# Time Symmetry Breaking and Stochasticity in Hamiltonian Physics

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Experience shows us that irreversible processes should be part of our description of nature. Still it is often stated that these aspects are due to approximations. We have followed an opposite direction and tried to relate these properties to dynamics. According to Poincaré, we have to distinguish integrable systems from non-integrable systems. Integrable systems may be diagonalized through a unitary operator we shall call $U$. $U$ is a unitary operator. In contrast we have non-integrable dynamical systems. Our knowldege of non-integrable systems remains quite limited. In this paper, we shall consider the class of non-integrable dynamical systems where the unitary operator $U$ admits an analytical continuation we call $\Lambda$. $\Lambda$ is a star unitary operator as defined in the text. The unitary operator leads to well-defined units described by the so-called action variables. In classical dynamics, $\Lambda$ leads to a new type of unit which are of the type described by kinetic theory. They interact through collision operators and cross sections. The aim of this paper is to describe the transition from $U$ to $\Lambda$. This needs a convenient representation of $U$. In our earlier work we have introduced so-called kinetic operators such as creation operators, destruction operators, and so on. We have recently shown that the unitary operator, $U$ can be rigorously expressed in terms of these operators. The important point is that these operators admit analytic continuations. The mathematical properties of $U$ and $\Lambda$ are quite different. The main difference is that $U$ is distributive, $UAB = (UA)(UB)$ while $\Lambda$ is not distributive $\Lambda AB \neq (\Lambda A)(\Lambda B)$. This introduces fluctuations both in quantum and classical theory. In short, the two aspects, irreversibilty and stochasticity are both associated with non-integrability. This approach leads to a wide number of possible applications, such as the introduction of dressed excited states in quantum mechanics, interacting fields, dynamical foundation of stochasticity and derivation of white noise. Many of these aspects are treated in other papers in print.

## 1 Introduction

It is well known that classical dynamics and quantum mechanics leads to time reversible and deterministic laws. Still in many fields we discover situations where this picture is not applicable. There are two obvious examples: Kinetic theory (or non-equilibrium statistical mechanics) and thermodynamics. Kinetic theory deals with probabilities. As in thermodynamics, it includes a broken time symmetry. You find in many books that kinetic theory and thermodynamics are based on approximations introduced in Hamiltonian dynamics. But that is difficult to accept. Indeed to quote only one example, kinetic theory as developed by Boltzmann and others leads to predictions of transport coefficients for dilute gases, which are in complete agreement with observation. Also non-equilibrium thermodynamics leads to the prediction of coherent structures, which are in quantitative agreement with experiment. For this reason, our group has always been interested to formulate dynamics in such a way that probabilities and time symmetry breaking are included in the microscopic description.

Classical or quantum integrable systems are generally formulated in the Hilbert Space formalism. For such systems, dynamics can be reduced to a set of non-interacting modes by a canonical or

unitary transformation. On the other hand, we know since Poincaré, that most systems are non-integrable. As we shall see, these systems break time symmetry and cannot be described by unitary evolution in the Hilbert Space. Integrable systems and non-integrable systems seem to obey quite different laws. But we have now obtained a unified formulation of dynamics, which applies to both the integrable systems for which there exists a unitary operator $U$ leading to a diagonalization of the Hamiltonian, as well as to non-integrable systems outside the Hilbert Space. As we shall see, this involves the construction of a non-unitary transformation operator $\Lambda$ which corresponds to an analytic extension of $U$. We see therefore that kinetic theory and thermodynamics, far from based on any "falsification" of dynamics, are well-defined extensions of classical or quantum dynamics. For reasons we shall explain later, our form of dynamics can be called a dynamics of "correlations."

Many years ago, in a monograph [1], one of the authors has introduced the idea of "dynamics of correlations." The idea of dynamics of correlations is based on operators which changes the degree of correlation as defined later in this paper. For integrable systems, we shall show that the unitary operator $U$ can be expressed rigorously in terms of the kinetic operators. For non-integrable systems we can now formulate dynamics in terms of the analytic continuation of the kinetic operators. In this paper, we shall try to give a simple introduction to the physical ideas. Calculations can be followed in the original papers [2, 3, 4, 5, 6].

## 2 Poincaré's theorem

We consider systems with Hamiltonian

$$H = H_0 + \lambda V, \tag{1}$$

where $H_0$ is the unperturbed Hamiltonian describing non-interacting particles, and $V$ is the interaction. We assume the coupling constant $\lambda$ is dimensionless. For integrable systems, Poincaré has shown that the invariants associated with $H_0$ can be extended to $H$. The interaction $V$ can be eliminated. But Poincaré has also shown that for most classical systems there appear divergences in the construction of invariants of motion. We call these systems "non-integrable in the sense of Poincaré". More precisely, the divergences occur in the perturbation expansion (i.e. expansion in $\lambda^n$ with $n \geq 0$) of invariants of motion other than functions of the Hamiltonian. The divergences are due to vanishing denominators, which occur when the frequencies of the system obey relations called Poincaré resonances. Then the interactions cannot be eliminated by unitary (or canonical) analytical transformations.

An essential condition for the appearance of Poincaré resonances is that the frequencies are continuous functions of the momenta. This implies that quantum systems in a finite volume are integrable as they have discrete spectra. However, the situation changes when we consider systems in the limit of infinite volume [1]. Then we have a continuous spectrum and resonances.

We can still deal with the vanishing denominators if we interpret them as distributions, for example

$$\frac{1}{w} \Rightarrow \frac{1}{w \pm i\epsilon} = \mathcal{P}\frac{1}{w} \mp \pi i \delta(w), \tag{2}$$

where $\epsilon > 0$ is an infinitesimal. We shall come back later to the choice of $\mp i\epsilon$, which is crucial to obtain well-defined perturbation expansions. For such situations, the non-unitary transformation $\Lambda$ leads to new units that cannot be reduced to trajectory or wave-function descriptions.

Once the regularization of Poincaré's divergences is achieved, we find two unexpected new elements: the breaking of time symmetry and the appearance of stochasticity. We come to new units

[1] This avoids the introduction of boundary conditions. We never have isolated dynamical systems as correlations cross the boundaries.

or modes, which are no more invariants. They obey irreversible kinetic processes describing their mutual interactions.

The basic problem of classical mechanics or quantum mechanics is the diagonalization of $H$. This corresponds to the introduction of a unitary operator $U$, such that $H$ is diagonal. For integrable systems, we can always diagonalize $H$. For non-integrable systems, this is generally not possible. Our interest is in non-integrable systems.

We study transformation theory on the level of probability or distribution functions $\rho$ in the Liouville-von Neumann space. As is well-known we have

$$i\frac{\partial}{\partial t}\rho = L_H\rho, \qquad L_H \equiv [H,\ ] \tag{3}$$

in units with $\hbar = 1$. $L_H$ is a "superoperator" (in quantum mechanics, it acts on the density operator $\rho$). For integrable systems we may introduce $\tilde{\rho} = U\rho$. Then $H$ is diagonal and the evolution of $\rho$ is trivial. For non-integrable systems, we shall write $\tilde{\rho} = \Lambda\rho$. This leads to interacting units with broken time symmetry as they appear in kinetic theory. To introduce $\Lambda$ we need continous spectrum, however even for a discrete spectrum, our conclusions are valid for limited time. Since $L_H$ is a hermitian operator, the equation (3) has only real eigenvalues in the Hilbert Space for $\rho$. This is no more so for dissipative processes. Analytic continuation lead outside the Hilbert space [7]. Therefore the formulation of dynamics for non-integrable systems involves an extension of the statistical description outside the Hilbert space.

