Quantum Irreversibility and Gamov States

M. Gadella, J. Mateo
Departamento de Física Teórica, Facultad de Ciencias, Valladolid, Spain

We argue that Hilbert spaces are necessary in order to provide a mathematical formulation of Quantum Mechanics but not sufficient. The sufficient structure would be an equipment of Hilbert space called rigged Hilbert space. We show how rigged Hilbert spaces are the right construction valid for a mathematical implementation of the Dirac formulation of Quantum Mechanics, a rigorous definition of vector states for exponentially decaying resonances and a description of irreversibility in Quantum Mechanics produced by resonances. As an example, we apply rigged Hilbert spaces to the Friedrichs model.

1. Introduction

It became clear after the time of von Neumann that Hilbert space structure, was necessary for a mathematical formulation of Quantum Mechnics although not sufficient. It is necessary becuase it contains such important ingredients like scalar products (used in the definition of transition amplitudes and probabilities), self adjoint and unitary operators (suitable to define observables, time evolution and symmetries in Quantum Mechanics), etc. It is not sufficient as it does not include all the features of the Dirac formulation of Quantum Mechanics (widely used by physicists, see [10]), does not include resonance states and it does not allow for formulation of the irreversibilty at the microscopical level.

It is the purpose of the present paper to clarify all these points and to show that von Neumann formulation on Hilbert spaces should be completed so as to include the above mentioned features: Dirac formulation, quantum resonance states (also called Gamov vectors) and quantum irreversibility.

We shall see that along the Hilbert states, containing the pure states of the system under our study, we should have another two spaces making a triplet called *rigged Hilbert space* or also *Gelfand triplet*, honoring the russian mathematician who introduced this structure [14].

A rigged Hilbert space (henceforth RHS) is a tern of spaces of the form

$$\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times, \tag{1}$$

where:

i.) $\mathcal{H}$ is the Hilbert space of the pure states of the system under our consideration. Since the extension from Hilbert space to rigged Hilbert space is necessary when we study observables with continuous spectrum only (and henceforth complex systems), we shall assume that $\mathcal{H}$ is infinite dimensional (althought the property of separability of $\mathcal{H}$ must be kept).

ii.) $\mathbf{\Phi}$ is a subspace of $\mathcal{H}$ with two important properties. First of all, it is dense in $\mathcal{H}$. This means that for each vector $\psi \in \mathcal{H}$ there always exists a vector $\varphi \in \mathbf{\Phi}$ as close as we want to ψ with the topology on $\mathcal{H}$. Secondly, it has its own topology, $\tau_{\mathbf{\Phi}}$, which is stronger than the topology that $\mathbf{\Phi}$ has as subspace of $\mathcal{H}$. By "stronger", we meant that if $\mathcal{O} \subset \mathbf{\Phi}$ is an open set with $\tau_{\mathbf{\Phi}}$, it is also an open set with the Hilbert space topology (henceforth $\tau_{\mathcal{H}}$) and that there are open sets with $\tau_{\mathbf{\Phi}}$ that are not open sets with $\tau_{\mathcal{H}}$. The topology $\tau_{\mathbf{\Phi}}$ may have additional properties like nuclearity [14] that are too technical and hence, we do not want to discuss here.

iii.) An *antilinear functional* on $\mathbf{\Phi}$ is a mapping $F : \mathbf{\Phi} \longmapsto C$, of $\mathbf{\Phi}$ on the set C of complex numbers such that for any vectors φ, ψ in $\mathbf{\Phi}$ and any pair of complex numbers α, β, it holds

$$F(\alpha\varphi + \beta\psi) = \alpha^* F(\varphi) + \beta^* F(\psi),$$

where the star denotes complex conjugation. Here, we shall consider continuous antilinear functionals (with respect to the topology $\tau_{\mathbf{\Phi}}$ on $\mathbf{\Phi}$ and the topology on the set of complex numbers C) only. Continuous antilinear functionals on $\mathbf{\Phi}$ form a vector space, if we define their sum and their product by scalars as

$$(\alpha\, F + \beta\, G)(\psi) := \alpha\, F(\psi) + \beta\, G(\psi),$$

where F and G are continuous antilinear functionals, α and β complex numbers and ψ an arbitrary vector in $\mathbf{\Phi}$. Therefore, the set of continuous antilinear functionals on $\mathbf{\Phi}$ forms a vector space that we call the *antidual space* of $\mathbf{\Phi}$ and we denote as $\mathbf{\Phi}^\times$. As an example of vector in $\mathbf{\Phi}^\times$, take $\varphi \in \mathcal{H}$ and define the action of F_φ on any vector ψ in $\mathcal{H}$ as:

$$F_\varphi(\psi) = \langle\psi|\varphi\rangle,$$

where $\langle\cdot|\cdot\rangle$ represents the scalar product on $\mathcal{H}$. It is not difficult to show that F_φ is a vector in $\mathbf{\Phi}^\times$ and that the mapping from $\mathcal{H}$ into $\mathbf{\Phi}^\times$

$$\varphi \longmapsto F_\varphi$$

defined for each $\varphi \in \mathcal{H}$ is one to one. Thus, if we identify φ with F_φ for each $\varphi \in \mathcal{H}$, we conclude that $\mathcal{H} \subset \mathbf{\Phi}^\times$. There are however, functionals in $\mathbf{\Phi}^\times$ which do not correspond to any vector in $\mathcal{H}$, like Dirac deltas. Thus, summarizing, we have (1). We provide the space $\mathbf{\Phi}^\times$ with a topology. On $\mathbf{\Phi}$ this topology is weaker (has less open sets) than both $\tau_{\mathbf{\Phi}}$ and $\tau_{\mathcal{H}}$. We shall denote it as $\tau_\times$. The relation among these topologies, as all them are defined on the space $\mathbf{\Phi}$, is often described in written as $\tau_{\mathbf{\Phi}} \succ \tau_{\mathcal{H}} \succ \tau_\times$ [9]. Both $\mathbf{\Phi}$ and $\mathcal{H}$ are dense in $\mathbf{\Phi}^\times$ with the topology $\tau_\times$.

