

The reason for this may be as follows. In the symplectic approach, the vectors and space-time in its present meaning are to be understood not as the fundamental entities. Therefore, gravity treated as a generally covariant theory of the space-time distortions have to be just an effective theory. The latter admits a lot of free parameters, the choice of which should eventually be clarified by an underlying theory.

3. Next-to-ordinary symplectic space-time: l = 2

Coordinate space kinematics In this case there takes place the isomorphy $SO(5, C) \simeq Sp(4, C)/Z_2$. Cases $l = 1, 2$ are the only ones when the structure of the symplectic group gets simplified in terms of the complex orthogonal groups.

One can introduce the set of Clifford matrices $(\Sigma_I)_A{}^{\bar{B}}$, $I = 1, \ldots, 5$. Relative to the maximal compact subgroup $SO(5)$, they can be chosen both Hermitian $(\Sigma_I)_A{}^{\bar{B}} = [(\Sigma_I)_{\bar{B}}{}^A]^*$ and antisymmetric $(\Sigma_I)_{A\bar{B}} = -(\Sigma_I)_{\bar{B}A}$, similar to $(\Sigma_0)_{AB} = \epsilon_{AB}$. One can require that Σ_I are traceless. Thus under

$SO(5)$, six matrices Σ_0, Σ_I provide the complete independent set for the antisymmetric matrices in the four-dimensional spinor space.

After introducing matrices $\Sigma_{IJ} = -i/2(\Sigma_I\overline{\Sigma}_J - \Sigma_J\overline{\Sigma}_I)$, with $\bar{\Sigma} \equiv -\epsilon\Sigma^*\epsilon$, one gets the (anti)symmetry conditions for them: $\Sigma_{IJ} = -\Sigma_{JI}$ and $(\Sigma_{IJ})_{AB} = (\Sigma_{IJ})_{BA}$. Therefore, ten matrices Σ_{IJ} make up the complete set for the symmetric matrices in the spinor space. The matrices $(\Sigma_{IJ}, i\Sigma_{IJ})$ represent the $SO(5,C)$ generators $M_{IJ} = (L_{IJ}, K_{IJ})$ in the space of the first kind spinors.

With respect to maximal compact subgroup $SO(5)$, the Hermitian second-rank spin-tensor X may be decomposed in the complete set of matrices Σ_0, Σ_I and Σ_{IJ} with the real coefficients:

$$X = \frac{1}{2}\left(x_0\Sigma_0 + x_I\Sigma_I + \frac{1}{2}\,x_{IJ}\Sigma_{IJ}\right),$$

and thus $(X,X) = x_0^2 + x_I^2 - \frac{1}{2}x_{IJ}^2$. There is one more independent invariant combination of x_0, x_I and x_{IJ} originating from the invariant $\mathrm{tr}(X\bar{X})^2$ which is equivalent to $\det X$.

Relative to the embedding $SO(5,C) \supset SO(5)$ one has the following decomposition in the irreducible representations:

$$\underline{16} = \underline{1} \oplus \underline{5} \oplus \underline{10}\,.$$

Under the discrete transformations one can get:

$$\begin{aligned} P &: \; x_0 \to x_0,\; x_I \to x_I,\; x_{IJ} \to -x_{IJ}\,, \\ T &: \; x_0 \to -x_0,\; x_I \to -x_I,\; x_{IJ} \to x_{IJ}\,. \end{aligned}$$

From the point of view of the rotational subgroup $SO(5)$, extra times x_I constitutes the axial vector, whereas the spatial coordinates x_{IJ} constitutes the pseudo-tensor.

The rank of the algebra of $Sp(4,C)$ is $l = 2$. Hence an arbitrary irreducible representation of the noncompact group $Sp(4,C)$ is uniquely characterized by two complex Casimir operators I_2 and I_4 of the second and the forth order, respectively, i.e. by four real quantum numbers.

Otherwise, an irreducible representation of $Sp(4,C)$ can be described by the mixed spin-tensor $\Psi^{\bar{B}_1\cdots}_{A_1\cdots}$ of a proper rank. This spin-tensor should be traceless in any pair of the indices of the same kind, and its symmetry in each kind of indices should correspond to a two-row Young scheme. Therefore, an irreducible representation of $Sp(4,C)$ may unambiguously be characterized by a set of four integers $(r_1, r_2; \bar{r}_1, \bar{r}_2)$, $r_1 \geq r_2 \geq 0$ and $\bar{r}_1 \geq \bar{r}_2 \geq 0$.

The rank of the maximal compact subgroup $SO(5) \simeq Sp(4)/Z_2$ is equal to $l = 2$. Hence a state in a representation is additionally characterized by two additive quantum numbers, namely, the eigenvalues of the mutually commuting momentum components of L_{IJ} in two different planes, say, L_{12} and L_{45}.

