Universal Shroedinger's Uncertainties Relations "Coordinate–Moment" and "Energy–Time"

A.D. Sukhanov*

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1. The formation of main quantum mechanical postulates has passed already 75 years ago. However discussion on account of its physical interpretation does not subside. The most important element of these debates is connected with problem of simultaneous measurability of physical characteristics.

As is well known, the structure of algebras of observables does not change at the transition from classical to quantum mechanics, but observables as themselves become operators submitting to known commutative relations. This fundamental thesis has predetermined a special role of commutators in the traditional quantum mechanical structure. It is not surprising, that the base of its physical interpretation has formed by Heisenberg's uncertainties relations (URs). Their right-hand side contains only the contribution of average of corresponding commutator.

However this type of statistical dependency of observables is not single possible in quantum mechanics. How Shroedinger and Robertson have shown as far back as 1930, in the most general case there are URs, in which the right-hand side contains the contributions from commutator but anticommutator of observables too. From this point of view, known Heisenberg's URs as themselves make sense only as private version of Shroedinger's-Robertson's URs (briefly Shroedinger's URs), importance of which is not appreciated hitherto.Shroedinger's URs are universal since they are even non-trivial in a quasiclassical limit. This fact allows to make good use their analogues outside of quantum mechanics.

2. Shroedinger's URs for arbitrary observables are a direct consequence of Koshi-Bunyakovsky-Schwartz' unequality in Hilbert space. We shall suppose that we consider observable A and B, to which Hermitian operators $\widehat{A}$ and $\widehat{B}$ are put in correspondence. In quantum state $|\psi\rangle \equiv |\ \rangle$, which is not eigenstate for one of them, it is possible to describe observables A and B by averages $\overline{A}$ and $\overline{B}$ and dispersions (variances); for instance,

$$(\Delta A)^2 \equiv \langle|(\Delta\widehat{A})^2|\rangle = \langle|\Delta\widehat{A}\cdot\Delta\widehat{A}|\rangle = \langle\Delta A|\Delta A\rangle; \tag{1}$$

$$\Delta\widehat{A} \equiv \widehat{A} - \overline{A}; \qquad \Delta\widehat{A}\ |\ \rangle \equiv |\Delta A\rangle. \tag{2}$$

Then the Shroedinger's URs for arbitrary observables A and B look like

$$(\Delta A)^2(\Delta B)^2 \geqslant |\widetilde{R}_{AB}|^2 \equiv |\langle\Delta A|\Delta B\rangle|^2 = |\ \langle\ |\Delta\widehat{A}\cdot\Delta\widehat{B}|\ \rangle\ |^2. \tag{3}$$

The generalized correlator (covariance) R_{AB} in the Shroedinger's URs (3) is

$$|\widetilde{R}_{AB}| = \sqrt{\sigma_{AB}^2 + c_{AB}^2}; \tag{4}$$

*Peoples' Friendship University of Russia, 117198, Moscow, Russia. E-mail:ogol@oldi.ru

$$\sigma_{AB} = \frac{1}{2}\langle|\{\Delta\widehat{A}, \Delta\widehat{B}\}|\rangle; \quad c_{AB} = \frac{1}{2}\left|\langle|[\widehat{A}, \widehat{B}]|\rangle\right| \tag{5}$$

are contributions in $|\widetilde{R}_{AB}|$ from commutator and anticommutator of operators $\Delta\widehat{A}$ $\Delta\widehat{B}$ correspondingly.

It is not difficult to see that generalized correlator $|\widetilde{R}_{AB}|$ differs from quantities σ_{AB} and c_{AB} separately since it is an invariant for unitary transformations in Hilbert space. Just this quantity as a whole describes the nature of correlation of fluctuations of observable A and B in arbitrary states.