The idea that interactions may kick the system out from the Hilbert space was already introduced by Dirac in the frame of field theory [8].

## 3 Extension of unitary transformations

In order to have a common formulation for both integrable and non-integrable systems, we study the dynamics in the Liouville space. We shall consider quantum mechanics. The time evolution is given by the Liouville-von Neumann equation (3). As in Eq. (1) we have $L_H = L_0 + \lambda L_V$.

Let us consider first the case of non-interacting particles, with $\lambda = 0$. The unperturbed Hamiltonian $H_0$ has a complete set of eigenstates,

$$H_0|\alpha\rangle = \omega_\alpha|\alpha\rangle, \tag{4}$$

$$\langle\alpha'|\alpha\rangle = \delta_{\alpha'\alpha}, \qquad \sum_\alpha |\alpha\rangle\langle\alpha| = 1. \tag{5}$$

The eigenstates of $L_0$ are dyads of eigenstates of $H_0$ and its eigenvalues are differences of eigenvalues of $H_0$,

$$L_0|\alpha;\alpha'\rangle\rangle = (\omega_\alpha - \omega'_\alpha)|\alpha;\alpha'\rangle\rangle. \tag{6}$$

We have represented dyadic operators as $|\alpha;\alpha'\rangle\rangle \equiv |\alpha\rangle\langle\alpha'|$. These operators and their duals $\langle\langle\alpha;\alpha'| = |\alpha'\rangle\langle\alpha|$ form a Hilbert space with inner product

$$\langle\langle B|A\rangle\rangle = \mathrm{Tr}(B^\dagger A). \tag{7}$$

We decompose the density operator $\rho$ into independent components

$$\rho = \sum_\nu P^{(\nu)}\rho, \tag{8}$$

where $P^{(\nu)}$ are projectors to the orthogonal eigenspaces of $L_0$. We have

$$L_0P^{(\nu)} = P^{(\nu)}L_0 = w^{(\nu)}P^{(\nu)}, \tag{9}$$

where $w^{(\nu)}$ are the real eigenvalues of $L_0$ in Eq. (6). For example, if $\alpha$ is a one-dimensional variable, the projection operators are written explicitly as

$$\begin{aligned} P^{(0)} &\equiv \sum_{\alpha} |\alpha;\alpha\rangle\rangle\langle\langle\alpha;\alpha|, \\ P^{(\alpha\beta)} &\equiv |\alpha;\beta\rangle\rangle\langle\langle\alpha;\beta| \qquad (\alpha \neq \beta). \end{aligned} \tag{10}$$

They are orthogonal and complete:

$$P^{(\mu)}P^{(\nu)} = P^{(\mu)}\delta_{\mu\nu}, \qquad \sum_{\nu} P^{(\nu)} = 1 \tag{11}$$

with $(\nu) = (0)$ or $(\alpha\beta)$. We have

$$w^{(0)} = 0, \qquad w^{(\alpha\beta)} = \omega_\alpha - \omega_\beta. \tag{12}$$

The unperturbed Liouville equation is now decomposed into a set of independent equations,

$$i\frac{\partial}{\partial t}P^{(\nu)}\rho = w^{(\nu)}P^{(\nu)}\rho. \tag{13}$$

We associate the diagonal component of $\rho$ (in the basis eigenstates of $H_0$) with $\nu = 0$. We have $w^{(0)} = 0$, i.e., the diagonal density matrices are invariants of motion in the unperturbed case. The off-diagonal components with $\nu \neq 0$ simply oscillate with frequencies $w^{(\nu)}$.

Next we consider the interacting case, with $\lambda \neq 0$. We first assume that the system is integrable in the sense of Poincaré. This means that we can construct (by perturbation expansion or otherwise) a superoperator $U$ that puts the dynamics into the same form as in the unperturbed case. We have

$$i\frac{\partial}{\partial t}U\rho = (UL_HU^{-1})U\rho. \tag{14}$$

This corresponds to a change of representation,

$$\rho \Rightarrow \bar{\rho} \equiv U\rho, \qquad L_H \Rightarrow \bar{\Theta} \equiv UL_HU^{-1} \tag{15}$$

(hereafter we use bars to denote operators defined for integrable systems). The transformed Liouville operator $\bar{\Theta}$ is diagonal in the unperturbed basis, i.e., we have

$$\bar{\Theta}P^{(\nu)} = P^{(\nu)}\bar{\Theta} = \bar{\theta}^{(\nu)}, \tag{16}$$

where

$$\bar{\theta}^{(\nu)} = \bar{w}^{(\nu)}P^{(\nu)} \tag{17}$$

and $\bar{w}^{(\nu)}$ are the real eigenvalues of $\bar{\Theta}$, corresponding to $w^{(\nu)}$ shifted by the interaction. As a consequence of the change of representation, dynamics is reduced to the set of equations

$$i\frac{\partial}{\partial t}P^{(\nu)}\bar{\rho} = \bar{w}^{(\nu)}P^{(\nu)}\bar{\rho}, \tag{18}$$

which are analogous to Eq. (13).

For the integrable case the problem of diagonalization of $L_H$ is reducible to the problem of diagonalization of the Hamiltonian $H$. If we have a complete set of eigenstates of $H$.

$$H|\bar{\phi}_\alpha\rangle = \bar{\omega}_\alpha|\bar{\phi}_\alpha\rangle. \tag{19}$$

$$\langle\bar{\phi}'_\alpha|\bar{\phi}_\alpha\rangle = \delta_{\alpha'\alpha}, \qquad \sum_\alpha |\bar{\phi}_\alpha\rangle\langle\bar{\phi}_\alpha| = 1, \tag{20}$$

then, analogous to the unperturbed case, the eigenstates eigenstates of $L_H$ are dyads of the eigenstates of $H$ and its eigenvalues are differences of eigenvalues of $H$,

$$L_H|\bar{\phi}_\alpha;\bar{\phi}'_\alpha\rangle\rangle = (\bar{\omega}_\alpha - \bar{\omega}'_\alpha)|\bar{\phi}_\alpha;\bar{\phi}'_\alpha\rangle\rangle. \tag{21}$$

We have

$$\bar{w}^{(0)} \equiv 0, \qquad \bar{w}^{(\alpha\alpha')} \equiv \bar{\omega}_\alpha - \bar{\omega}_{\alpha'}. \tag{22}$$

This means that $U$ is factorizable as

$$U = u \times u^{-1}, \tag{23}$$

where $u$ is the transformation that diagonalizes $H$ (we use the notation $(A \times B)\rho = A\rho B$ to denote factorizable superoperators). We have

$$|\bar{\phi}_\alpha\rangle = u^{-1}|\alpha\rangle, \qquad |\bar{\phi}_\alpha;\bar{\phi}_{\alpha'}\rangle\rangle = U^{-1}|\alpha;\alpha'\rangle\rangle. \tag{24}$$

The states with $\nu = 0$ correspond to the perturbed invariants of motion with $\bar{w}^{(0)} = 0$. Eq. (24) with $\alpha = \alpha'$ shows that for integrable systems there is a one-to-one correspondence (through the transformation $U$) between the unperturbed and the perturbed invariants of motion.