Remark. So far, we have denoted the action of the functional $F \in \mathbf{\Phi}^\times$ on the vector $\varphi \in \mathbf{\Phi}$ as $F(\varphi)$. In order to keep pace with the Dirac notation, we shall henceforth use

$$\langle\varphi|F\rangle := F(\varphi).$$

Thus, the bracket $\langle\varphi|F\rangle$ is linear to the right ($\langle\varphi|\alpha F + \beta G\rangle = \alpha\langle\varphi|F\rangle + \beta\langle\varphi|G\rangle$) and antilinear to the left ($\langle\alpha\varphi + \beta\psi|F\rangle = \alpha^*\langle\varphi|F\rangle + \beta^*\langle\psi|F\rangle$) as in the Dirac notation. Note that $\langle\varphi|F\rangle$ is indeed a sesquilinear scalar product between $\mathbf{\Phi}$ and $\mathbf{\Phi}^\times$.

Rigged Hilbert spaces are of wide use in physics. Among their applications, we can select:

1. Giving a rigorous meaning to the Dirac formulation of Quantum Mechanics [8].

2. Provide a rigorous meaning to the spectral decompositions of the Frobenius Perron operator for certain chaotic systems (Exact and Kolmogorov) in terms of Pollicot-Ruelle resonances [6].

3. Give a rigorous definition of Gamov vectors or vector states for resonances. Gamov vectors have been defined by this russian scientist in 1928 in order to define a wave function for exponentially decaying states [13]. This construction had serious difficulties to be correctly interpreted as a state function since it is arbitrarily big on a neighborhood of the infinity. The formulation of quantum mechanics in RHS, allows to define these objects in a natural manner in the dual space of a certain RHS.

4. Extend Quantum Mechanics to show the *Intrinsecally Irreversible* character of certain quantum processes, like decay. In presence of resonances the evolution operator group splits into two semigroups, one with $t \leq 0$ and the other with $t \geq 0$. This second semigroup will represent the decay process and the former its time reversal. Thus, future is distinguished from the past giving an arrow of time. This fact has been used by Prigogine and coworkers to formulate a theory of irreversible process, for which the irreversibility comes from the dynamics. Although the theory of Prigogine is controversial, we consider it very appealing. From the mathematical point of view the splitting of the unitary group of time evolution in the case of quantum systems with resonances and the baker map has been well studied with full rigor [5, 6, 9].

5. Extending the formalism of statistical mechanics in order to include generalized states and singular structures on it. This can be achieved with the use of the rigged Liouville spaces (RLS) [3].

6. Defining some elements that appear in the axiomatic theory of Quantum fields: Wightman Functional, Borchers Algebra, generalized states, etc. This aspect is beautifully discussed in [7].

7. Dealing with physical problems requiring the use of distributions. As is well known, distributions are objects in the dual of a nuclear locally convex space (space of the type $\mathbf{\Phi}^\times$) [14].

It is not our purpose to discuss all these aspects along this paper, because this would exceed largely of the possible containt of a Conference paper. Instead, we shall comment the following aspects: In section 2, we briefly present the Dirac formulation of Quantum Mechanics for operator with continuous spectra in terms of RHS. In section 3, we give the basis of the analysis of resonance scattering and time asymmetry in quantum mechanics using RHS. Finally, in section 4, we discuss an interesting example of quantum system with resonances: the Friedrichs model and show the procedure to obtain the Gamov vectors on this model.

2. Dirac formulation of Quantum Mechanics

According to Dirac, any observable must have a *complete set* of eigenstates with respective eigenvalues in the spectrum of the observable. This spectrum is looked as the set of all posible results of the measurement of this observable. This characterization of an observable is not possible in the von Neumann mathematical formulation of quantum mechanics, as we here illustrate with an example.

Let P be the one dimensional momentum operator. This is a self adjoint operator on the space of complex square integrable functions on the real line. Its spectrum covers the whole real line. In order to find the eigenvectors, with eigenvalues in the spectrum of P, we have to solve the spectral equation: $P\psi = -i\, d/dx(\psi) = \lambda\, \psi$. This gives $\psi(x) = e^{i\lambda x}$, which is not a square integrable function and therefore, not a state, according to von Neumann formulation. Therefore, P has not square integrable eigenfunctions and hence, cannot have a complete system of eigenstates. The same happens with the momentum operator in higher dimensions, with the position operator and with all operators with continuous spectrum like the Hamiltonians of the form $\mathbf{p}^2/2m + V(\mathbf{x})$.

Therefore, the Dirac formulation of quantum mechanics is not implementable in Hilbert space for observables with continuous spectrum. In fact, we need to extend the Hilbert space so as to include in the formalism the eigenvectors of an observable with eigenvalues in the continuous spectrum. This extension is indeed the antidual $\mathbf{\Phi}^\times$ in a rigged Hilbert space (RHS) of the form (1). In the case of the one dimensional momentum operator, the suitable RHS is $\mathcal{S} \subset L^2(R) \subset \mathcal{S}^\times$, where S is the Schwartz space, i.e., the space of all real valued complex functions that admit continous derivatives at all orders and they and all their derivatives vanish at the infinite faster than the inverse of any polynomial. The antidual $\mathcal{S}^\times$ is the space of tempered distributions[1].