In states, for which $\sigma_{AB} \equiv 0$, but $c_{AB} \neq 0$, Shroedinger's URs become in their special version - Heisenberg's URs

$$(\Delta A)^2(\Delta B)^2 \geqslant c^2_{AB}. \tag{6}$$

For states, in which $c_{AB} \equiv 0$, but $\sigma_{AB} \neq 0$, Shroedinger's URs become in their another version - in the quantum analogue of Einstein's URs in the statistical thermodynamics

$$(\Delta A)^2(\Delta B)^2 \geqslant \sigma^2_{AB}. \tag{7}$$

It is essential to note that this fact has not only a place in the case, when the commutator $[\widehat{A}, \widehat{B}] = 0$ by $\hbar \neq 0$. This is valid in that case, when average of nontrivial commutator turns to zero. Eventually, if $\sigma_{AB} \neq 0$, but $\sigma_{AB} \ll c_{AB}$ or if $c_{AB} \neq 0$, but $\sigma_{AB} \gg c_{AB}$, we may say, that the right-hand side of Shroedinger's URs coincides approximately with the right-hand side of Heisenberg's URs or with one of Einstein's URs quantum analogue.

In a quasiclassical limit when $\hbar \to 0$, the commutator contribution c_{AB} in Shroedinger's UR turns to zero. At the same time the contribution of anticommutator approaches to correlator of fluctuations of c-number quantities and is not equal to zero, if a statistical dependence exists between them. As a result, in this limit Shroedinger's URs take the form, which is similar to Einstein's URs in the equilibrium statistical thermodynamics, that is known from the classical theory of probabilities

$$\overline{(\Delta A)^2} \cdot \overline{(\Delta B)^2} \geqslant \overline{(\Delta A \cdot \Delta B)}^{\,2}. \tag{8}$$

3. We shall analyze in detail Shroedinger's URs "coordinate-moment" for microparticle making one-dimensional motion. In this case $\widehat{A} = \widehat{q}$ and $\widehat{B} = \widehat{p}$, so Shroedinger's UR takes the form

$$(\Delta q)^2(\Delta p)^2 \geqslant \left|\widetilde{R}_{qp}\right|^2 = \sigma^2_{qp} + \frac{\hbar^2}{4}. \tag{9}$$

As a matter of convenience of further analysis we shall present quantity σ_{qp} in following form

$$\sigma_{qp} = m\int dq(q - \overline{q})j(q) \equiv mI, \tag{10}$$

where $j(q)$ is the density of probability current. Correspondingly the quantity I makes sense of the first moment of the density of probability current. In quasiclassical limit it become to the first moment of the density of particles number current.

It is not difficult to see that Shroedinger's URs become to corresponding Heisenberg's URs, if the quantity I and in the same time a contribution σ_{qp} are equal to zero. It is necessary for it: either $j \equiv 0$ or the function $j(q)$ by $j \neq 0$ is even. From this point of view, the all of wave functions describing finite movement (standing de Broglie's waves) are real and give $\sigma_{qp} = 0$ Similarly harmonic wave function describing motion of free microparticle in idealized model (running de Broglie's wave) gives $\sigma_{qp} = 0$ also as for it j =const.

Finally we shall note that if $\sigma_{qp} \neq 0$, that it stays different from zero and in quasiclassical limit. Then it is possible to neglect by contribution $c_{qp} = \frac{\hbar}{2}$ in the right-hand side of URs. So Shroedinger's URs approximate complies with quantum analogue of Einstein's URs. For unstationary states is always $\sigma_{qp} \gg c_{qp}$ under $t \to \infty$.

4. We shall address now to UR "energy-time". Although formula of form $\delta\varepsilon \cdot \delta t \gtrsim \hbar$ have been involved in quantum dynamics to different degrees since the end of the 1920s, the question of the mathematical meaning and physical interpretation of the corresponding quantities remains open in general. Until recently, the answer to this question has been sought in following direction. With the lack of a time operator, it is proposed to give a reasonable meaning to the concept of time uncertainty while formally staying within quantum physics and, in particular, preserving the operator description for the energy. The most successful attempt of this type was due to Mandelstam and Tamm.

