On the Generalization of the Fundamental Field Equations for Locally Anisotropic Space-Time *

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It is shown that the Lorentz symmetry is not the only possible realization of the relativistic symmetry. Apart from conventional Lorentz transformations, the relativistic symmetry can be also represented by the so-called generalized Lorentz transformations which in such a case play a role of the relativistic symmetry transformations of a flat locally anisotropic event space. By upholding Einstein's special principle of relativity this result reduces the widely debated problem of Lorentz symmetry violation to the problem of existence of local anisotropy of space-time. Proceeding from the assumption that space-time does indeed possess local anisotropy, the equations of the relativistic point mechanics for the locally anisotropic space-time have been generalized and are reproduced here. Next we formulate some initial guiding principles for the corresponding generalization of relativistic field theory. The efficiency of these principles is tested in a generalization of the Dirac equations and of the Lagrangians of Klein-Gordon type.

1 Introduction

At present there exist some empirical indications in favour of the view that in nature Lorentz symmetry may be broken. When pointing to experimental/observational evidences of such a violation of Lorentz symmetry, we first of all have in mind the following phenomena:
(i) a breaking of the discrete space-time symmetries in weak interactions;
(ii) an anisotropy of the relic background radiation filling the Universe;
and in particular
(iii) the absence of the GZK cutoff.
The GZK cutoff of the spectrum of primary ultra-high energy cosmic protons due to inelastic collisions of the protons with cosmic background radiation photons (photo-production of pions), was predicted [1, 2] on the basis of conventional relativistic theory. Today, the available data show that this prediction contradicts the observed behaviour of the spectrum: a cutoff does not exist at proton energies $\sim 5 \times 10^{19} eV$ (see Fig. 1).

However, still is it unclear how such ultra-high energy events could be explained. Both astrophysical scenarios, the acceleration of lower energy particles in special astrophysical environements, or the decay of hypothetical supermassive particles do run into difficulties [3], [4]. The possible violation of Lorentz invariance has also been adressed through modified kinematical constraints with more or less ad-hoc changes in the dispersion relation [5–8]. Here, we proceed from the idea that

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Lorentz symmetry serve only as an approximation to a stricter relativistic symmetry based on a group of the generalized Lorentz transformations.

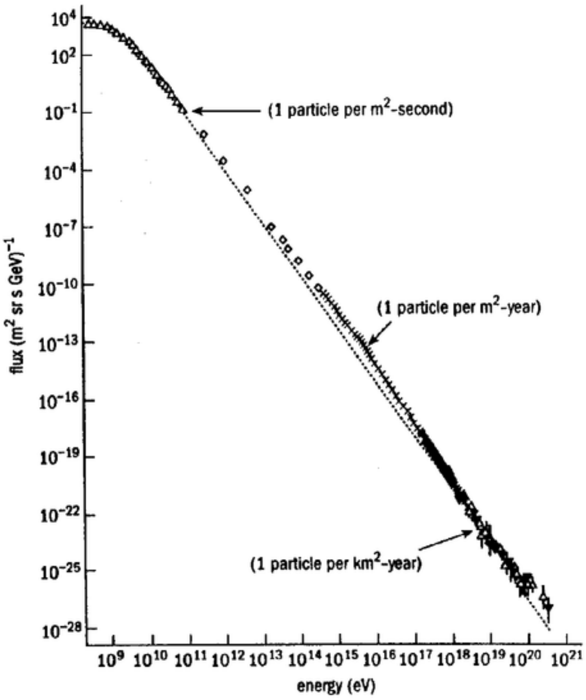

Fig. 1. The energy spectrum of cosmic rays. For more details see [3].

Thus, the main idea which will hopefully allow us to remove the above-mentioned discrepancy between theory and experiment consists in the following: a correct calculation of the GZK cutoff must be performed on the basis of relativistic field theory modified in accordance with the requirement of invariance of the fundamental field equations with regard to the group of generalized Lorentz transformations. Since the generalized Lorentz symmetry with necessity leads to local anisotropy of space-time, we are actually in need of a relativistic field theory generalized for locally anisotropic space-time.

2 Generalized Lorentz symmetry as the relativistic symmetry of a flat anisotropic event space

Let us proceed from the assumption [5] that at relative velocities of inertial frames extremely close to the velocity of light the conventional Lorentz transformations do indeed differ markedly from some new relativistic symmetry transformations. In order to find such novel or, as they have been named in [9], the generalized Lorentz transformations we first confine ourselves to a 2D space for more clarity and write down the conventional ones:

$$\left\{ \begin{array}{lll} x_0' & = & x_0 \cosh\alpha - x\sinh\alpha \\ x' & =- & x_0\sinh\alpha + x\cosh\alpha\,; \qquad \tanh\alpha = v/c\,. \end{array} \right. \tag{1}$$

Now, attempting to remain within the framework of Einstein's special relativity principle, we shall try to generalize (1) in such a way that the new linear transformations would also form a group with the group parameter α and would keep invariant the interval $ds^2 = 0 = dx_0^2 - dx^2$ for the events relating to propagation of massless particles. It is easy to see that apart from (1) all these requirements are also satisfied by the transformations

$$\begin{cases} x_0' = e^{-r\alpha} \left(x_0 \cosh\alpha - x \sinh\alpha \right) \\ x' = e^{-r\alpha} \left(- x_0 \sinh\alpha + x \cosh\alpha \right), \end{cases} \tag{2}$$

where r is a dimensionless parameter of the additional scale transformation. Since according to (2) the relation of the group parameter α to the velocity v of the primed frame remains unchanged, i.e. $\tanh\alpha = v/c$, (2) can be rewritten as follows

$$\begin{cases} x_0' = \left(\frac{1-v/c}{1+v/c}\right)^{r/2} \frac{x_0-(v/c)x}{\sqrt{1-v^2/c^2}} \\ x' = \left(\frac{1-v/c}{1+v/c}\right)^{r/2} \frac{x-(v/c)x_0}{\sqrt{1-v^2/c^2}}. \end{cases} \tag{3}$$

These are just the 2D generalized Lorentz transformations we were aspiring to reproduce here.