Let us briefly review the RHS implementation of the Dirac formulation of quantum mechanics for the simplest case of an operator A with pure continuous spectrum without degeneration. Then, let us assume that A is a quantum observable with purely continuous spectrum (for instance the momentum operator). In the Dirac formulation of Quantum Mechanics, there exists a set of eigenvectors $|\lambda\rangle$ of A,

$$A\,|\lambda\rangle = \lambda\,|\lambda\rangle$$

such that for any pure state $|\varphi\rangle$, we have

$$|\varphi\rangle = \int_{\sigma(A)} \varphi(\lambda)\,|\lambda\rangle\, d\lambda$$

[1]Considered as continuous *antilinear* functionals on $\mathcal{S}$.

with

$$\varphi(\lambda) = \langle\lambda|\varphi\rangle^* =: \langle\varphi|\lambda\rangle. \qquad (2)$$

Here, $\sigma(A)$ is the spectrum of A. For a function $f(\lambda)$,

$$f(A)\,|\varphi\rangle = \int_{\sigma(A)} f(\lambda)\,\varphi(\lambda)\,|\lambda\rangle\,d\lambda.$$

We know that if a real number λ belongs to the continuous spectrum of A, there is **no** eigenvector of A with eigenvalue λ on the Hilbert space on which we define A. Therefore $|\lambda\rangle$ must be outside of Hilbert space!! This is only possible if we extend $\mathcal{H}$ with a space of the form $\mathbf{\Phi}^\times$, i.e., if we equip $\mathcal{H}$ so as to become a RHS of the form (1).

Our problem is now how to implement the Dirac requirement according to which any observable (implemented as a self adjoint [Hermitian] operator on $\mathcal{H}$) has a complete set of eigenvectors. This is a nontrivial mathematical problem for which the solution was given by means of a theorem, proven by the russian mathematician I.M. Gelfand [14] and completed by the polish mathematician K. Maurin [17]. This result is the following:

Theorem by Gelfand and Maurin. Let A be a quantum observable with continuous spectrum $\sigma(A)$. Then, there exists a RHS, $\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^\times$ such that

i.) $A\mathbf{\Phi} \subset \mathbf{\Phi}$ and A is continuous on $\mathbf{\Phi}$.

ii.) The operator A can be extended into $\mathbf{\Phi}^\times$ by means of the duality formula:

$$\langle A\,\varphi|F\rangle = \langle\varphi|A\,F\rangle\,, \qquad \forall\,\varphi \in \mathbf{\Phi},\, F \in \mathbf{\Phi}^\times,$$

where $\langle\varphi|F\rangle = F(\varphi)$ is the action of the functional $F \in \mathbf{\Phi}^\times$ on $\varphi \in \mathbf{\Phi}$.

iii.) For (almost with respect to the Lebesgue measure on the real line) all $\lambda \in \sigma(A)$, there exists a $|\lambda\rangle \in \mathbf{\Phi}^\times$, such that

$$A|\lambda\rangle = \lambda\,|\lambda\rangle.$$

iv.) For each pair $\varphi, \psi \in \mathbf{\Phi}$, we have that

$$\langle\psi|\varphi\rangle = \int_{\sigma(A)} \langle\psi|\lambda\rangle\langle\lambda|\varphi\rangle\,d\lambda.$$

If we omit the arbitrary $\psi \in \mathbf{\Phi}$, the previous formula reads:

$$|\varphi\rangle = \int_{\sigma(A)} |\lambda\rangle\langle\lambda|\varphi\rangle\,d\lambda$$

Note that $\langle\lambda|\varphi\rangle = \varphi(\lambda)$.

v.) For a function $f(\lambda)$, we have

$$f(A)|\varphi\rangle = \int_{\sigma(A)} f(\lambda)\,|\lambda\rangle\langle\lambda|\varphi\rangle\,d\lambda = \int_{\sigma(A)} f(\lambda)\,\varphi(\lambda)\,|\lambda\rangle\,d\lambda,$$

as required by Dirac. This formula can be extended when A is replaced by a c.s.c.o. as well as for self adjoint operators with more complicated spectrum.

Let us go back to the example of the one dimensional momentum operator, and define the action of the distribution $e^{i\lambda x}$ on the regular function $\psi(x) \in \mathcal{S}$ as usual in the theory of distributions[2] by

$$\langle\psi(x)|e^{i\lambda x}\rangle := \int_{-\infty}^{\infty} \psi^*(x)\,e^{i\lambda x}\,dx = [\widehat{\psi}(\lambda)]^*\,, \qquad (3)$$

[2]The complex conjugate in formula (3) comes from the fact that we are here looking distributions as *antilinear* functionals.

where $\widehat{\psi}(\lambda)$ is the Fourier transform of $\psi(x)$. The complex conjugate of (3) is given by

$$\langle e^{i\lambda x}|\psi(x)\rangle := \langle \psi(x)|e^{i\lambda x}\rangle^* = \int_{-\infty}^{\infty} \psi(x)\, e^{-i\lambda x}\, dx = \widehat{\psi}(\lambda). \tag{4}$$

Let us take two functions in $\mathcal{S}$, say $\psi(x)$ and $\varphi(x)$ and write the following scalar product

$$\langle \psi(x)|P\,\varphi(x)\rangle = \int_{-\infty}^{\infty} [\psi(x)]^* \left(-i\frac{d}{dx}\right) \varphi(x)\, dx. \tag{5}$$

The Plancherel theorem [19] gives the following expression for (5):

$$\langle \psi(x)|P\,\varphi(x)\rangle = \int_{-\infty}^{\infty} [\widehat{\psi}(\lambda)]^*\, \lambda\, \widehat{\varphi}(\lambda)\, d\lambda, \tag{6}$$

which is equal to (writing $|\lambda\rangle := |e^{i\lambda x}\rangle$, which implies that $A|\lambda\rangle = \lambda\,|\lambda\rangle$):

$$\langle \psi(x)|P\,\varphi(x)\rangle = \int_{-\infty}^{\infty} \lambda\, \langle \psi|\lambda\rangle\langle \lambda|\varphi\rangle\, d\lambda. \tag{7}$$

The inverse Fourier transform gives[3]

$$\psi(x) = \int_{-\infty}^{\infty} \widehat{\psi}(\lambda)\, e^{i\lambda x}\, d\lambda = \int_{-\infty}^{\infty} \langle e^{i\lambda x}|\psi(x)\rangle\, e^{i\lambda x}\, d\lambda = \int_{-\infty}^{\infty} \langle \lambda|\psi\rangle\, e^{i\lambda x}\, d\lambda. \tag{8}$$

Since λ covers the whole real line, (7) and (8) show the Dirac completitude relation for the momentum operator.