The idea we have followed since long is that we need an extension of $U$ to describe non-integrable dynamical systems. Indeed in the non-integrable case we can no more construct $U$, due to Poincaré's resonances. However, as mentioned before, we can extend the construction of $U$ on the level of distribution functions if we interpret the denominators as distributions with suitable analytic continuations. We shall present an example below.

Our construction leads to a non-unitary operator $\Lambda$, which is an extension of $U$ to non-integrable systems. We have now the transformations (see Eq. (15))

$$\rho \Rightarrow \tilde{\rho} = \Lambda\rho, \qquad L_H \Rightarrow \tilde{\Theta} \equiv \Lambda L_H \Lambda^{-1}. \tag{25}$$

The transformed Liouville equation is then

$$i\frac{\partial}{\partial t}\tilde{\rho} = \tilde{\Theta}\tilde{\rho}. \tag{26}$$

The transformed Liouvillian $\tilde{\Theta}$ (now called the "collision operator" in kinetic theory) is no more diagonal in the basis of projectors $P^{(\nu)}$ (this would bring us back to the integrable case; see Eq. (17)). However, we keep the commutation with the unperturbed projectors, as

$$\tilde{\Theta}P^{(\nu)} = P^{(\nu)}\tilde{\Theta} \equiv \tilde{\theta}^{(\nu)}. \tag{27}$$

This means that $\tilde{\Theta}$ is block-diagonal in the basis of projectors $P^{(\nu)}$. It leads to transitions *inside* each $P^{(\nu)}$ subspace.

We can introduce the complete set of $P^{(\nu)}$ projectors and express $\Lambda$ and $\tilde{\Theta}$ as,

$$\Lambda = \sum_\nu P^{(\nu)}\Lambda, \qquad \tilde{\Theta} = \sum_\nu P^{(\nu)}\tilde{\Theta} = \sum_\nu \tilde{\theta}^{(\nu)}. \tag{28}$$

Let us note that Eq. (27) can be written as

$$L_H\Pi^{(\nu)} = \Pi^{(\nu)}L_H, \tag{29}$$

where

$$\Pi^{(\nu)} = \Lambda^{-1} P^{(\nu)} \Lambda. \tag{30}$$

The projectors $\Pi^{(\nu)}$ have been used extensively in our approach [9]. The dynamics (includeing interactions) is decomposed into a set of independent "subdynamics".

Instead a set of equations with only invariant or oscillating solutions (as was the case for Eq. (18)), we obtain now, as will be shown, a set of Markovian kinetic equations

$$i\frac{\partial}{\partial t} P^{(\nu)} \tilde{\rho} = \tilde{\theta}^{(\nu)} P^{(\nu)} \tilde{\rho}. \tag{31}$$

There is an important point: in the construction of $\Lambda$ (instead of $U$) time-symmetry is broken as a consequence of the analytic continuation of the denominators. As a consequence, the operator $\tilde{\Theta}$ is not hermitian and leads to complex eigenvalues. It becomes a dissipative operator that describes dissipative processes such as decay or diffusion. To lowest order in the coupling constant it reduces to Pauli's collision operator of quantum mechanics. In classical mechanics it leads to the Fokker-Planck equation. From the similitude relation (25) of the operator $\tilde{\Theta}$ with the Liouville operator, we see that $L_H$ has the same (complex) eigenvalues as $\tilde{\Theta}$. This is only possible if the domain of $L_H$ is extended beyond the Hilbert space. To introduce time irreversibility we need to go outside the Hilbert space: the density operators $\rho$ are then non-factorizable "distributions" (generalized functions). Furthermore, the appearance of kinetic equations means that the wave-function description is not preserved by the transformation $\Lambda$. Indeed, in contrast to $U$, $\Lambda$ is a *nonfactorizable* superoperator. In this representation for non-integrable systems, dynamics is described in the Liouville-von Neumann space, and not in terms of wave functions.

In both integrable and non-integrable cases, we transform the Liouville equation into a set of independent equations. But as already mentioned in the integrable case the meaning is quite different from the non-integrable case. Indeed, Eq. (18) corresponds to oscillations, while Eq. (31) corresponds to Markovian kinetic equations. Note that all non-Markovian memory effects are eliminated in the representation $\tilde{\rho} = \Lambda\rho$, which describes interacting dressed particles or modes [10].

Before we go further, let us define more precisely the meaning of the projection operators $P^{(\nu)}$. As an example consider a model of a particle interacting with a field, the Friedrichs model with Hamiltonian

$$H = \omega_1 |1\rangle\langle 1| + \sum_k \omega_k |k\rangle\langle k| + \lambda \sum_k V_k (|k\rangle\langle 1| + |1\rangle\langle k|). \tag{32}$$

The state $|1\rangle$ represents the bare particle (or atom) in its excited level and no field present, while the state $|k\rangle$ represents a bare field mode of momentum $k$ together with the particle in its ground state. The interaction describes transitions between these states, corresponding to absorption and emission processes.

For density matrices the diagonal elements give the probability to find the particle in the state $|1\rangle$ or the field in a mode $|k\rangle$, while the off-diagonal elements give information on the quantum correlations between particle and field, or among field modes. The interaction changes the state of the correlations. Hence, in the density matrix formulation, there appears naturally a "dynamics of correlations" [1]. To formulate this more precisely, let us first introduce the concept of the "vacuum-of-correlations subspace" that is the set of diagonal dyads $|\alpha\rangle\langle\alpha|$ with $\alpha = 1, k$. We then introduce an integer $d$ that specifies the degree of correlation. This is defined as the minimum number $d$ of successive interactions $\lambda L_V$ by which a given dyadic state can reach the vacuum of correlation. For example, the dyadic states $|1\rangle\langle k|$ and $|k\rangle\langle 1|$ corresponding to particle-field correlations have $d = 1$, while the dyads $|k\rangle\langle k'|$ corresponding to field-field correlations have $d = 2$. For the Friedrichs model $d = 2$ is the maximum value of the degree of correlation.

The projector $P^{(0)}$ corresponds to the vacuum of correlations subspace, while the projectors $P^{(k1)}$ and $P^{(1k)}$ correspond to the $d = 1$ subspace and $P^{(kk')}$ to the $d = 2$ subspace. The complement projectors $Q^{(\nu)}$ are defined by

$$P^{(\nu)} + Q^{(\nu)} = 1. \tag{33}$$

They are orthogonal to $P^{(\nu)}$, i.e., $Q^{(\nu)}P^{(\nu)} = P^{(\nu)}Q^{(\nu)} = 0$, and satisfy $[Q^{(\nu)}]^2 = Q^{(\nu)}$.