Obviously, in contrast to (1), the generalized Lorentz transformations (2) or (3) do not leave invariant the pseudo-Euclidean metric $ds^2 = dx_0^2 - dx^2$ but conformally modify it. Therefore, the question arises as to what the metric of an event space invariant under such generalized Lorentz transformations is. The rigorous solution to this problem is

$$ds^2 = \left[\frac{(dx_0 - dx)^2}{dx_0^2 - dx^2}\right]^r (dx_0^2 - dx^2). \tag{4}$$

Not being a quadratic form but a homogeneous function of the coordinate differentials of degree two, the metric (4) falls into the category of Finslerian metrics [10]. It describes a flat but anisotropic event space. As long as we deal with 2D anisotropic space, its anisotropy manifests itself in the noninvariance of the metric (4) under the reflections $x_0 \to -x_0$ or $x \to -x$. If $r = 0$, then the anisotropy disappears. In this case, the event space becomes isotropic while the generalized Lorentz transformations (3) reduce to the usual Lorentz transformations. However, if $r \neq 0$ characterizing the magnitude of space anisotropy is sufficiently small, then the additional dilatation of space-time, which distinguishes the generalized Lorentz transformations from the usual ones, becomes markedly different from unity only at relative velocities of the inertial frames extremely close to the velocity of light. In the physics of ultra-high energy cosmic rays we deal with precisely such a situation. Therefore, the use of the generalized Lorentz transformations instead of the usual ones makes it possible, in principle, to remove the discrepancy between theory and experiment in this field; this may be regarded as a hint towards a local anisotropy of space.

Certainly, the 2D metric (4) is of methodical interest only and must be generalized to the 4D case. The corresponding 4D metric can be obtained from (4) by means of the substitution:

$$(dx_0^2 - dx^2) \to (dx_0^2 - d\boldsymbol{x}^2); \quad (dx_0 - dx) \to (dx_0 - \boldsymbol{\nu} d\boldsymbol{x}),$$

where $\boldsymbol{\nu}^2 = 1$. As a result we arrive at

$$ds^2 = \left[\frac{(dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2}{dx_0^2 - d\boldsymbol{x}^2}\right]^r (dx_0^2 - d\boldsymbol{x}^2). \tag{5}$$

This Finslerian metric depends on two constant parameters r and $\boldsymbol{\nu}$ and describes a flat anisotropic space-time with partially broken rotational symmetry[1]. Instead of the 3-parameter group of rotations of the pseudo-Euclidean space, the 4D space-time (5) admits only the 1-parameter group of

[1] A flat relativistically invariant 4D space-time with entirely broken 3D isotropy has been considered in [11].

rotations around the unit vector $\boldsymbol{\nu}$, which indicates a preferred direction in 3D space. No changes occur for translational symmetry: space-time translations leave the metric (5) invariant. As regards the transformations linking the various inertial frames, the usual Lorentz boosts modify the metric (5). Therefore, they do not belong to the isometry group of the space-time (5). By proper use of them, however, invariance transformations for the metric (5) can be constructed. The corresponding 4D generalized Lorentz transformations will be the following

$$x'^i = D(\boldsymbol{v},\boldsymbol{\nu})\, R^i_j(\boldsymbol{v},\boldsymbol{\nu})\, L^j_k(\boldsymbol{v})\, x^k\,, \tag{6}$$

where $\boldsymbol{v}$ is the velocity of the moving reference frame, $L^j_k(\boldsymbol{v})$ is the ordinary Lorentz boost, $R^i_j(\boldsymbol{v},\boldsymbol{\nu})$ is the further rotation of the spatial axes of the moving frame around the vector $[\boldsymbol{v}\boldsymbol{\nu}]$ through an angle

$$\varphi = \arccos\left\{1 - \frac{(1-\sqrt{1-\boldsymbol{v}^2/c^2})[\boldsymbol{v}\boldsymbol{\nu}]^2}{(1-\boldsymbol{v}\boldsymbol{\nu}/c)\boldsymbol{v}^2}\right\} \tag{7}$$

of relativistic aberration of $\boldsymbol{\nu}$, and $D(\boldsymbol{v},\boldsymbol{\nu})$ is a dilatational transformation of the space-time:

$$D(\boldsymbol{v},\boldsymbol{\nu}) = \left(\frac{1-\boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1-\boldsymbol{v}^2/c^2}}\right)^r I\,. \tag{8}$$

In the latter formula I is the unit matrix.

In contrast to Lorentz boosts, the generalized transformations (6) make up a 3-parameter noncompact group with generators $X_1\,, X_2\,, X_3$. Thus, with the inclusion of the 1-parameter group of rotations around the preferred direction $\boldsymbol{\nu}$ and of the 4-parameter group of translations, the inhomogeneous group of isometries of the space (5) turns out to have 8-parameters. In order to obtain the simplest representation for its generators, it is sufficient to choose the third space axis along $\boldsymbol{\nu}$ and then to make use of the infinitesimal form of the transformations (6). As a result,

$$\begin{aligned}
X_1 &= -(x^1p_0 + x^0p_1) - (x^1p_3 - x^3p_1)\,,\\
X_2 &= -(x^2p_0 + x^0p_2) + (x^3p_2 - x^2p_3)\,,\\
X_3 &= -rx^ip_i - (x^3p_0 + x^0p_3)\,,\\
R_3 &= x^2p_1 - x^1p_2\,; \qquad\qquad p_i = \partial/\partial x^i\,.
\end{aligned} \tag{9}$$