This result can be generalized for observables with a degenerate or singular continuous spectrum.

3. Resonance scattering

Resonance scattering is a model for quantum resonance processes. Resonance scattering implies the existence of two dynamics: a "free" dynamics characterized by the "free" Hamiltonian H_0 and a "perturbed" dynamics, where the perturbation is given by a potential V, so that this dynamics is governed by a "total" Hamiltonian of the form $H := H_0 + V$. To have an intuitive image, we may assume that the potential V is spherically symmetric and vanishes outside a bounded region. Then, a "free" state $\varphi^{\rm in}(t)$ is prepared in the remote past and evolves with H_0 until it reaches the interaction region, where V acts. This state represents a particle. If this particle stays in the interaction region a much higher time than the time it would have stayed if no interaction were present, we say that a quasistationary state or resonance has been created. After a certain time the particle abandons the interaction region (i.e., the resonance decays) and evolves freely again [8].

From this point of view, decaying processes may be looked as the decaying part of a resonance scattering process where the creation of the quasistationary state (or capture process) is ignored [8].

Thus and following reference [8], we assume that resonances are produced in resonance scattering. Resonances are characterized by the resonance energy E_R and the width Γ.

In order to produce a resonance scattering we need two dynamics. The *free* dynamics characterized by H_0 and the *perturbed* dynamics characterized by $H = H_0 + V$, where V is the potential.

Under certain rather general conditions, resonances are characterized by these three equivalent situations [8]:

i.) Large time delay.

ii.) A sudden change of the order of π in the phase shift.

[3] In the Fourier transform and its inverse we have omitted the irrelevant $(2\pi)^{-1/2}$ term.

iii.) A pole of the analytic continuation through the positive semiaxis, from above to below, of the S-operator in the energy representation, $S(E)$, at the point

$$z_R = E_R - i\Gamma/2$$

(or in the language of the Riemann surface on the lower half plane of the second sheet. Note that, in the energy representation, there is another pole at z_R^* on the upper half plane of the second sheet).

3.1. Some general ideas concerning decaying states

The classical decay law for radiactive particles has been established experimentally for most of decaying processes as

$$N(t) = N(0)\exp\{-t/\tau\},$$

where τ is the lifetime of the radiactive particle. Quantum mechanically, we define the *survival probability* of the decaying state $\psi \equiv \psi(0)$ after a time $t > 0$, as

$$P(t) := |A(t)|^2,$$

where

$$A(t) := \langle\psi|e^{-itH}|\psi\rangle.$$

If we have a sample with $N(0)$ particles at the time $t = 0$, after a time t we shall have

$$N(t) = N(0)\,P(t)$$

undecayed particles. In the energy representation, we can write the nondecay probability $P(t)$ as

$$P(t) = \int_{\sigma(H)} e^{-itE}\,|\psi(E)|^2\,dE,$$

where $\psi(E)$ is the wave function for the state ψ in the energy representation. As the density

$$|\psi(E)|^2$$

is integrable, the *Riemann-Lebesgue lemma* [19] says that the following limit exists:

$$\lim_{t\to\infty} P(t) = 0.$$

Now the question can be posed as follows: *is $P(t)$ exponential at all values of time* as in the classical case?

Answer: **NO** [12].

A necessary and sufficient condition for $P(t)$ to have an exponential behaviour for all values of $t > 0$ is that ψ have a Breit Wigner energy distribution. This means that, the square of the modulus of the state ψ in the energy representation is given by

$$|\psi(E)|^2 = \frac{1}{2\pi\tau}\,\frac{1}{(E-E_R)^2 + 1/(2\tau)^2}$$

for $-\infty < E < \infty$. Here E_R is the resonant energy.

This is incompatible with the fact that physically realizable Hamiltonians are, in general, bounded from below, since wave functions in the energy representation must vanish identically below the spectrum of H. Therefore, if a vector state has to represent an exponentially decay state, it cannot belong to the Hilbert space of pure states.

3.2. Gamov vectors and decaying states

Up to the present degree of accuracy, experimental facts tend to confirm the existence of unstable quantum states that decay exponentially, at least in a wide range of values of time. If we want to assign vector states to these unstable quantum systems, we know, after the previous subsection, that these vector states are not normalizable vectors in Hilbert space. Roughly speaking, we can define *Gamov vectors* as state vectors for exponentially decaying resonances. As only normalizable state vectors can be prepared, we conclude that Gamov states cannot be prepared.

Thus, Gamov vectors, $|f\rangle$ should fulfill a relation of the type $e^{-itH}|f\rangle \sim e^{-\Gamma t/2}|f\rangle$ and this idea yielded to define them as eigenvectors of the Hamiltonian, with complex eigenvalues at the points $E_R - i\Gamma/2$ which characterize the resonance [18]. Again, the self adjointness of the Hamiltonian prohibites the existence of Gamov vectors in Hilbert space.