The introduction of this concept is based on the following procedure. For an arbitrary observable A described by a Hermitian operator $\widehat{A}$ and for microsystem energy ε described by the Hamiltonian $\widehat{H}$, the original UR is

$$\Delta\varepsilon \cdot \Delta A \geqslant c_{AH} \equiv \frac{1}{2}\left|\langle|[\widehat{A},\widehat{H}]|\rangle\right|, \tag{11}$$

where $\Delta\varepsilon$ and ΔA are the variances of ε and A. The commutator in the right-hand side of (11) can be expressed from the Heisenberg's equation for $\widehat{A}$. Then we obtain

$$\Delta\varepsilon \cdot \Delta A \geqslant \frac{\hbar}{2}\left|\langle|\frac{d\widehat{A}}{dt}|\rangle\right|. \tag{12}$$

Following Mandelstam and Tamm, the time uncertainty is defined as

$$\Delta t_A \equiv \frac{\Delta A}{\left|\langle|\frac{d\widehat{A}}{dt}|\rangle\right|}. \tag{13}$$

From the physical standpoint, this quantity has the meaning of the time interval during which the deviation of A from the mean $\overline{A}$ reaches ΔA. After the transition from ΔA to Δt_A, UR (12) takes a known form, but involving only rigorously defined quantities

$$\Delta\varepsilon \cdot \Delta t_A \geqslant \frac{\hbar}{2}. \tag{14}$$

The definition of Δt_A in Eq. (13) has the advantage that it allows introducing a reasonable notion of time uncertainty. At the same time, it has a number of serious disadvantages. First, it has ambiguous because it allows introducing several time intervals depending on different observables A_i for the same microsystem. Second, it loses its meaning in the case where $\Delta\varepsilon \neq 0$ and either $\Delta A \neq 0$ or the commutator $[\widehat{A},\widehat{H}]$ itself, or its mean, vanishes.

5. Staying within the Mandelstam-Tamm approach to the energy-time UR, we propose defining the time uncertainty starting not with Heisenberg's URs but with the universal Shroedinger's URs.

We start with Shroedinger's URs (3) and set $\widehat{B} \equiv \widehat{H}$. Instead of (3) and (4)–(5), then we obtain

$$\Delta\varepsilon \cdot \Delta A \geqslant |\widetilde{R}_{AH}|, \tag{15}$$

$$|\widetilde{R}_{AH}|^2 = \sigma_{AH}^2 + c_{AH}^2 = \frac{1}{4}\langle|\{\Delta\widehat{A},\Delta\widehat{H}\}|\rangle^2 + \frac{1}{4}\left|\langle|[\widehat{A},\widehat{H}]|\rangle\right|^2. \tag{16}$$

Now we generalize Mandelstam-Tamm's formula (13) and define the generalized time uncertainty as

$$\Delta t_A^* \equiv \frac{\hbar \cdot \Delta A}{2R_{AH}}. \tag{17}$$

The generalized UR "energy-time" (14) then becomes

$$\Delta\varepsilon \cdot \Delta t_A^* \geqslant \frac{\hbar}{2} \tag{18}$$

Our definition of t_A^* in Eq. (17) is unambiguous (i.e. is not related to any observable) and has no singularity when the contribution of the commutator c_{AH} vanishes.

6. Our analysis shows following:
(a) Heisenberg's URs are private version of universal and invariant Shroedinger's URs.
(b) The contributions of anticommutator and commutator in Shroedinger's URs are important equally.
(c) A role of the first one of them in Shroedinger's UR "coordinate-moment" grows with time and in a quasiclassical limit.
(d) Shroedinger's URs allow to generalize UR "energy-time" by Mandelstam and Tamm, excluded from it possible ambiguities and singularities.