The generators (9) satisfy the commutation relations

$$\begin{array}{llll}
[X_1X_2] = 0\,, & [R_3X_3] = 0\,, & & \\
[X_3X_1] = X_1\,, & [R_3X_1] = X_2\,, & & \\
[X_3X_2] = X_2\,, & [R_3X_2] = -X_1\,; & & \\
[p_ip_j] = 0\,; & & & \\
[X_1p_0] = p_1\,, & [X_2p_0] = p_2\,, & [X_3p_0] = rp_0 + p_3\,, & [R_3p_0] = 0\,,\\
[X_1p_1] = p_0 + p_3\,, & [X_2p_1] = 0\,, & [X_3p_1] = rp_1\,, & [R_3p_1] = p_2\,,\\
[X_1p_2] = 0\,, & [X_2p_2] = p_0 + p_3\,, & [X_3p_2] = rp_2\,, & [R_3p_2] = -p_1\,,\\
[X_1p_3] = -p_1\,, & [X_2p_3] = -p_2\,, & [X_3p_3] = rp_3 + p_0\,, & [R_3p_3] = 0\,.
\end{array} \tag{10}$$

From (10), we conclude in particular that the homogeneous isometry group of the space (5) contains 4 parameters (the generators X_1, X_2, X_3, R_3). Being a subgroup of the conformal group, it is isomorphic to the corresponding 4-parameter subgroup (with the generators X_1, X_2, $X_3|_{r=0}$, R_3) of the homogeneous Lorentz group. Since the 6-parameter homogeneous Lorentz group has no 5-parameter subgroup [12] while the 4-parameter subgroup is unique (up to isomorphisms), the transition from pseudo-Euclidean space to the event space (5) implies a minimum of symmetry-breaking of the Lorentz symmetry, in which case the relativistic symmetry represented now by the generalized Lorentz transformations remains preserved[2].

At last note that due to nonunimodularity of the matrices $\mathcal{L}_k^i = DR_j^i L_k^j$ representing the generalized Lorentz transformations (6), the transformational properties of some geometric entities turn out to be changed as compared with conventional special relativity theory. For instance, a 4-volume element $dx^0 d^3x$ is no longer invariant but is a scalar density of weight -1, i.e. it transforms as follows: $dx'^0 d^3x' = J^{-1} dx^0 d^3x$, where J is the Jacobian, $J = |\partial x^k/\partial x'^j| = |\mathcal{L}^{-1}{}_j^k| = D^{-4}$. Similarly, matrices η_{ik} and η^{ik} having the identical forms $\eta_{ik} = diag\,(1,-1,-1,-1)$ and $\eta^{ik} = diag\,(1,-1,-1,-1)$ in all frames of reference related by the transformations (6) are no longer invariant tensors but are, respectively, a covariant tensor density of weight $-1/2$ and a contravariant tensor density of weight $1/2$. This statement signifies that $\eta'_{ik} = J^{-1/2}\mathcal{L}^{-1}{}_i^l \mathcal{L}^{-1}{}_k^m \eta_{lm} = \eta_{ik}$ and $\eta'^{ik} = J^{1/2}\mathcal{L}_l^i \mathcal{L}_m^k \eta^{lm} = \eta^{ik}$. Then it is clear that $\eta_{ik}\eta^{kl} = \delta_i^l$ is a unit tensor. Later on we shall be using η_{ik} and η^{ik} to lower and raise indices. The process, however, will be accompanied by a change in weight. We shall be also in need of an entity ν^i which indicates a preferred direction in 4D space-time and whose components have the same values, $\left\{\nu^0 = 1\,,\, \boldsymbol{\nu}\right\}$, in all frames of reference related by the transformations (6). It is easy to verify that $\nu'^i = J^{(1+r)/(4r)}\mathcal{L}_k^i \nu^k = \nu^i$, i.e., that ν^i is a contravariant vector density of weight $(1+r)/(4r)$.

Our next task consists in the following: we must try to modify the standard equations of relativistic mechanics so that they become invariant under the group of generalized Lorentz transformations.

3 Generalization of the relativistic point mechanics. An influence of local space anisotropy on the inertia of particles

In order to generalize conventional relativistic point mechanics in accordance with the requirement of invariance of the corresponding equations under the group of generalized Lorentz transformations it is sufficient in the action integral

$$S = -mc\int_a^b ds \tag{11}$$

to replace the pseudo-Euclidean expression for ds by the Finslerian expression (5)[3]. As a result, the Lagrangian function corresponding to a free particle in the locally anisotropic space, takes the form

$$L = -mc^2\left(\frac{1-\boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1-\boldsymbol{v}^2/c^2}}\right)^r \sqrt{1-\boldsymbol{v}^2/c^2}\,. \tag{12}$$

This Lagrangian function leads to the following expressions for the momentum $\boldsymbol{p} = \partial L/\partial \boldsymbol{v}$ and the energy $E = \boldsymbol{p}\boldsymbol{v} - L$ of a relativistic particle

$$E = \frac{mc^2}{\sqrt{1-\boldsymbol{v}^2/c^2}}\left(\frac{1-\boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1-\boldsymbol{v}^2/c^2}}\right)^r \left[1-r+r\,\frac{1-\boldsymbol{v}^2/c^2}{1-\boldsymbol{v}\boldsymbol{\nu}/c}\right], \tag{13}$$

$$\boldsymbol{p} = \frac{mc}{\sqrt{1-\boldsymbol{v}^2/c^2}}\left(\frac{1-\boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1-\boldsymbol{v}^2/c^2}}\right)^r \left[(1-r)\boldsymbol{v}/c + r\boldsymbol{\nu}\,\frac{1-\boldsymbol{v}^2/c^2}{1-\boldsymbol{v}\boldsymbol{\nu}/c}\right]. \tag{14}$$

[2]Some examples of Finslerian spaces with a partial and entirely broken relativistic symmetry are considered in [13].