# 4 Dynamics of Dissipative Systems

We come now to what we may call the backbone of our approach. For integrable systems, we have seen that the central problem is the construction of the unitary operator $U$ in the Liouville-von Neuman space. For non-integrable systems, we introduced in our previous papers [5, 6, 7] new operators $C^{(\nu)}$, $D^{(\nu)}$, $\chi^{(\nu)}$ corresponding to the dynamics of correlations. The superoperator $C^{(\nu)}$ is an "off-diagonal" superoperator, as it describes off-diagonal transitions $C^{(\nu)} = Q^{(\nu)}C^{(\nu)}P^{(\nu)}$ from the $P^{(\nu)}$ correlation subspace to the $Q^{(\nu)}$ subspace. By operating $C^{(\nu)}$ on the $\nu$ correlation subspace $P^{(\nu)}$, this operator creates correlations other than the $\nu$ correlation. In particular $C^{(0)}$ creates higher correlations from the vacuum of correlations. For this reason the $C^{(\nu)}$ are generally called "creation-of-correlations" superoperators, or creation operators in short. Conversely, the $D^{(\nu)} = P^{(\nu)}D^{(\nu)}Q^{(\nu)}$ are called destruction operators. The superoperator $\chi^{(\nu)} = P^{(\nu)}\chi^{(\nu)}P^{(\nu)}$ is "diagonal," as it describes a diagonal transition between states belonging to the same subspace $P^{(\nu)}$ (see Appendix A).

In terms of these operators, we may indeed consider dynamics as a dynamics of correlations. We start by expressing $U$ in terms of the kinetic operators $C$, $D$ and $\chi$. The kinetic operators for integrable systems are given by the relations (see [5]).

$$\begin{aligned} &\bar{\chi}^{(\nu)} \equiv P^{(\nu)}U^{-1}P^{(\nu)} \\ &\bar{C}^{(\nu)}\bar{\chi}^{(\nu)} \equiv Q^{(\nu)}U^{-1}P^{(\nu)}. \end{aligned} \tag{34}$$

We have as well the hermitian conjugate components

$$\begin{aligned} &[\bar{\chi}^{(\nu)}]^\dagger \equiv P^{(\nu)}UP^{(\nu)}, \\ &[\bar{\chi}^{(\nu)}]^\dagger\bar{D}^{(\nu)} \equiv P^{(\nu)}UQ^{(\nu)}, \end{aligned} \tag{35}$$

where $\bar{D}^{(\nu)} \equiv [\bar{C}^{(\nu)}]^\dagger$. The diagonalization of the Hamiltonian starting with the projectors $P^{(\nu)}$ is equivalent to the dynamics of correlations. Using Eq. (33) we obtain

$$\begin{aligned} U^{-1}P^{(\nu)} &= (P^{(\nu)} + \bar{C}^{(\nu)})\bar{\chi}^{(\nu)}, \\ P^{(\nu)}U &= [\bar{\chi}^{(\nu)}]^\dagger(P^{(\nu)} + \bar{D}^{(\nu)}). \end{aligned} \tag{36}$$

For integrable systems, we may deduce from $U$ the kinetic operators $C, D, \chi$.

One can also write the eigenfunctions of $L_H$ for integrable systems in terms of the kinetic operators. Here we use the notation [5, 7]

$$|\bar{F}^0_\alpha\rangle\!\rangle \equiv |\bar{\phi}_\alpha;\bar{\phi}_\alpha\rangle\!\rangle, \qquad |\bar{F}^{\alpha\beta}\rangle\!\rangle \equiv |\bar{\phi}_\alpha;\bar{\phi}_\beta\rangle\!\rangle \quad (\alpha \neq \beta). \tag{37}$$

Then we have

$$L_H|\bar{F}^\nu_j\rangle\!\rangle = \bar{w}^{(\nu)}|\bar{F}^\nu_j\rangle\!\rangle. \tag{38}$$

From Eq. (36) we obtain

$$|\bar{F}^\nu_j\rangle\!\rangle = (P^{(\nu)} + \bar{C}^{(\nu)})|f^\nu_j\rangle\!\rangle, \quad \langle\!\langle\bar{F}^\nu_j| = \langle\!\langle f^\nu_j|(P^{(\nu)} + \bar{D}^{(\nu)}), \tag{39}$$

where $|f^\nu_j\rangle\!\rangle \equiv \bar{\chi}^{(\nu)}|\nu_j\rangle\!\rangle$, $j$ is a degeneracy index ($j = \alpha$ for $\nu = 0$) and $|0_\alpha\rangle\!\rangle \equiv |\alpha;\alpha\rangle\!\rangle$

Note that $P^{(\nu)}|\bar{F}_j^\nu\rangle\!\rangle = |f_j^\nu\rangle\!\rangle$ and $Q^{(\nu)}|\bar{F}_j^\nu\rangle\!\rangle = \bar{C}^{(\nu)}|f_j^\nu\rangle\!\rangle$. Hence the $Q^{(\nu)}$ component of $|\bar{F}_j^\nu\rangle\!\rangle$ is a functional of the $P^{(\nu)}$ component,

$$Q^{(\nu)}|\bar{F}_j^\nu\rangle\!\rangle = \bar{C}^{(\nu)}P^{(\nu)}|\bar{F}_j^\nu\rangle\!\rangle. \tag{40}$$

Similarly for the left eigenstates of $L_H$ we have

$$\langle\!\langle\bar{F}_j^\nu|Q^{(\nu)} = \langle\!\langle\bar{F}_j^\nu|P^{(\nu)}\bar{D}^{(\nu)}. \tag{41}$$

As we have shown in previous papers, for integrable systems Eqs. (40) and (41) can be deduced from $U$ as indicated in Eqs. (34) and (35). They lead to a closed equations for the creation and destruction operators [5].

The above construction allows us to give an explicit expression to the operator in Eq. (17) in terms of the kinetic operators [7]

$$\bar{\theta}^{(\nu)} = P^{(\nu)}w^{(\nu)} + [\bar{\chi}^{(\nu)}]^{-1}\lambda L_V\bar{C}^{(\nu)}\bar{\chi}^{(\nu)}. \tag{42}$$

The time evolution of the density matrix in the unperturbed representation depends on the correlations. These correlations replace here the interaction $V$. The situation changes for integrable systems when we go to the unitary representation. Then both interactions and correlations are eliminated (as the kinetic equation reduces to Eq. (13)). The elimination of the interactions or the correlations are equivalent problems for integrable systems (see ref. [2]). Eq. (42) becomes the collision equation for non-integrable systems.

The main result is that the dyadic formulation of quantum mechanics can be expressed in terms of the kinetic operators. This is the starting point for our transition from integrable to non-integrable systems.

For non-integrable system we have to eliminate Poincaré's divergences (see Eq. (2)). This is done by analytic continuation of the resonances which appear in the kinetic operators $C$, $D$, $\chi$. The key point is to choose the sign of $i\epsilon$ depending on whether we have a transition to higher, equal or lower correlations, in each term of the perturbation expansion [4, 7, 9]. We assume the same formal expression (36) (as used for $U$) for $\Lambda$ in terms of kinetic operators

$$\begin{aligned} \Lambda^{-1}P^{(\nu)} &= (P^{(\nu)} + C^{(\nu)})\chi^{(\nu)}, \\ P^{(\nu)}\Lambda &= [\chi^{(\nu)}]^*(P^{(\nu)} + D^{(\nu)}). \end{aligned} \tag{43}$$

However because of Poincaré resonances, we have to proceed to the analtyic continuation of $C^{(\nu)}U^{(\nu)}$ $D^{(\nu)}\chi^{(\nu)}$. This procedure seems to lead to units as close as possible from the units deduced in the unitary representation. However here already this analytic continuation breaks the unitarity of the transformation $\Lambda$. $\Lambda$ has a new property we called star-unitarity [9], which is an extension of unitarity. We have $\Lambda^{-1} = \Lambda^*$ where $*$ denotes star conjugation. Star conjugation means hermitian conjugation plus a change in the role of higher and lower correlations.