Thus, Gamov vectors lie outside the Hilbert space $\mathcal{H}$. It has been shown that Gamov vectors belong to the antidual space $\mathbf{\Phi}_+^\times$ of a rigged Hilbert space $\mathbf{\Phi}_+ \subset \mathcal{H} \subset \mathbf{\Phi}_+^\times$. With the $\tau_\times$ topology on $\mathbf{\Phi}_+^\times$, the Hilbert space $\mathcal{H}$ is dense in $\mathbf{\Phi}_+^\times$. Thus, we can find in $\mathcal{H}$, and even in $\mathbf{\Phi}$, a normalizable vector ψ (or $|\psi\rangle$) as close to the Gamov vector as we want in the weak topology $\tau_\times$. Then, ψ would represent the (preparable) decaying state. If we represent the Gamov vector as $|f^D\rangle$, we should have:

$$|\psi\rangle = |f^D\rangle + \psi^{\mathrm{BG}}. \tag{9}$$

Neither $|f^D\rangle$ nor ψ^{BG} are normalizable vectors in Hilbert space, for if ψ^{BG} would be normalizable, then $|f^D\rangle$ would have been a linear combination of two normalizable vectors and hence normalizable[4]. Since we are assuming that we know the Hilbert space $\mathcal{H}$, finding the above mentiones RHS is equivalent to finding the space $\mathbf{\Phi}_+$, since then, the antidual $\mathbf{\Phi}_+^\times$ is given automathically. The space $\mathbf{\Phi}_+^\times$ must have the following properties [9]:

i.) $H\mathbf{\Phi}_+ \subset \mathbf{\Phi}_+$.

ii.) The space $\mathbf{\Phi}_+$ reduces the semigroup e^{itH} for $t > 0$, i.e.,

$$e^{itH}\mathbf{\Phi}_+ \subset \mathbf{\Phi}_+ , \qquad \forall\, t > 0.$$

However, for any $t < 0$, we have that $e^{itH}\mathbf{\Phi}_+$ is not contained in $\mathbf{\Phi}_+$ [9].

The space $\mathbf{\Phi}_+$ can be mathematically defined to have this properties with the help of the Hardy functions on a half plane[5].

[4]Observe the difference of roles played by the vectors $|\lambda\rangle = e^{i\lambda x}$ and the Gamov vectors. Any $\psi \in \mathcal{H}$ can be written in terms of a superposition of the eigenvectors of the momentum operator as shown on formula (8) (this is also true for the eigenvectors of any self adjoint operator with eigenvalues in the continous spectrum, as shown by formula (2)). However, no vector in $\mathcal{H}$ can be written as a superposition of Gamov vectors.

[5]Hardy functions on the upper half plane are complex analytic functions $f(z)$ on the *open* upper half plane (Im $z > 0$) such that there exists a positive number K with

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2\, dx < K.$$

As a consequence, the function $f(x)$ given by the boundary values of $f(z)$ on the real axis is square inetgrable with L^2 norm smaller than $\sqrt{K}$. The boundary function $f(x)$ uniquely determines $f(z)$ and viceversa. Thus, Hardy functions on the upper half plane are square integrable functions. Moreover, they form a (closed) subspace of $L^2(R)$ and hence a Hilbert space, that we denote as $\mathcal{H}_+^2$. Hardy functions are defined analogously on the lower half plane. The space of Hardy functions on the lower half plane is denoted as $\mathcal{H}_-^2$. We have that $L^2(R) = \mathcal{H}_+^2 \oplus \mathcal{H}_-^2$. For the properties of Hardy functions, see [15].

The precise construction of $\mathbf{\Phi}_+$ is given in the next footnote. Nevertheless can be sketchet as follows [9]: Let V be the unitary operator that diagonalizes H (that we assume to have purely nondegenerate continuous spectrum equal to $R^+ := [0, \infty)$). Let S be the Schwartz space, defined in section 2. Consider the intersection $S \cap \mathcal{H}_+^2$. This is a space of complex functions on the real axis R, which is a subspace of the Hilbert space of square integrable functions

The corresponding properties for the antidual $\mathbf{\Phi}_+^\times$ can be obtained with the help of the duality formulas:

$$\langle H\psi|F\rangle = \langle\psi|HF\rangle \quad \text{and} \quad \langle e^{itH}\,\psi|F\rangle = \langle\psi|e^{-itH}\,F\rangle\,, \tag{10}$$

that extend both H and the evolution operator into the antiduals. In particular, we have that:

i.) $H\mathbf{\Phi}_+^\times \subset \mathbf{\Phi}_+^\times$.

ii.) Time evolution on $\mathbf{\Phi}_+^\times$ is defined if and only if $t > 0$ and then,

$$e^{-itH}\,\mathbf{\Phi}_+^\times \subset \mathbf{\Phi}_+^\times. \tag{11}$$

This is an obvious consequence of the property ii.) for $\mathbf{\Phi}_+$ and the duality formula (10).

3.3. The Gamov vectors

We have defined heuristically the Gamov vectors as state vectors that decay exponentially. A formal definition of this concept requires some mathematical preparation and is introduced in [9][6]. In this presentation, we shall restrict ourselves in giving their most characteristic properties. These are:

i.) The Gamov vector $|f^D\rangle$ is in $\mathbf{\Phi}_+^\times$.

ii.) The Gamov vector $|f^D\rangle$ is an eigenvector of H with complex eigenvalue:

$$H\,|f^D\rangle = \left(E_R - i\frac{\Gamma}{2}\right)\,|f^D\rangle\,, \tag{12}$$

where E_R represents the resonant energy and Γ the inverse of the mean life.

iii.) The Gamov vector decays exponentially for all positive times:

$$e^{-itH}\,|f^D\rangle = e^{-itE_R}\,e^{-\Gamma t}\,|f^D\rangle\,, \quad \forall\, t > 0. \tag{13}$$

---

on R, $L^2(R)$. The space of the restrictions of the functions in $S\cap\mathcal{H}_+^2$ to the positive real axis R^+, is denoted here as $S\cap\mathcal{H}_+^2\big|_{R^+}$. One can prove that $S\cap\mathcal{H}_+^2\big|_{R^+}$ is dense in $L^2(R^+)$ and that VHV^{-1} is the multiplication operator on $L^2(R^+)$. Properties i.) and ii.) hold for VHV^{-1} on $S\cap\mathcal{H}_+^2\big|_{R^+}$. Therefore, it is natural to define

$$\mathbf{\Phi}_+ := V^{-1}\left[S\cap\mathcal{H}_+^2\big|_{R^+}\right].$$

This space $\mathbf{\Phi}_+$ fulfill the required properties.