[3]In [14] it has been proved that the action S, generalized in such a way, is the correct one because it remains relativistically invariant and does reach a minimum on a straight world line connecting points a and b.

It can be verified by direct substitution that energy and momentum are related by the relation

$$\left[\frac{(p_0 - \boldsymbol{p\nu})^2}{p_0^2 - \boldsymbol{p}^2}\right]^{-r} (p_0^2 - \boldsymbol{p}^2) = (mc)^2 (1-r)^{(1-r)} (1+r)^{(1+r)} \,. \tag{15}$$

This relation determines the square of the Finslerian length of the 4-momentum p. In passing from one inertial frame to another its components $p^0 = E/c$ and $\boldsymbol{p}$ must transform such as to guarantee invariance of the form (15). We have noted above that the invariance of the Finslerian metric (5) is established by the generalized Lorentz transformations (6). From the comparison of (15) and (5), the invariance of (15) results from the transformations

$$p'^i = D^{-1} R^i_j L^j_k p^k \,, \tag{16}$$

where the matrices L^j_k and R^i_j are the same as in (6), while

$$D^{-1} = \left(\frac{1 - \boldsymbol{v\nu}/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}}\right)^{-r} I \,. \tag{17}$$

Thus, under generalized Lorentz transformations the scale transformation (17) for momenta is inverse to the corresponding scale transformation (8) for the coordinates of events. Consequently, the phase of a plane wave is an invariant of the generalized Lorentz transformations.

Eq. (13) determines the dependence of the energy E of a free particle, present in the anisotropic space, on both the magnitude and the direction of its velocity $\boldsymbol{v}$. At $\boldsymbol{v} = 0$ the energy reaches its absolute minimum $E_0 = mc^2$. As regards the momentum $\boldsymbol{p}$, its direction, according to (14), does not coincide with the direction of the velocity of a massive particle. Even in the case $\boldsymbol{v} = 0$, the momentum of a particle does not vanish; there remains a "rest momentum" $\boldsymbol{p}_0 = rmc\boldsymbol{\nu}$. Massless particles have no such property; for them, as in conventional special theory of relativity, $v = c$ and $E^2/c^2 - \boldsymbol{p}^2 = 0$.

In the space of 4-momenta p the relation (15) is the equation of mass shell. It appears as a deformed two-sheeted hyperboloid inscribed into a cone $p^{0\,2} - \boldsymbol{p}^2 = 0$. For the upper sheet of such a "hyperboloid" p^0 reaches its absolute minimum $p^0_{min} = E_0/c = mc$ at $\boldsymbol{p} = \boldsymbol{p}_0 = rmc\boldsymbol{\nu}$. For the lower sheet, p^0 reaches its absolute maximum $p^0_{max} = -mc$ at $\boldsymbol{p} = -rmc\boldsymbol{\nu}$. In order to display the mass shell graphically (see Fig. 2), we: (i) introduced the dynamic 4-velocity $u = p/mc$ in place of p; (ii) put $c = 1$ and chose the coordinate axes such that $\boldsymbol{\nu} = (1, 0, 0)$; (iii) confined our consideration to the case of two-dimensional motion and used polar coordinates, in which $\boldsymbol{v} = (v\cos\alpha\,, v\sin\alpha\,, 0)$; (iv) realized eqs. (13) and (14) as the equations of the mass shell (15) given in a parametric form with the parameters v and α.

Being an intrinsic property of space, anisotropy is independent of the magnitude of relative velocities. Therefore, also nonrelativistic mechanics as a whole is different from the Newtonian case. In fact, in the nonrelativistic limit the following expressions are obtained from (13) and (14)

$$E = mc^2 + (1-r)\frac{m\boldsymbol{v}^2}{2} + r(1-r)\frac{m(\boldsymbol{v\nu})^2}{2} \,, \tag{18}$$

$$\boldsymbol{p} = rmc\boldsymbol{\nu} + (1-r)m\boldsymbol{v} + r(1-r)m(\boldsymbol{v\nu})\boldsymbol{\nu} \,. \tag{19}$$

Since within the framework of nonrelativistic mechanics the rest mass m is an additive quantity, the occurrence of the constant terms mc^2 and $rmc\boldsymbol{\nu}$ in (18) and (19) does not affect the conservation laws and the equations of motion. As a result, these terms can be omitted, and the kinetic energy and kinetic momentum, read off from (18) and (19), are

$$T = \frac{1}{2}\mathcal{M}_{\alpha\beta}\, v^\alpha v^\beta \,, \qquad p_\alpha = \mathcal{M}_{\alpha\beta}\, v^\beta \,, \tag{20}$$