Instead of $U\rho$, we now consider $\Lambda\rho$, which formally satisfies the same equation as $U\rho$. However now $\tilde{\Theta}$ is the dissipative collision operator of kinetic theory (compare Eq. (18) and Eq. (31)).

As an example, we consider again the Friedrichs model. We assume $\omega_k \geq 0$. The state $|1\rangle$ is either unstable or stable depending on whether its energy $\omega_1$ is above or below a certain positive threshold energy, respectively [5]. This threshold depends on the coupling constant and the potential. We first restrict ourselves to situations where $\omega_1 < 0$. This condition ensures that the state $|1\rangle$ is stable and also that all terms in the perturbation expansion are well defined (i.e. we have integrability in the sense of Poincaré' [11]).

In addition to the bare states, we can construct dressed states $|\bar{\phi}_\alpha\rangle$ that are eigenstates of $H$. For example for the one particle state we have

$$H|\bar{\phi}_1\rangle = \bar{\omega}_1|\bar{\phi}_1\rangle, \tag{44}$$

where $\bar{\omega}_1$ is the (real) shifted energy of the discrete state.

The exact state $|\bar{\phi}_1\rangle$ is known and is expandable in perturbation series (see Appendix A and [5, 11]). To first order in $\lambda$ we have

$$|\bar{\phi}_1\rangle = |1\rangle - \sum_k \frac{\lambda V_k}{\omega_k - \omega_1}|k\rangle + O(\lambda^2), \tag{45}$$

which is a superposition of states $|1\rangle$ and $|k\rangle$.

The bare and dressed dyads are related as (see Eq. (24))

$$U^{-1}|1;1\rangle\rangle = |\bar{\phi}_1;\bar{\phi}_1\rangle\rangle. \tag{46}$$

The remarkable point is that, as we have already noticed, $U$ can be expressed in terms of the kinetic operators ($\bar{C}$, $\bar{D}$, $\bar{\chi}$) we have introduced. This leads to a closed link between quantum mechanics (or classical mechanics) and kinetic theory.

Now we turn to the case where the energy of the bare particle $\omega_1$ is above its threshold of stability. In this case the state $|1\rangle$ becomes unstable, and decays emitting photons. For this case, it is well-known that the state $|\bar{\phi}_1\rangle$ disappears due to "Poincaré resonances" at $\omega_k = \omega_1$ in Eq. (45) [3, 11]. In other words there is no eigenstate of $H$ that can be obtained by a unitary transformation acting on the bare state $|1\rangle$. The disappearance of $|\bar{\phi}_1\rangle$ may be interpreted as the disappearance of one of the invariants of motion (i.e., $|\bar{\phi}_1\rangle\langle\bar{\phi}_1|$). The system is non-integrable in the sense of Poincaré. We come here to a basic unsolved problem of quantum mechanics [12, 13]: how to define a *dressed* unstable state. We have of course the state $|1;1\rangle\rangle$ as well as the dressed states $|\bar{\phi}_1;\bar{\phi}_1\rangle\rangle$ for the integrable case. In spite of the considerable literature this problem is not solved. However, on the level of the Liouville-von Neumann dynamics we can introduce a dressed particle state through the non-unitary transformation $\Lambda$ obtained by the analytic continuation of $U$:

$$\Lambda^{-1}|1;1\rangle\rangle = \rho_{11}^p \tag{47}$$

$\rho_{11}^p$ corresponds to the *dressed* unstable particle defined in the Liouville space. The superscript "$p$" stands for the perturbed state. This state is outside the Hilbert space as analytical continuation leads to generalized functions. This is in agreement with our remark that dissipation is only meaningful outside the Hilbert space. The properties of $\rho_{11}^p$ have been studied in a recent paper [5], where the analytic continuation of $\Lambda$ is given. $\rho_{11}^p$ obeys a strict Markov equation, while wave functions present deviations from Markovian behavior. These deviations are difficult to accept as they would destroy indiscernibility. Our method separates effects due to the preparation of the unstable state from the decay. According to the preparation we have different short time behavior. This corresponds to the "Zeno time" [14] as well as other effects. In contrast, the behavior of $\rho_{11}^p$ is independent of the preparation. Note also that $\rho_{11}^p$ leads no more to the well known Lorentz shape, but to a distribution of photons with finite dispersion [5].

In order to show the relation between $U$ and $\Lambda$ we present first, as an example, specific components of these operators up to second order in $\lambda$ (see Appendix A). We have

$$\langle\langle k;k|U^{-1}|1;1\rangle\rangle = \frac{\lambda^2 V_k^2}{(\omega_1 - \omega_k)^2} + O(\lambda^4) \tag{48}$$

for the integrable case, and

$$\langle\langle k;k|\Lambda^{-1}|1;1\rangle\rangle = \frac{1}{2}\Big[\frac{\lambda^2 V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}+\text{c.c.}\Big]+O(\lambda^4) \tag{49}$$

for the non-integrable case. Eq. (48) corresponds to the well known Rayleigh-Schrödinger expansion (in Liouville space), while Eq. (49) corresponds to an extension of the Rayleigh-Schrödinger expansion to non-integrable dynamical systems. Note that if we insist on keeping a unitary transformation for the non-integrable case we would obtain a diverging distribution as

$$\begin{aligned}\langle\langle k;k|U^{-1}|1;1\rangle\rangle_{\text{nonint}} &= \frac{\lambda^2 V_k^2}{|\omega_1-\omega_k+i\epsilon)|2}+O(\lambda^4)\\ &\propto \frac{1}{\epsilon}\delta(\omega_1-\omega_k)\propto\frac{1}{\epsilon}\tilde{\theta}^{(0)}.\end{aligned} \tag{50}$$

This is an example of Poincaré's divergences. It occurs at the resonance $\omega_k=\omega_1$. It is related to the collision operator $\tilde{\theta}^{(0)}$ [3]. However, for the non-unitary transformation (49) we avoid the divergence by our analytic continuation. In Appendix A we present the fourth order terms of both $U^{-1}$ and $\Lambda^{-1}$.

The expansion in terms of $\lambda$ can be pursued to all orders in $\lambda$. While the radius of convergence of the series is generally not known, for the Friedrichs model one can obtain compact exact expressions for any value of $\lambda$ [5]. As already mentioned the analytic continuation has been done separating the transitions from higher correlations from the transitions to lower correlations [9, 4]. The analytical continuation is not unique. In each term we could replace $i\epsilon$ by $-i\epsilon$. The possibility of two different extensions corresponds to the inversion between past and future, and is the basis for dissipative processes. Our method leads to a separation of processes that lead to equilibrium in the future from processes that lead to equilibrium in the past. The main point is that we can separate these processes in terms of two different "semigroups." Which semigroup to choose is a question of coherence. In the universe as known to us, all dissipative processes have the same direction. This is *by definition* the direction from past to future. Anyway irreversible processes appear as a result of analytic continuation, and not due to any falsification of dynamics. However, the mathematics of irreversible processes is highly non trivial. For example we have in the integrable case

$$UH^2=(UH)^2. \tag{51}$$

The operator $U$ is "distributive." In contrast, due to its nonfactorizability, $\Lambda$ is non-distributive[4, 5]

$$\Lambda H^2\neq(\Lambda H)^2. \tag{52}$$

This difference indicates that there are fluctuations in energy. Calculations presented in Appendix B show that these fluctuations lead to the well-known uncertainty relation between energy and lifetime,

$$\Delta E\Delta\tau\geq 1/2, \tag{53}$$

where $\Delta\tau$ is the lifetime of the unstable state and $\Delta E$ is given by the difference [4, 5]

$$(\Delta E)^2=\langle\Lambda H^2\rangle-\langle(\Lambda H)^2\rangle, \tag{54}$$

The time energy uncertainty relation thus takes here a clear meaning.