[6] If resonances are produced in resonance scattering, we have two Hamiltonians, the free Hamiltonian H_0 and the perturbed Hamiltonian H. We assume that the Møller wave operators $\Omega_- := \Omega_{IN}$ and $\Omega_+ := \Omega_{OUT}$ exists and are asymptotically complete [1]. We further assume that the absolutely continuous spectrum of H_0 and H is nondegenerate and that coincides with the positive semiaxis.

Then, there exists a unitary operator U that diagonalizes H_0. This means that $U:\mathcal{H}\longmapsto L^2(R^+)$ and that UH_0U^{-1} is the multiplication operator on $L^2(R^+)$. Take, $V_\pm := U\Omega_\pm^{-1}$. Then, both $V_\pm HV_\pm^{-1}$ represent the multiplication operator on $L^2(R^+)$ and hence, diagonalize H.

Now define:

$$\mathbf{\Phi}_+ := V_+^{-1}\left[S\cap\mathcal{H}_+^2\big|_{R^+}\right] \quad \text{and} \quad \mathbf{\Phi}_- := V_-^{-1}\left[S\cap\mathcal{H}_-^2\big|_{R^+}\right]$$

Take, $\varphi_\pm \in \mathbf{\Phi}_\pm$ and assume that $z_R = E_R - i\Gamma/2$ is the resonance pole for the resonance under our consideration. Take $\varphi_\pm(E) := V_\pm\varphi_\pm$ for any $\varphi_\pm \in \mathbf{\Phi}_\pm$. The function $\varphi_\pm(E)$ is in $S\cap\mathcal{H}_\pm^2\big|_{R^+}$. Then, we define

$$\langle\varphi_+|f^D\rangle := [\varphi_+(z_R^*)]^* \quad \text{and} \quad \langle\varphi_-|f^G\rangle := [\varphi_-(z_R)]^*$$

where the star denotes the complex conjugate. This is the formal definition for the Gamov vectors corresponding to a resonance represented as a sigle pole of the analytic continuation of the S-matrix [8].

However, for negative times, this time evolution is not even defined. Thus, the Gamov vector $|f^D\rangle$ admits a semigroup time evolution only[7]. This semigroup evolution can be considered as the mathematical implementation of the idea that quantum systems with resonances are *irreversible systems* [8] in the sense that the evolution to the future is separated from the evolution to the past. This interpretation has been advocated by the Brussels school [20, 5].

iv.) The background decays slower than exponential

$$e^{-itH}\,\psi^{\rm BG} \longrightarrow 0 \quad \text{as} \quad t \to 0$$

and $\psi^{\rm BG} \in \mathbf{\Phi}_+^\times$.

Along the decaying Gamov vectors, there exists their time reversal, also called the growing Gamov vectors. The growing Gamov vectors belong to the antidual, $\mathbf{\Phi}_-^\times$, of a RHS $\mathbf{\Phi}_- \subset \mathcal{H} \subset \mathbf{\Phi}_-^\times$, which is the time reversal of $\mathbf{\Phi}_+ \subset \mathcal{H} \subset \mathbf{\Phi}_+^\times$. Now, the state $|\psi\rangle$ in (9) admits the following decomposition:

$$|\psi\rangle = |f^G\rangle + \psi^{\rm BG*}, \tag{14}$$

where

i.) The vectors $|f^G\rangle$ and $\psi^{\rm BG*}$ are in $\mathbf{\Phi}_-^\times$.

ii.) $H\,|f^G\rangle = (E_R + i\Gamma/2)\,|f^G\rangle$.

iii.) The Gamov vector grows exponentially up to $t = 0$:

$$e^{-itH}\,|f^G\rangle = e^{-itE_R}\,e^{\Gamma t}\,|f^G\rangle\,, \qquad t < 0$$

iv.) The background $\psi^{\rm BG*}$ also grows up to $t = 0$.

v.) If T is the time reversal operator, we have:

$$T|f^D\rangle = |f^G\rangle\,, \quad T|f^G\rangle = |f^D\rangle\,, \quad T\psi^{\rm BG} = \psi^{\rm BG*}\,, \quad T\mathbf{\Phi}_\pm = \mathbf{\Phi}_\mp\,, \quad T\mathbf{\Phi}_\pm^\times = \mathbf{\Phi}_\mp^\times.$$

The space $\mathbf{\Phi}_-$ is constructed exactly as $\mathbf{\Phi}_+$ by replacing Hardy functions on the upper half plane by Hardy functions on the lower half plane and $\mathbf{\Omega}_+$ by $\mathbf{\Omega}_-$. See footnotes 6 and 7.

4. Example: The Friedrichs model

The simplest form of the Friedrichs model includes a free Hamiltonian H_0 with a simple continuous spectrum, which is $R^+ \equiv [0, \infty)$, plus an eigenvalue, ω_0, imbedded in this continuous spectrum ($\omega_0 > 0$). Then, H_0 reads:

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega|\omega\rangle\langle\omega|\,d\omega \tag{15}$$

with

$$H_0\,|\omega\rangle = \omega|\omega\rangle, \quad \omega \in\ R^+ \quad \text{and} \quad H_0|1\rangle = \omega_0\,|1\rangle.$$

---

[7]Unfortunately, this semigroup evolution depends on the definition of $\mathbf{\Phi}_+$, which relies on the properties of Hardy functions. Should we define $\mathbf{\Phi}_+$ differently, without the use of Hardy functions and the semigroup decay law not longer holds. For instance, take $\mathcal{D}(R)$, the space of indefinitely differentiable functions from R into C with compact support. Take the Fourier transform of these functions and we get a space $\Delta := \mathcal{F}(\mathcal{D}(R))$. Then, we can define

$$\mathbf{\Phi}_+ := \Delta\big|_{R^+}.$$

This space has the property that $e^{itH}\mathbf{\Phi}_+ \subset \mathbf{\Phi}_+$ for all real values of t and, therefore, we have that $e^{-itH}\mathbf{\Phi}_+^\times \subset \mathbf{\Phi}_+^\times$. With this choice, Gamov vectors are defined as above.