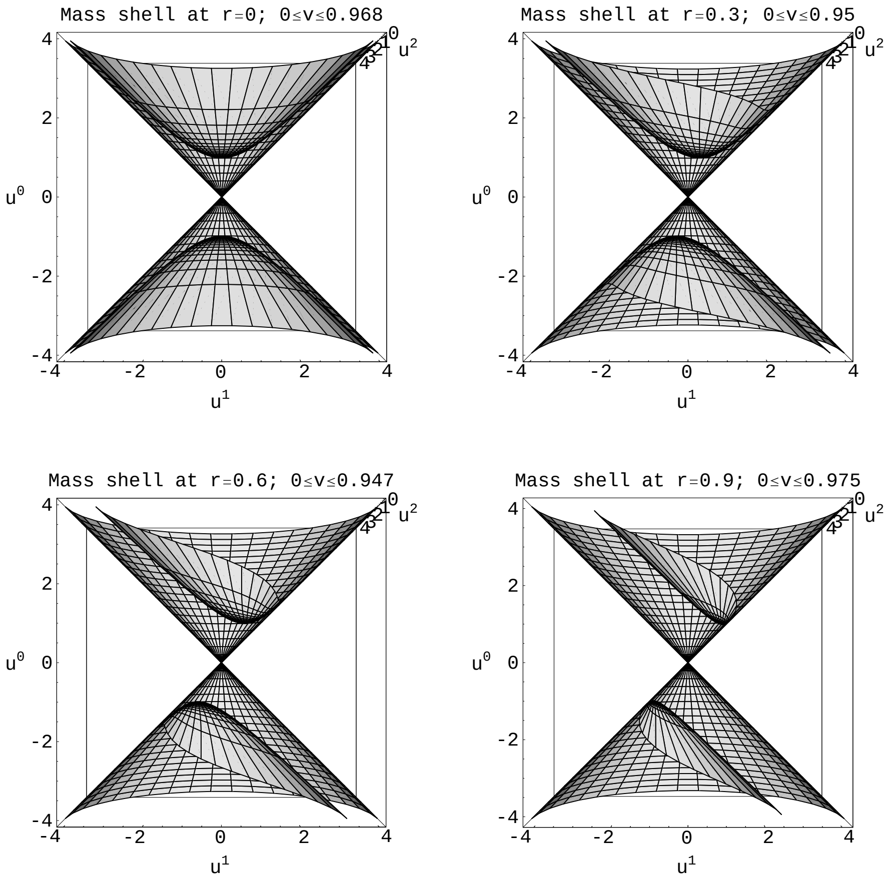

Fig. 2. Parametric 3D plots illustrating the dependence of deformation of a two-sheeted hyperboloid on the magnitude r of space anisotropy. Any of the deformed hyperboloids remains inscribed into a light cone and like a light cone it is an invariant of the generalized Lorentz transformations (16).

where

$$\mathcal{M}_{\alpha\beta} = m(1-r)(\delta_{\alpha\beta} + r\nu_\alpha\,\nu_\beta)\,. \tag{21}$$

Respectively, Newton's second law in anisotropic space takes the form

$$\mathcal{M}_{\alpha\beta}\,a^\beta = F_\alpha\,; \qquad (\alpha = 1,2,3)\,. \tag{22}$$

As we see the inertial properties of a nonrelativistic particle in anisotropic space are specified by a tensor of inertial mass (21).

While discussing the physical nature of inertia, Mach arrived at the conclusion that it is unreasonable to speak of the acceleration of a body relative to empty space. Inertia of bodies should be regarded as their ability to resist acceleration relative to external matter. Since external matter is distributed nonuniformly, inertia and inertial forces arising from acceleration should depend on the localization of a body and on the direction of its acceleration. Consequently, inertial mass should be a quantity represented by *a tensor field* over space-time. When this conclusion is compared with the fact that inertial mass in anisotropic space is represented by a tensor, such a comparison suggests that the parameters r and $\boldsymbol{\nu}$, in terms of which the inertial mass (21) is expressed, should be regarded not as constants but as fields over space-time with a matter distribution as their source. Consequently, we should also consider a space-time with local anisotropy varying from point to point. Then, due to the dependence on the fields r and $\boldsymbol{\nu}$ characterizing the local anisotropy of what will turn out to be a curved Finslerian space-time, the inertial mass (21) will acquire the character of a tensor field in correspondence with Mach's principle.

Within the framework of such a scheme[4] the flat anisotropic spaces (5) with various values of the parameters r and $\boldsymbol{\nu}$ play a role of the tangent spaces to a curved Finslerian space-time, in which case for each of the tangent spaces its proper intrinsic anisotropy is determined by its own values of the parameters r and $\boldsymbol{\nu}$. These values of the parameters are none other than the local values of the corresponding fields r and $\boldsymbol{\nu}$ in a tangent point.

Certainly, the discovery of an anisotropy of inertia would be a direct evidence for a local anisotropy of space. In the experiments made to this end [16, 17] the following upper limit on the anisotropy of inert mass was obtained: $\Delta m/m < 10^{-22}$. Such a strong limitation significantly decreased interest in the problem of local anisotropy and up to now is considered by many investigators as speaking for local isotropy of 3D space. At the same time it has long been pointed out [18, 19] that the conventional experimental estimate of 3D anisotropy at the level 10^{-22} is not correct. And as a reliable upper bound of anisotropy one should take a value 10^{-10} which results from measuring the transversal Doppler effect with the aid of the Mössbauer effect [20, 21].

The latter estimate essentially gives the upper limit for a mean value of the field r in "our region" of space-time. However we have to consider all permissible local values of the field r. It follows from (5) that the maximum permissible value of r is a value of $r = 1$. In this limiting case the linear element of the flat anisotropic space-time (5) degenerates into the total differential

$$ds = dx_0 - \boldsymbol{\nu} d\boldsymbol{x}\,, \tag{23}$$

and, consequently, the action (11) for a free particle of mass m is no longer dependent on the shape of the world line connecting points a and b. All this means that at $r = 1$ a massive particle loses its inertia. This can be illustrated by Eq. (21) which determines the inertial mass tensor $\mathcal{M}_{\alpha\beta}$, and also by Eqs. (13) and (14) which determine the dependence of the energy E and the momentum $\boldsymbol{p}$ on the particle velocity $\boldsymbol{v}$. From these formulae, at $r = 1$, it follows that $\mathcal{M}_{\alpha\beta} = 0$ while E and $\boldsymbol{p}$ become no longer dependent on $\boldsymbol{v}$ and become equal to the corresponding constants mc^2 and $mc\boldsymbol{\nu}$. At $r = 1$, apart from inertia, the notion of spatial extension disappears, which is due to the absence of a light cone and, hence, of the possibility itself for determining spatial distances with the aid of exchange of light signals. As a result, in the space-time (23) there remains a single physical characteristic — time duration ds and it should be regarded as an interval of absolute time.