Since $\rho_{11}^p$ is not an eigenstate of $L_H$, its average energy is different from Green's function energy (for weak coupling the difference os $O(\lambda^4)$. Furthermore to $\rho_{11}^p$ corresponds the nonvanishing energy fluctuation $\Delta E$. The line shape of $\rho_{11}^p$ differs radically from the Lorentz shape. A line shape close

to the line shape of $\rho_{11}^p$ could be observed by preparing a Hilbert-space state that approximates the state $\rho_{11}^p$ [6].

Of course the fluctuations due to the non-distributivity of $\Lambda$ appear in many problems. We have recently studied its effect in statistical mechanics and radiation damping, but we don't want to discuss these problems here because of space limitations.

# 5 Concluding Remarks

The arrow of time expressing the non-integrability changes the basic structure of space-time. We can now make the link between dynamics and thermodynamics. Indeed through the $\Lambda$ transformation one can define the positive Lyapounov operator $\Lambda^\dagger\Lambda$ which decreases monotonically in time. This operator may be associated with a "microscopic entropy." In contrast, the operator $U^\dagger U = 1$ is obviously constant. Therefore the introduction of the star operator leads to a microscopic formulation of thermodynamics [9, 10].

Our method also applies to thermodynamic systems with $N$ particles in a volume $V$ in the limit $N \to \infty$, $V \to \infty$ with the density $c = N/V$ finite. These systems are in general non-integrable. The same applies to field theory. Free fields are integrable systems. But in general interacting fields are not integrable. The interactions between fields lead again to dissipation and require an extension of dynamics outside the Hilbert space [15].

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# A Construction of $\Lambda^{-1}$

In this appendix we show an example of a perturbative construction of the star-unitary operator $\Lambda^{-1}$ for the Friedrichs model. This operator is obtained from an extension of the unitary operator $U^{-1}$ from the integrable case to the nonintegrable case.

We will consider the perturbation expansions up to $O(\lambda^4)$. In our example we will consider the operator $\Lambda^{-1}$ acting on the dyad $|1;1\rangle\rangle$. This dyad is of special interest, as it represents the bare excited state. Let us note that the off-diagonal ($Q^{(0)}$) elements of $\Lambda^{-1}|1;1\rangle\rangle$ are functionals of the diagonal ($P^{(0)}$) matrix elements as (see Eq. (43))

$$Q^{(0)}\Lambda^{-1}|1;1\rangle\rangle = C^{(0)}P^{(0)}\Lambda^{-1}|1;1\rangle\rangle. \tag{55}$$

We shall restrict our attention to the diagonal matrix elements. We start by writing the diagonal matrix elements of $U^{-1}|1;1\rangle\rangle$ in the integrable case. They are obtained from the eigenstates of the Hamiltonian $H$ associated with the bare discrete state. We have

$$H_0|1\rangle = \omega_1|1\rangle, \qquad H|\bar{\phi}_1\rangle = \bar{\omega}_1|\bar{\phi}_1\rangle. \tag{56}$$

Applying the Rayleigh-Schroedinger perturbation expansion we get

$$|\bar{\phi}_1\rangle = \bar{N}_1^{1/2}\Big[1 + g\lambda V + (g\lambda V)^2 + (g\lambda V)^3 - g^2\lambda V p_1 \lambda V g \lambda V\Big]|1\rangle + O(\lambda^4), \tag{57}$$

where

$$g = q_1 \frac{1}{\omega_1 - H_0}, \qquad q_1 = 1 - p_1, \quad p_1 = |1\rangle\langle 1|, \tag{58}$$

and $\bar{N}_1 = \langle\bar{\phi}_1|p_1|\bar{\phi}_1\rangle$ is a normalization constant. This may be found (up to $\lambda^4$ terms) from Eq. (57) and the relation $\bar{N}_1 + \langle\bar{\phi}_1|q_1|\bar{\phi}_1\rangle = 1$. Then, using

$$U^{-1}|1;1\rangle\!\rangle = |\bar{\phi}_1;\bar{\phi}_1\rangle\!\rangle \tag{59}$$

we obtain the matrix elements

$$\begin{aligned}
\langle\!\langle 1;1|U^{-1}|1;1\rangle\!\rangle &= 1 - \lambda^2 \sum_k \frac{V_k^2}{(\omega_1 - \omega_k)^2} \\
&+ \lambda^4 \sum_k \frac{V_k^2}{(\omega_1 - \omega_k)^2} \sum_l \frac{V_l^2}{(\omega_1 - \omega_l)^2} \\
&+ 2\lambda^4 \sum_k \frac{V_k^2}{(\omega_1 - \omega_k)^3} \sum_l \frac{V_l^2}{\omega_1 - \omega_l} + O(\lambda^6),
\end{aligned} \tag{60}$$

$$\begin{aligned}
\langle\!\langle k;k|U^{-1}|1;1\rangle\!\rangle &= \lambda^2 \frac{V_k^2}{(\omega_1 - \omega_k)^2}, \\
&- \lambda^4 \frac{V_k^2}{(\omega_1 - \omega_k)^2} \sum_l \frac{V_l^2}{(\omega_1 - \omega_l)^2} \\
&- 2\lambda^4 \frac{V_k^2}{(\omega_1 - \omega_k)^3} \sum_l \frac{V_l^2}{\omega_1 - \omega_l} + O(\lambda^6),
\end{aligned} \tag{61}$$

These matrix elements satisfy the relation

$$\langle\!\langle 1;1|U^{-1}|1;1\rangle\!\rangle + \sum_k \langle\!\langle k;k|U^{-1}|1;1\rangle\!\rangle = 1. \tag{62}$$

This is the property of trace preservation of the unitary transformation $U$. This property leads to a correspondence between the perturbation terms in Eqs. (60) and (61). For example, the last term in Eq. (60) corresponds to the last term in Eq. (61). Denoting these terms by $f_1$ and $f_k$, respectively, the correspondence is

$$f_1 = -\sum_k f_k. \tag{63}$$

A similar correspondence exists for the other terms.

In the integrable case ($\omega_1 < 0$) the denominators in Eqs. (60) and (61) are nonvanishing, as there are no resonances, i.e., we have $\omega_1 \neq \omega_k$ for all $\omega_k$.