[8]For a discussion of this matter from the physical point of view, see [10, 16].

The perturbation is given by

$$V = \int_0^\infty (f^*(\omega)\,|\omega\rangle\langle 1| + f(\omega)\,|1\rangle\langle\omega|)\,d\omega, \tag{16}$$

where $f(\omega)$ is a complex function on the positive semiaxis R^+. The total Hamiltonian is $H = H_0+V$. An arbitrary vector in the Hilbert space $(L^2(R^+) \oplus C^2)$ is given by

$$\psi = \alpha|1\rangle + \int_0^\infty \varphi(\omega)\,|\omega\rangle\,d\omega\,, \quad \text{with} \quad \varphi(\omega) \in L^2(R^+).$$

When the interaction is switched on, the eigenvalue $|1\rangle$ is *dissolved* into the continuum and a resonance is produced. In order to obtain this resonance and its corresponding Gamov vector, we consider the *reduced resolvent*:

$$|1\rangle\langle 1|\frac{1}{z-H}|1\rangle\langle 1| = \frac{1}{\eta(z)}\,|1\rangle\langle 1|$$

with [11]

$$\eta(z) = z - \omega_0 - \lambda^2 \int_0^\infty \frac{|f(\omega)|^2}{z-\omega}\,d\omega.$$

Under certain conditions, $\eta(z)$ is a complex analytic function with no singularities on the complex plane other than a branch cut coinciding with the positive semiaxis R^+ provided that $\eta(0) > 0$. It admits respective analytic continuations through the cut from above to below (from the upper to the lower half plane), $\eta_+(z)$, and from below to above (from the lower to the upper half plane), $\eta_-(z)$. The continuation $\eta_+(z)$ has a *simple* zero at z_0 with $\text{Im}\{z_0\} < 0$, which is an analytic function on the coupling parameter λ on a neighborhood of zero. Analogously, $\eta_-(z)$ admits a *simple* zero at z_0^*, which is also an analytic function of λ. Obviously, these zeroes are simple poles of the reduced resolvent.

It is important to remark that the Møller wave operators, and therefore the S-matrix, do exists in the Friedrichs model. Therefore, the S-matrix in the energy representation, i.e., the function $S(E)$ (we recall that in the Friedrichs model the spectrum of H is simple), exists. Its analytic continuations through R^+, both from above to below and from below to above, do exists and have simple poles at z_0 and z_0^* respectively. Thus, poles on the continuation of the reduced resolvent and poles on the continuation of $S(E)$ coincide [4].

Choice of the RHS. We have two possibilities:

$$\mathbf{\Phi}_\pm := C \oplus \left(S \cap \mathcal{H}^2_\pm \Big|_{R^+} \right),$$

so that

$$C \oplus \left(S \cap \mathcal{H}^2_\pm \Big|_{R^+} \right) \subset C \oplus L^2(R^+) \subset C \oplus \left(S \cap \mathcal{H}^2_\pm \Big|_{R^+} \right)^\times$$

is a RHS, where

$$\mathbf{\Phi}^\times_\pm := C \oplus \left(S \cap \mathcal{H}^2_\pm \Big|_{R^+} \right)^\times,$$

where C is the set of complex numbers. The form factor $f(\omega)$ is chosen so that $H = H_0 + V$ is an operator on $\mathbf{\Phi}^\times_\pm$.

Remark. Although we have considered here the existence of simple resonance poles only, it is importsnt to remark that for certain choices of the form factor the multiplicity of z_0 and z_0^* may change, although both must have the same multiplicity. See [2] for double pole resonances in a Friedrichs model.

4.1. The Gamov vectors

To obtain the Gamow vector [2], consider the following eigenvalue equation valid for any $x > 0$:

$$(H - x)\,\mathbf{\Psi}(x) = 0, \tag{17}$$

where $\mathbf{\Psi}(x)$ is the eigenvector of H for the eigenvalue x. As the eigenvectors $|1\rangle$ and $|\omega\rangle$ form a complete system, then,

$$\mathbf{\Psi}(x) = \psi(x)\,|1\rangle + \int_0^\infty \psi(x,\omega)\,|\omega\rangle\,d\omega. \tag{18}$$

If we carry (18) into (17), we obtain the following system of two equations:

$$(\omega_0 - x)\psi(\omega) + \lambda \int_0^\infty \psi(x,\omega)\,f^*(\omega)\,d\omega \quad = \quad 0 \tag{19}$$

$$(\omega - x)\,\psi(x,\omega) + \lambda f(\omega)\,\psi(\omega) \quad = \quad 0. \tag{20}$$

To solve this system, we write $\psi(\omega)$ in terms of $\psi(x,\omega)$ using (20) and carry the result to (19). We obtain an integral equation, which gives as a solution:

$$\mathbf{\Psi}_+(x) = |x\rangle + \lambda f^*(x)\,\frac{1}{\eta_+(x)}\left\{|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{x - \omega + i0}\,|\omega\rangle\,d\omega\right\}. \tag{21}$$