In the view of what has been said before we now can formulate some guiding principles which will be used in a generalization of the fundamental field equations for locally anisotropic space-time.

4 Initial guiding principles for a generalization of relativistic field theory

As shown above, Einstein's special principle of relativity can be extended to a flat locally anisotropic space-time. Since in the anisotropic space (5) different inertial frames are linked by the generalized Lorentz transformations, the relativity principle requires that

1. Generalized field equations must be invariant under the transformations belonging to the 8-parameter inhomogeneous group of relativistic symmetry of the anisotropic event space.

[4]For more details see [14, 15].

Returning to the equation (15) of the mass shell in the anisotropic space and using the 4D quantities ν^i and η_{ik} introduced in section 2, we can rewrite (15) in a 4D form:

$$\left[\frac{(\nu^i p_i)^2}{p^j p_j}\right]^{-r} p_k p^k = m^2(1-r)^{(1-r)}(1+r)^{(1+r)}\,. \tag{24}$$

Having put $c = \hbar = 1$, in what follows we consider (24) as an invariant dispersion relation for a wave vector p_k . Thus we arrive at the next guiding principle:

2. Generalized field equations must admit solutions in the form of plane waves of the type $e^{ip_k x^k}$, in which case a wave vector p_k must satisfy the dispersion relation (24).

Equation (24) allows us to look for some more guiding principles. A trivial one is

3. Generalized field equations at $r = 0$ (absence of anisotropy) must take the form of the corresponding standard equations. Actually, it is a correspondence principle.

As (in accordance with (24)) the motion of free massless particles in anisotropic space is similar to their motion in isotropic space, i. e. massless particles do not perceive the space anisotropy,

4. Generalized field equations must be constructed so that at $m = 0$ they would take the form of standard equations describing the corresponding massless particles.

As has been pointed out before, the influence of space anisotropy on inertia of a massive particle is such that at $r = 1$ the inertia of the particle disappears. In this case the mass of the particle loses its physical meaning of a measure of inertia and becomes unobservable. As a result we arrive at the final requirement:

5. Generalized field Lagrangians, in the limit $r = 1$, must be reducible (up to a 4-divergence) to the standard Lagrangians of the corresponding massless fields.

As a test of the given guiding principles we try to solve with their help a simple problem, viz. the generalization of the Klein-Gordon equation.

5 Generalization of Lagrangians of Klein-Gordon's type

Due to irrationality of the relation (24), generalized Klein-Gordon equation cannot be obtained by means of the substitution $p_i \to i\partial/\partial x^i$. In order to evade this difficulty we shall be using Lagrange's formalism. For a start let us consider a charged massive scalar field $\varphi(x)$. Writing the corresponding standard Lagrangian where the semicolon stands for partial derivation

$$\mathcal{L} = \varphi^*_{;n}\varphi^{;n} - m^2\varphi^*\varphi \tag{25}$$

we try to generalize it so that the new Lagrangian would lead to field equations invariant under the generalized Lorentz transformations. To achieve this goal we need to bear in mind that, since the action S must be relativistically invariant (in our case this means its invariance under the generalized Lorentz transformations) and the 4-volume $dx^0 d^3x$ is a scalar density of weight -1 (cf. section 2), a suitable Lagrangian must be a scalar density of weight 1. The invariance requirement itself, i. e.

requirement 1 of section 4 is not yet sufficient for fixing the Lagrangian in an unambiguous form; the reason is that the homogeneous group of generalized Lorentz transformations is not 6-parametric but is 4-parametric. In order to illustrate this assertion consider the following Lagrangians:

$$\mathcal{L}_1 = \left[\frac{(\nu^k j_k)^2}{j_k j^k}\right]^r \varphi^*_{;n}\varphi^{;n} - m^2(1-r)^{(1-r)}(1+r)^{(1+r)}\left[\frac{(\nu^k j_k)^2}{j_k j^k}\right]^{2r}\varphi^*\varphi\,, \tag{26}$$

where φ is a scalar and

$$j_k = i(\varphi^*\varphi_{;k} - \varphi\varphi^*_{;k})\,; \tag{27}$$

$$\mathcal{L}_2 = \left[\frac{\nu^l\nu^k\varphi^*_{;l}\varphi_{;k}}{\varphi^*_{;n}\varphi^{;n}}\right]^{-r}\varphi^*_{;n}\varphi^{;n} - m^2(1+r)^{(1+r)}(1-r)^{(1-r)}\varphi^*\varphi\,, \tag{28}$$

where φ is a scalar density of weight $1/2$.

The Lagrangians (26) and (28) are scalar densities of weight 1. Moreover, they satisfy the requirements 1 – 3 of section 4. However, they do not satisfy reqs. 4 and 5 and therefore must be rejected. The only suitable generalized Lagrangian has the form

$$\mathcal{L} = \varphi^*_{;n}\varphi^{;n} - \tilde{m}^2\left(\frac{\nu_k j^k}{\tilde{m}\varphi^*\varphi}\right)^{\frac{2r}{1+r}}\varphi^*\varphi\,, \tag{29}$$

where φ is a scalar density of weight $1/4$, $\tilde{m}$ is constant quantity with the dimension of a mass and j_k determined by (27) is a covariant vector density of weight $1/2$. At $r=1$ the second term in (29) takes the form

$$-\tilde{m}\nu_k j^k \equiv -(\tilde{m}\nu_n x^n j^k)_{;k} + \tilde{m}\nu_n x^n j^k_{;k}\ . \tag{30}$$

As j_k determined by (27) is a preserved current corresponding (in particular) to the massless Lagrangian $\mathcal{L} = \varphi^*_{;n}\varphi^{;n}$, we see (in virtue of (30)) that the generalized Lagrangian (29) does satisfy req. 5 of section 4.