Let us now consider the nonintegrable case ($\omega_1 > 0$). Now there appear resonances and the denominators in Eqs. (60) and (61) can vanish. Still, we can regularize them as

$$\frac{1}{\omega_1 - \omega_k} \Rightarrow \frac{1}{\omega_1 - \omega_k \pm i\epsilon}. \tag{64}$$

The extension of $U$ (integrable case) to $\Lambda$ (nonintegrable case) involves the regularization of the denominators in Eqs. (60) and (61). The regularization (either $+i\epsilon$ or $-i\epsilon$) in each denominator should be chosen so that all denominators or products of denominators in the $\lambda$ expansion can be interpreted as distributions. Furthermore the $\Lambda$ transformation should lead to a complete set of independent projectors $\Pi^{(\nu)}$ that commute with the Liouvillian (see Eq. (29)).

These requirements lead to the "$i\epsilon$ rule" of analytic continuation of the denominators: the sign of $i\epsilon$ depends on the types of transitions that the interaction ($L_V$) induces in the perturbation expansion. For example, in the construction of the $C^{(\nu)}$ operators, transitions from lower to higher correlations or same correlations are associated with $-i\epsilon$ as they correspond to relaxation or approach to equilibrium in the future ($t > 0$), while transitions from higher to lower correlations are associated with $+i\epsilon$ (for more details see [5] and references therein; see also the paragraph before Eq. (51)).

For the operators in the $\nu = 0$ subspace corresponding to the vaccum of correlations, the transitions always lead to higher correlations. The commutation relation (27) then leads to

$$C^{(0)} = G_C\lambda L_V P^{(0)} + (G_C\lambda L_V)^2 P^{(0)} + (G_C\lambda L_V)^3 P^{(0)} - G_C^2\lambda L_V P^{(0)}\lambda L_V G_C\lambda L_V P^{(0)} + O(\lambda^4), \quad (65)$$

where

$$G_C = Q^{(0)}\frac{-1}{L_0 - i\epsilon}. \quad (66)$$

This formula can be obtained directly through a "Lippman-Schwinger"-type of equation for $C^{(0)}$ [5]. Similarly we have

$$D^{(0)} = P^{(0)}\lambda L_V G_D + P^{(0)}(\lambda L_V G_D)^2 + P^{(0)}(\lambda L_V G_D)^3 - P^{(0)}\lambda L_V G_D\lambda L_V P^{(0)}\lambda L_V G_D^2 + O(\lambda^4), \quad (67)$$

where

$$G_D = \frac{1}{-i\epsilon - L_0}Q^{(0)}. \quad (68)$$

The knowledge of $C^{(0)}$ and $D^{(0)}$ allows us to determine one of the matrix elements of $\Lambda^{-1}$. This matrix element is $\langle\langle 1;1|\Lambda^{-1}|1;1\rangle\rangle$. Indeed, as we show now, this satisfies the relation

$$\langle\langle 1;1|\Lambda^{-1}|1;1\rangle\rangle = \sqrt{\langle\langle 1;1|A^{(0)}|1;1\rangle\rangle} + O(1/L), \quad (69)$$

where $A^{(0)} = [P^{(0)} + D^{(0)}C^{(0)}]^{-1}$. To prove Eq. (69) we start with the relation

$$\langle\langle 1;1|\Lambda\Lambda^{-1}|1;1\rangle\rangle = 1. \quad (70)$$

We have $\langle\langle 1;1|\Lambda = \langle\langle 1;1|[\chi^{(0)}]^*(P^{(0)} + D^{(0)})$ and $\Lambda^{-1}|1;1\rangle\rangle = (P^{(0)} + C^{(0)})\chi^{(0)}|1;1\rangle\rangle$. Replacing this in Eq. (70) we get

$$\langle\langle 1;1|[\chi^{(0)}]^*(P^{(0)} + D^{(0)}C^{(0)})\chi^{(0)}|1;1\rangle\rangle = 1, \quad (71)$$

where we used that $P^{(0)}C^{(0)} = D^{(0)}P^{(0)} = 0$. Inserting complete sets of unperturbed dyads we get

$$\sum_{\alpha,\alpha'}\langle\langle 1;1|[\chi^{(0)}]^*|\alpha;\alpha\rangle\rangle\langle\langle\alpha;\alpha|(P^{(0)} + D^{(0)}C^{(0)})|\alpha';\alpha'\rangle\rangle\langle\langle\alpha';\alpha'|\chi^{(0)}|1;1\rangle\rangle = 1. \quad (72)$$

Taking into account the volume dependence of the interaction one can show that only the intermediate states with $\alpha = \alpha' = 1$ give an order $O(L^0)$ contribution, while the other intermediate states give $O(1/L)$ contributions. Then, using the relation $\langle\langle 1;1|[\chi^{(0)}]^*|1;1\rangle\rangle = \langle\langle 1;1|\chi^{(0)}|1;1\rangle\rangle$, which follows from the definition of star conjugation (see [5]), we obtain the desired result Eq. (69).

Using the expansion

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots \tag{73}$$

for $x = \langle\langle 1;1|D^{(0)}C^{(0)}|1;1\rangle\rangle$, we obtain from Eq. (69)

$$\langle\langle 1;1|\Lambda^{-1}|1;1\rangle\rangle = 1 - \frac{1}{2}\langle\langle 1;1|D^{(0)}C^{(0)}|1;1\rangle\rangle + \frac{3}{8}[\langle\langle 1;1|D^{(0)}C^{(0)}|1;1\rangle\rangle]^2 + \cdots. \tag{74}$$

The perturbation expansions of $C^{(0)}$ and $D^{(0)}$ in Eqs. (65) and (67) then lead to

$$\begin{aligned}
\langle\langle 1;1|\Lambda^{-1}|1;1\rangle\rangle &= 1 - \frac{\lambda^2}{2}\sum_k \Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2} + \text{c.c.}\Big], \\
&+ \sum_{k,l}\frac{\lambda^4}{8}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}\frac{V_l^2}{(\omega_1-\omega_l-i\epsilon)^2} + \text{c.c.}\Big], \\
&+ \sum_{k,l}\frac{3\lambda^4}{8}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}\frac{V_l^2}{(\omega_1-\omega_l+i\epsilon)^2} + \text{c.c.}\Big], \\
&+ \lambda^4\sum_{k,l}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^3}\frac{V_l^2}{\omega_1-\omega_l+i\epsilon} + \text{c.c.}\Big] + O(\lambda^6).
\end{aligned} \tag{75}$$

Using this result, we can now directly obtain the matrix elements $\langle\langle k;k|\Lambda^{-1}|1;1\rangle\rangle$: we simply extend the correspondence (63) to the nonintegrable case, for each of the terms in Eq. (75). We obtain

$$\begin{aligned}
\langle\langle k;k|\Lambda^{-1}|1;1\rangle\rangle &= \frac{\lambda^2}{2}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2} + \text{c.c.}\Big], \\
&- \sum_{l}\frac{\lambda^4}{8}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}\frac{V_l^2}{(\omega_1-\omega_l-i\epsilon)^2} + \text{c.c.}\Big], \\
&- \sum_{l}\frac{3\lambda^4}{8}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}\frac{V_l^2}{(\omega_1-\omega_l+i\epsilon)^2} + \text{c.c.}\Big], \\
&- \lambda^4\sum_{l}\Big[\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^3}\frac{V_l^2}{\omega_1-\omega_l+i\epsilon} + \text{c.c.}\Big] + O(\lambda^6).
\end{aligned} \tag{76}$$

In this way we have completed the extension of the matrix elements (60) and (61) from the integrable to the nonintegrable case. A nontrivial part was the extension of the term

$$-\lambda^4\frac{V_k^2}{(\omega_1-\omega_k)^2}\sum_l\frac{V_l^2}{(\omega_1-\omega_l)^2} \tag{77}$$

in $U$, to

$$\Big[-\frac{\lambda^4}{8}\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}\sum_l\frac{V_l^2}{(\omega_1-\omega_l-i\epsilon)^2} - \frac{3\lambda^4}{8}\frac{V_k^2}{(\omega_1-\omega_k+i\epsilon)^2}\sum_l\frac{V_l^2}{(\omega_1-\omega_l+i\epsilon)^2}\Big] + \text{c.c.} \tag{78}$$

in $\Lambda$. The coefficients $1/8$ and $3/8$ come from the expansion of the square root in Eq. (74).