This is a functional in $\mathbf{\Phi}_+^\times$. When applied to a vector in $\mathbf{\Phi}_+$ gives an analytic function on the lower half plane. We say that $\mathbf{\Psi}_+(x)$ admits analytic continuation to the lower half plane (in a weak sense). This continuation has a simple pole at z_0 so that we can write on a neighborhood of z_0:

$$\mathbf{\Psi}_+(z) = \frac{C}{z - z_0} + o(z). \tag{22}$$

From (17) and (22), we get:

$$0 = (H - z)\mathbf{\Psi}_+(z) = \frac{1}{z - z_0}\,(H - z)\,C + (H - z)\,o(z), \tag{23}$$

which gives

$$(H - z_0)\,C = 0 \Longrightarrow HC = z_0 C.$$

Now, to calculate the Gamow vector we get the residue of (22) at z_0. This is simple, since on a neighborhood of z_0, we have:

$$\mathbf{\Psi}_+(z) \approx \frac{\text{constant}}{(\mathrm{z} - \mathrm{z}_0)}\left\{|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z - \omega + i0}\,|\omega\rangle\,d\omega\right\} + \mathrm{RT}, \tag{24}$$

where RT means "regular terms". Since

$$\frac{1}{z - \omega + i0} = \frac{1}{z_0 - \omega + i0} - \frac{z - z_0}{(z_0 - \omega + i0)^2} + o(z)$$

we have that

$$\mathbf{\Psi}_+(z) \approx \frac{\text{constant}}{(\mathrm{z} - \mathrm{z}_0)}\left\{|1\rangle + \lambda \int_0^\infty \frac{f(\omega)}{z_0 - \omega + i0}\,|\omega\rangle\,d\omega\right\} + \mathrm{RT}. \tag{25}$$

Therefore, save for an irrelevant constant, we have that the decaying Gamov vector is

$$C = |f^D\rangle = |1\rangle + \int_0^\infty \frac{\lambda\,f(\omega)}{z_0 - \omega + i0}\,|\omega\rangle\,d\omega. \tag{26}$$

Analogously, the growing Gamov vector is

$$|f^G\rangle = |1\rangle + \int_0^\infty \frac{\lambda\, f^*(\omega)}{z_0^* - \omega - i0}\, |\omega\rangle\, d\omega. \tag{27}$$

Therefore, we have obtained the Gamov vectors for the Friedrichs model with simple resonance poles[9]. Of course $|f^D\rangle \in \mathbf{\Phi}_+^\times$ and $|f^G\rangle \in \mathbf{\Phi}_-^\times$.

Acknowledgements

We want to express our gratitude to the organizers of the *XXIV International Workshop on the Fundamental Problems of High Energy Physics and Field Theory* in Protvino for having given us the opportunity of contribute to the workshop. We are particularly indebt to Prof. G.P. Pronko for his nice hospitality in Protvino and his continuous interest in the present research.
Partial finantial support is acknowledge to Junta de Castilla y León Project PC02/99, DGCYT PB98-0370 and DGICYT PB98-0370.

References

[1] W.O. Amrein, J. Jauch and B. Sinha, *Scattering Theory in Quantum Mechanics*, Benjamin, Reading MA (1977).

[2] I.E. Antoniou, M. Gadella, G.P. Pronko, J. Math. Phys., **39**, 2459-2475 (1998).

[3] I.E. Antoniou, M. Gadella, Z. Suchanecki, *Properties of the Liouville Operator* in *Irreversibility and Causality*, A. Bohm, H-D Doebner, P. Kielanowski, Eds., Springer Lecture Notes in Physics vol **504**, Springer Verlag, Berlin and New York (1998).

[4] I.E. Antoniou, Yu. Melnikov, *Quantum scattering resonances: Poles of a continued S-matrix and poles of an extended resolvent* in *Irreversibility and Causality. Semigroups and Rigged Hilbert Spaces.* A. Bohm, H-D Doebner and P. Kielanowski, Eds. Springer Lecture Notes in Physics vol. **504**. Springer Verlag, Berlin and New York (1998).

[5] I.E. Antoniou, I. Prigogine, Physica, **192A**, 443 (1993).

[6] I.E. Antoniou, S. Tasaki, Int. J. Quant. Chem., **44**, 425 (1993); Physica **190A**, 303 (1992).

[7] N.N. Bogolubov, A.A. Logunov, I.T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, Bejamin (1975).

[8] A. Bohm, *Quantum Mechanics. Foundations and Applications*, Springer Verlag, Berlin and New York (1993).

[9] A. Bohm, M. Gadella, *Dirac kets, Gamow vectors and Gelfand triplets*, Springer Lecture Notes in Physics, vol 348. Springer Verlag, Berlin and New York (1989).

[10] C. Cohen-Tannoudji, F. Diu, F. Laloe, *Quantum Mechanics.* French Edition: Hermann, Paris (1973).

[11] P. Exner, *Open Quantum Systems and Feynman Integrals*, Reidel, Dordrech (1985).

[9] We want to add here that resonance poles may be of any finite order and that a Friedrichs model with double poles have been defined in [2].

[12] L. Fonda, G.C. Ghirardi, A. Rimini, Rep. Prog. Phys., **41**, 587 (1978).

[13] G. Gamov, Z. Phys., **51**, 204 (1928).

[14] I.M. Gelfand and G.E. Shilov, *Generalized Functions*, vol I, Academic Press, New York (1964). I.M. Gelfand, N. Y. Vilenkin, *Generalized Functions*, vol IV, Academic Press, New York (1964).

[15] K. Hoffmann, *Banach Spaces of Analytic Functions*, (Prentice Hall, New Jersey, 1962). P. Koosis, *Introduction to H^p Spaces*, (Cambridge, UK, 1980). P. Koosis, *The Logaritmic Integral*, (Cambridge, UK, 1990).

[16] T.D. Lee, *Particle Physics and Introduction to Field Theory*, Harwood Academic, New York (1981).

[17] K. Maurin, *General Eigenfunction Expansions and Unitary Representations of Topological Groups*, Polish Scientific Publishers, Warsawa (1968).

[18] N. Nakanishi, Progr. theor. Phys., **19**, 607 (1958).

[19] M. Reed, B. Simon, *Fourier Analysis, Self Adjointness*, Academic Press, New York (1975).

[20] I. Prigogine, *From Being to Becoming*, Freeman, New York (1980).

[21] T. Petroski, I. Prigogine, Physica, **175A**, 146 (1991); T. Petroski, I. Prigogine, S. Tasaki, Physica, **173A**, 175 (1991).