If we now substitute $\varphi = ae^{ip_k x^k}$ with a constant into the field equations corresponding to Lagrangian (29), we arrive at the following conclusion: requirement 2 can be satisfied provided that

$$\tilde{m} = m2^{-r}(1+r)^{(1+r)/2}(1-r)^{(1-r)/2}\,. \tag{31}$$

Thus all requirements 1 – 5 have been satisfied. Therefore, in the following the Lagrangian

$$\mathcal{L} = \varphi^*_{;n}\varphi^{;n} - m^2(1+r)(1-r)^{(\frac{1-r}{1+r})}\left(\frac{\nu^k j_k}{2m\varphi^*\varphi}\right)^{(\frac{2r}{1+r})}\varphi^*\varphi \tag{32}$$

will be considered as a correct one for a free charged massive field $\varphi(x)$ in flat locally anisotropic space (5).

If we work in a similar manner and try to write as a generalized Lagrangian for a neutral massive field (analogous to (29)),

$$\mathcal{L} = \frac{1}{2}\varphi_{;n}\varphi^{;n} - \frac{\tilde{m}^2}{2}\left(\frac{\nu^k\varphi_{;k}}{\tilde{m}\varphi}\right)^{(\frac{2r}{1+r})}\varphi^2\,, \tag{33}$$

then one can see that such a Lagrangian satisfies all the above-mentioned requirements with the exception of req. 2. This means that in the locally anisotropic space the real massive field does not exist as a free field, but it exists as neutral component φ_2 of the isotopic triplet $\boldsymbol{\varphi}(x) = \{\varphi_1(x), \varphi_2(x), -\varphi_1^*(x)\}$, the generalized Lagrangian for which has the form

$$\mathcal{L} = \varphi^*_{1;n}\varphi_1^{;n} + \frac{1}{2}\varphi_{2;n}\varphi_2^{;n} - \frac{m^2}{2}(1-r^2)\left[\frac{\nu^k j_k}{(1-r)m(2\varphi_1^*\varphi_1 + \varphi_2^2)}\right]^{(\frac{2r}{1+r})}(2\varphi_1^*\varphi_1 + \varphi_2^2)\,, \tag{34}$$

where $j_k = i(\varphi_1^*\varphi_{1;k} - \varphi_1\varphi_{1;k}^*)$. For $\varphi_2(x) = 0$, the Lagrangian (34) takes the form (32). Otherwise there is an interaction between the components of the isotopic triplet; the assumption $\varphi_2^2/(\varphi_1^*\varphi_1) < 1$ allows the use of perturbation theory.

6 An exact solution of the generalized field equations for a charged massive field in locally anisotropic space-time

A system of field equations resulting from the generalized Lagrangian (32), in which j_k is determined by (27), is essentially nonlinear and, in addition, too cumbersome to be reproduced here. Therefore note only its main property: in the limit $r = 0$ (the absence of anisotropy), the system of generalized K-G equations reduces to the system of standard K-G equations

$$\begin{aligned} \varphi_{;k}^{;k} + m^2\varphi &= 0\,, \\ \varphi_{;k}^{*;k} + m^2\varphi^* &= 0\,, \end{aligned} \tag{35}$$

whereas, in the limit $r = 1$ (the maximum attainable anisotropy) it takes the form

$$\begin{aligned} \varphi_{;k}^{;k} + i2m\varphi_{;k}\nu^k &= 0\,, \\ \varphi_{;k}^{*;k} - i2m\varphi_{;k}^*\nu^k &= 0\,. \end{aligned} \tag{36}$$

By means of the substitution $\varphi = \tilde{\varphi}e^{-im\nu_k x^k}$, $\varphi^* = \tilde{\varphi}^* e^{im\nu_k x^k}$ we arrive at massless K-G equations

$$\begin{aligned} \tilde{\varphi}_{;k}^{;k} &= 0\,, \\ \tilde{\varphi}_{;k}^{*;k} &= 0\,. \end{aligned} \tag{37}$$

It is just the result we were aspiring to reach: while, at $r = 1$, the mass m turns out to be unobservable, the massive field becomes the massless one.

The structure of the generalized Lagrangian (32) is such, that module $\varrho(x)$ and phase $\alpha(x)$ of the complex field $\varphi(x)$ serve as more natural independent field variables. Introducing these variables with the help of the substitution $\varphi = \varrho e^{-i\alpha}$, we find that the Lagrangian (32) takes the form

$$\mathcal{L} = \varrho_{;n}\varrho^{;n} + \varrho^2\alpha_{;n}\alpha^{;n} - m^2(1+r)(1-r)^{\left(\frac{1-r}{1+r}\right)}\left(\frac{\nu^n\alpha_{;n}}{m}\right)^{\left(\frac{2r}{1+r}\right)}\varrho^2\,. \tag{38}$$