Momentum space kinematics The spin-tensor of the particle momentum looks like:

$$\Pi = \frac{1}{2}\left(p_0\Sigma_0 + p_I\Sigma_I + \frac{1}{2}\,P_{IJ}\Sigma_{IJ}\right).$$

There are two independent invariants built of Π:

$$\begin{aligned} C_1 &\equiv \mathrm{tr}\,\Pi\bar{\Pi} = p_0^2 + {p_I}^2 - \frac{1}{2}\,{P_{IJ}}^2\,, \\ C_2 &\equiv \mathrm{tr}\,(\Pi\bar{\Pi})^2 = \frac{1}{4}(p_0^2 + {p_I}^2)^2 + p_0^2\,{p_I}^2 + \mathcal{O}(P^2)\,, \quad P \to 0\,. \end{aligned}$$

The solution to the above constraints determines the dispersion law for a massive particle:

$$\begin{aligned} p_0^2 &= m^2 + K\,, \\ {p_I}^2 &= C_1 - m^2 - K + \frac{1}{2}\,{P_{IJ}}^2\,. \end{aligned}$$

Here

$$m^2 = \frac{1}{2}\Big[C_1 \pm \Big(2C_1^2 - C_2\Big)^{1/2}\Big]$$

is the rest energy squared, $K(m^2, C_1, P, \hat{p})$ is the kinetic term, $\hat{p}$ is the unit orientation vector of the time-momentum. One has $K = \mathcal{O}(P^2)$ at $P \to 0$. Reality of m and K proves to require $0 \leq m^2 \leq C_1$ which results in the subsequent restriction (cf. Fig. 1)

$$C_1^2 \leq C_2 < 2C_1^2\,.$$

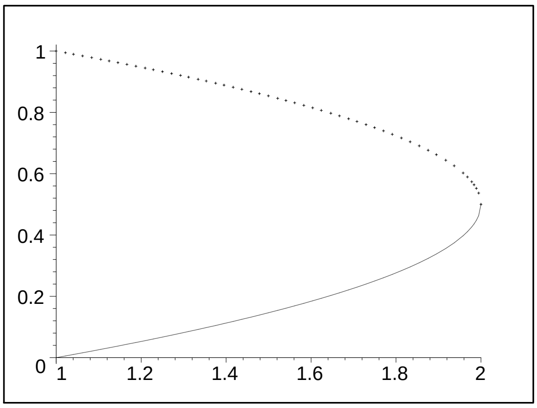

Figure 1: Two-valued plot of the normalized mass squared m^2/C_1 vs normalized second invariant C_2/C_1^2.

Reduction: $\mathbf{l \to 1}$ The ultimate unit of dimensionality in the symplectic approach is the discrete number $l = 1, 2, \ldots$ corresponding to the dimensionality $2l$ of the spinor space. The dimensionality $d = 4l^2$ of the space-time emerges as a secondary quantity. The extended space-time with $l > 1$ should eventually compactify by means of the symplectic gravity via discrete (quantum) transitions:

$$\begin{gathered} \ldots \\ \downarrow \\ l = 2 \\ \downarrow \\ l = 1 \\ \downarrow \\ l = 0\,. \end{gathered}$$

The case $l = 0$ corresponds to the (hypothetical) annihilation of the world into “nothing”.

For $l = 2$, depending on the spinor decomposition relative to the embedding $Sp(4, C) \supset Sp(2, C)$, three generic inequivalent types of the space-time decomposition are conceivable:

$$\begin{aligned} \underline{16} &= 4 \cdot \underline{4}\,, \\ \underline{16} &= 2 \cdot \underline{4} \oplus (\underline{3} + \text{h.c.}) \oplus 2 \cdot \underline{1}\,, \\ \underline{16} &= \underline{4} \oplus (2 \cdot \underline{2} + \text{h.c.}) \oplus 4 \cdot \underline{1}\,. \end{aligned}$$

Note that due to compactification, the last case can result in the violation of the spin-statistics connection in four space-time dimensions.

4. Summary

To summarize: the hypothesis that the symplectic structure of space-time is superior to the metric one provides, in particular, the rationale for the four-dimensionality and the $1+3$ signature of the ordinary space-time.

When looking for the space-times with extra dimensions, the hypothesis predicts the one-parametric discrete series of the metric space-times of the peculiar dimensionalities and signatures, with the spatial and time extra dimensions in a definite proportion. Because one of the time directions remains rotationally invariant under fixed boosts, there emerges the (non-relativistic) causality despite the presence of extra times.

The symplectic approach provides an unorthodox alternative to the pseudo-orthogonal space-times and inspires a lot of new opportunities for the physics of extra dimensions. But beyond the physical adequacy of the extended space-times as such, by generalizing from the basic symplectic case $l = 1$ to its counterpart for general $l > 1$, a deeper insight into the nature of the very four-dimensional space-time may be attained.