Our construction of $\Lambda$ coincides with the expressions obtained previously in [4, 5]. The advantadge of the method presented here is that it shows in a transparent way the correspondence betwen the transformations $U$ and $\Lambda$ associated with integrable and nonintegrable systems, respectively.

## B Energy fluctuation

In this appendix we show that

$$\Lambda H^2 \neq (\Lambda H)^2 \tag{79}$$

for the Hamiltonian Eq. (32) (see also [4, 5]). In the nonintegrable (unstable) case this Hamiltonian has a complex spectral representation [11] in terms of the Gamow vectors

$$H = z_1|\phi_1\rangle\langle\tilde{\phi}_1| + \sum_k \omega_k|\phi_k\rangle\langle\tilde{\phi}_k|, \tag{80}$$

where

$$z_1 = \tilde{\omega}_1 - i\gamma \tag{81}$$

and $(2\gamma)^{-1}$ is the lifetime of the unstable state.

In order to show Eq. (79) we consider the expectation values (for $n = 1, 2$)

$$\langle 1|\Lambda H^n|1\rangle = \langle\!\langle 1;1|\Lambda|H^n\rangle\!\rangle. \tag{82}$$

The state $\langle\!\langle 1;1|\Lambda$ is the dual of the dressed unstable state $\Lambda^{-1}|1;1\rangle\!\rangle$. As shown in [5] these states are given by superpositions of eigenstates of the Liouvillian

$$\Lambda^{-1}|1;1\rangle\!\rangle = |F_1\rangle\!\rangle + \sum_k b_k|F_k\rangle\!\rangle, \tag{83}$$

$$\langle\!\langle 1;1|\Lambda = \langle\!\langle\tilde{F}_1| + \sum_k b_k\langle\!\langle\tilde{F}_k|, \tag{84}$$

where $b_k$ is the line shape of emitted photons, and

$$|F_1\rangle\!\rangle = |\phi_1;\phi_1\rangle\!\rangle, \qquad \langle\!\langle\tilde{F}_1| = \langle\!\langle\tilde{\phi}_1;\tilde{\phi}_1| \tag{85}$$

$$|F_k\rangle\!\rangle = |\tilde{\phi}_k;\tilde{\phi}_k\rangle\!\rangle, \qquad \langle\!\langle\tilde{F}_k| = \langle\!\langle\tilde{\phi}_k^{\text{c.c.}};\tilde{\phi}_k^{\text{c.c.}}| \tag{86}$$

These eigenstates are products of Gamov vectors. In a previous paper [11] we have shown that

$$\begin{aligned}\langle\!\langle\tilde{F}_1|H^n\rangle\!\rangle &= \langle\tilde{\phi}_1|H^n|\tilde{\phi}_1\rangle \\ &= z_1^n\langle\tilde{\phi}_1|\tilde{\phi}_1\rangle = 0\end{aligned} \tag{87}$$

$$\begin{aligned}\langle\!\langle\tilde{F}_k|H^n\rangle\!\rangle &= \langle\tilde{\phi}_k^{\text{c.c.}}|H^n|\tilde{\phi}_k^{\text{c.c.}}\rangle \\ &= \omega_k^n\langle\tilde{\phi}_k^{\text{c.c.}}|\tilde{\phi}_k^{\text{c.c.}}\rangle = \omega_k^n\end{aligned} \tag{88}$$

So we obtain from Eqs. (82) and (84)

$$\begin{aligned}\langle 1|\Lambda H^n|1\rangle &= \langle\!\langle\tilde{F}_1|H^n\rangle\!\rangle + \sum_k b_k\langle\!\langle\tilde{F}_k|H^n\rangle\!\rangle \\ &= 0 + \sum_k b_k\omega_k^n\end{aligned} \tag{89}$$

$$\tag{90}$$

Therefore for

$$\Delta E^2 \equiv \Lambda H^2 - (\Lambda H)^2 \tag{91}$$

we have

$$\langle 1|\Delta E^2|1\rangle = \sum_k b_k \omega_k^2 - \langle 1|(\Lambda H)^2|1\rangle. \tag{92}$$

As we show now for the last term we have the relation

$$\langle 1|(\Lambda H)^2|1\rangle = \langle 1|\Lambda H|1\rangle^2. \tag{93}$$

To prove this relation we insert a complete set of bare states in the l.h.s.,

$$\langle 1|(\Lambda H)^2|1\rangle = \langle 1|\Lambda H|1\rangle\langle 1|\Lambda H|1\rangle + \sum_k \langle 1|\Lambda H|k\rangle\langle k|\Lambda H|1\rangle. \tag{94}$$

The Hamiltonian, being an invariant of motion, belongs to the $\Pi^{(0)}$ subspace [2]. So we have

$$H = \Pi^{(0)} H = \Lambda^{-1} P^{(0)} \Lambda H. \tag{95}$$

This leads to the relation $\Lambda H = P^{(0)}\Lambda H$, which means $\Lambda H$ is diagonal in the unperturbed basis. Therefore the second term in the r.h.s. of Eq. (94) is identically zero. This proves the relation Eq. (93).

Replacing Eq. (93) in Eq. (92) we obtain

$$\langle 1|\Delta E^2|1\rangle = \sum_k b_k \omega_k^2 - (\sum_k b_k \omega_k)^2. \tag{96}$$

This is the mean square deviation of the energy associated with the line shape $b_k$. For weak coupling we have [5]

$$b_k \approx (\frac{2\pi}{L})\frac{1}{\pi}\frac{\gamma^3}{[(\omega_k - \tilde{\omega}_1)^2 + \gamma^2]^2}. \tag{97}$$

Then in the continous limit we obtain

$$\langle 1|\Delta E^2|1\rangle \approx \int_{-\infty}^{\infty} \frac{dk}{\pi}\frac{\gamma^3\omega_k^2}{[(\omega_k - \tilde{\omega}_1)^2 + \gamma^2]^2} - \Big[\int_{-\infty}^{\infty} \frac{dk}{\pi}\frac{\gamma^3\omega_k}{[(\omega_k - \tilde{\omega}_1)^2 + \gamma^2]^2}\Big]^2. \tag{98}$$

For $\omega_k = |k|$ the integrals can be explicitly evaluated. We obtain

$$\langle 1|\Delta E^2|1\rangle \approx \gamma^2, \tag{99}$$

or

$$\langle 1|[\Lambda H^2 - (\Lambda H)^2]|1\rangle \approx \gamma^2. \tag{100}$$

This shows the nondistributivity (and nonfactorizability) of the $\Lambda$ transformation.

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