Now we are able to represent the field equations resulting from (38):

$$\begin{aligned} &\varrho_{;n}^{;n} - \varrho\left[\alpha_{;n}\alpha^{;n} - m^2(1-r^2)\left(\frac{\nu^n\alpha_{;n}}{m(1-r)}\right)^{\left(\frac{2r}{1+r}\right)}\right] = 0\,, \\ &\alpha_{;n}^{;n} + \frac{r}{(1+r)}\left(\frac{\nu^n\alpha_{;n}}{m(1-r)}\right)^{-\left(\frac{2}{1+r}\right)}\nu^j\nu^k\alpha_{;j;k} + \frac{2}{\varrho}\varrho_{;n}\alpha^{;n} - \\ &-\frac{2rm}{\varrho}\left(\frac{\nu^n\alpha_{;n}}{m(1-r)}\right)^{-\left(\frac{1-r}{1+r}\right)}\varrho_{;n}\nu^n = 0\,, \end{aligned} \tag{39}$$

in which case a preserved current and energy-momentum tensor appear as

$$\mathcal{J}^n = 2\varrho^2\left\{\alpha^{;n} - rm\left(\frac{\nu^k\alpha_{;k}}{m(1-r)}\right)^{\left(\frac{r-1}{r+1}\right)}\nu^n\right\}\,, \tag{40}$$

$$T^{lk} = 2\varrho^{;l}\varrho^{;k} + \mathcal{J}^k\alpha^{;l} - \mathcal{L}g^{lk} = 2\varrho^{;l}\varrho^{;k} + \alpha^{;l}\mathcal{J}^k +$$
$$+ \left[- \left(\varrho_{;n}\varrho^{;n} + \varrho^2\alpha_{;n}\alpha^{;n}\right) + m^2(1-r^2)\left(\frac{\nu^n\alpha_{;n}}{m(1-r)}\right)^{\left(\frac{2r}{1+r}\right)}\varrho^2\right] g^{lk} . \tag{41}$$

Regarding the problem of exact solutions for the system of nonlinear field equations (39) we were able to find a complete family of plane-wave-like solutions including, in particular, solutions of the type $ae^{ip_kx^k}$ with a constant amplitude a. In order to obtain such a family it is sufficient to use the conservation laws and the Ansatz:

$$\varrho = \varrho(p_kx^k)\,, \qquad \alpha = \alpha(p_kx^k)\,. \tag{42}$$

As a result, we arrive at the following system of ordinary differential equations:

$$(\dot{\varrho}^2 + \varrho^2\dot{\alpha}^2)p_np^n + [m(1-r)]^{\left(\frac{2}{1+r}\right)}(\nu^np_n\dot{\alpha})^{\left(\frac{2r}{1+r}\right)}\varrho^2 = \mathcal{E}\ , \tag{43}$$

$$\frac{2\varrho^2}{\dot{\alpha}}\left[\dot{\alpha}^2p_np^n - m^{\left(\frac{2}{1+r}\right)}r(1-r)^{\left(\frac{1-r}{1+r}\right)}(\nu^np_n\dot{\alpha})^{\left(\frac{2r}{1+r}\right)}\right] = \mathcal{C}\ , \tag{44}$$

where the dot denotes $\frac{d}{d(p_kx^k)}$, the quantities ν^np_n and p_np^n are related by the dispersion relation (24), and the constants $\mathcal{E}$ and $\mathcal{C}/p_0$ are respectively the energy and charge densities in a frame where $\boldsymbol{p} = 0$.
From the viewpoint of classical mechanics, eqs. (43) and (44) represent the integrals of motion, viz. the energy and angular momentum of a two-dimensional nonlinear oscillator. In particular, putting $\mathcal{C} = 0$ and taking into account (42), we find the corresponding field configuration

$$\varphi(x) = \sqrt{\frac{\mathcal{E}(1+r)^{(1+r)}}{(p_kp^k)r^r}}\sin\left\{\sqrt{\frac{r^r}{(1+r)^{(1+r)}}}\ p_k(x^k + \overset{\circ}{x}{}^k)\right\} e^{-i\sqrt{r}\left\{\sqrt{\frac{r^r}{(1+r)^{(1+r)}}}\ p_k(x^k+\overset{\circ}{x}{}^k)\right\}}$$

as an exact plane-wave-like solution of the generalized K-G equations.
With the help of this solution it can be observed how the local space anisotropy modifies the neutral solution, $\varphi(x) \propto \sin p_k(x^k + \overset{\circ}{x}{}^k)$, of the standard K-G equations (35).

7 Conclusion

Usually, relativistic symmetry as a necessary condition for the description of physics at high velocities is identified with Lorentz symmetry. However, as shown in sections 2 this assumption is incorrect: the group of generalized Lorentz transformations can be taken as another possibility. Thus, the debate about a possible violation of Lorentz symmetry can be carried through within a strictly relativistic framework. In particular, it should be investigated whether the GZK-cutoff for cosmic radiation at ultrahigh energies can be removed by the new relativistic symmetry. The prize to be paid for weakening the 10-parameter inhomogeneous Lorentz group to the 8-parameter inhomogeneous group of generalized Lorentz transformations is the loss of local isotropy of 3D space.

In this paper, we started an investigation of relativistic field theory in locally anisotropic space and obtained some reassuring first results concerning the wave equation for a massive scalar field. Although not discussed, we are also showing here the Dirac Lagrangian generalized for the flat locally anisptropic space-time (5):

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^n\psi_{;n} - \bar{\psi}_{;n}\gamma^n\psi\right) - m\left(\frac{\nu_n\bar{\psi}\gamma^n\psi}{\bar{\psi}\psi}\right)^r\bar{\psi}\psi\ .$$

At last remind that, at $r = 1$, there is a degenerated state of the space-time where the fundamental fields prove themselves as massless ones. Therefore it is not unlikely that such a state of space-time is unstable. The space-time may find itself in a stable state with $r < 1$ due to a spontaneous phase transition in its geometric structure. It should be accompanied by the corresponding breakdown of gauge symmetry of the matter fields. In such a case the field r should play the role of a Higgs field.
An extended version of this paper will be published elsewhere.

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