Time Operator in the Friedrichs Model

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A time superoperator T canonically conjugate to the Liouville superoperator is constructed for a model of unstable particles in the Liouville-von Neumann space of density matrices. While there is no time operator conjugate to the Hamiltonian in the wave function space due to positivity of energy, T may exist in the density matrix space as the spectrum of the Liouvillian covers all the real axIs. This is the first example of an observable that can only be formulated in the Liouville-von Neumann space of density matrices. In our example, the expectation value of T gives the lifetime of the unstable particle. On the basis of T, an entropy superoperator is obtained that manifests the irreversible behavior of unstable systems at the microscopic level.

1 Historical introduction

In quantum mechanics the position and momentum of a particle are represented by the non-commuting operators $\hat{x}$ (position) and $\hat{p}$ (momentum) with commutation relation

$$[\hat{p}, \hat{x}] = -iI\,, \quad \hbar = 1\,. \tag{1}$$

This commutation relation leads to the uncertainty principle

$$\Delta p \Delta x \geq 1/2, \tag{2}$$

that limits the precision of simultaneous measurement of position and momentum.

On the other hand, the time-energy uncertainty relation in quantum mechanics cannot be interpreted in the same way because, as noted by Pauli [1], for physical systems where the energy is bounded from below, it is impossible to construct a selfadjoint time operator $\hat{t}$ conjugate to the Hamiltionian H such that $[H, \hat{t}] = -iI$. The reason is that $\hat{t}$ would allow the existence of a displacement operator $\exp(i\Delta E\partial/\partial E)$ that could shift the energy of physical states to arbitrary negative values. The non-existence of $\hat{t}$ has deep implications, as it is related to the time-reversibility in the Hilbert-space formulation of quantum mechanics. If a time operator existed, an entropy operator M could be obtained, by taking M to be a monotonic function of $\hat{t}$. As mentioned by Prigogine [2]: "The impossibility of defining the entropy operator M, the non-existence of a time operator in quantum mechanics, and the problem of interpreting and justifying the time-energy uncertainty relation are thus linked together. Their common origin is the fact that in the usual formulation of quantum mechanics the generator H of the time translation group is identical with the energy operator of the system. To be able to define an entropy operator, it is thus necessary to overcome this degeneracy. The simplest way of achieving this is to go to the so-called Liouvillian formulation of (quantum) dynamics ..."

Even though the spectrum of energies of H is bounded from below, the spectrum of the Liouville superoperator $L_H = [H, \cdot\,]$ is unbounded, as the eigenvalues of L_H are differences of energies. As a result one may introduce [2],[3] a selfadjoint superoperator T that satisfies

$$[L_H, T] = -iI. \quad (3)$$

For classical discrete unstable maps such as the Baker transformation, Misra, Prigogine and Courbage [4],[5] have already introduced a time operator (see also [6]). The existence of a time operator in this case is related to the possibility of making explicit the irreversible aspect of the conservative chaotic dynamics. This requires the description at the level of probability distributions [7]. Time operator defines a new representation where the time-evolution is generated by a dissipative semigroup associated with Markov processes. Through the time operator T, Misra, Prigogine and Courbage [5] have defined the entropy operator M, which is as a monotonic function of T (Lyapunov function), analogous to Boltzmann's H function. The expectation value of the entropy operator in a given state indicates the "distance" of the of system to its final asymptotic state.

As in classical mechanics, in quantum mechanics we can construct an entropy superoperator once we have T. Time operators for relativistic free systems [8] have also been studied previously using Hamiltonians unbounded from below such as in the Klein-Gordon equation. Recently a time operator for non-conservative systems described by the diffusion equation [9] has been obtained.

Both the time and entropy superoperators do not preserve the purity of states [2]. This requires their formulation in the space of density operators. In addition, there is a close relation between the existence of the time superoperator and instabilities and resonances in the system. Stable systems have isolated points in the spectrum of the Hamiltonian that correspond to stable states. As we shall see these isolated points spoil the possibility of obtaining a complete set of the eigenstates of T operator.

We have constructed explicitly [10], [11], [12] a time superoperator for the Friedrichs model of unstable systems with conservative Hamiltonian dynamics (such as an unstable atom interacting with photons). In contrast to the models studied in [8], our model is unstable due to Poincaré resonances [13] and therefore it is a non-integrable system in the sense of Poincaré. Moreover the spectrum of the Hamiltonian of this model is bounded from below. Due to the simple structure of the Hamiltonian one can explicitly find the eigenstates of H. Thanks to this simplicity, the Friedrichs model has been extensively investigated to discuss the irreversibility associated with spontaneous emission [13], extensions of the Hilbert space that include decaying states in the spectral decomposition of the Hamiltonian [14]-[16], and the definition of unstable dressed particles in terms of density matrices [17]. A time operator acting on wave functions has been constructed by Lockhart [3] for the Friedrichs model with no lower bound on the energy. We add one more interesting feature of the Friedrichs model with Hamiltonian bounded from below that allows the explicit construction of the time superoperator in the Liouville-von Neumann space of density operators.

In Section II we present a general construction of the time super operator in the Liouville-von Neumann space based on the spectral representation of the Hamiltonian with continuous spectrum bounded from below. In Section III we obtain the eigenstates of the time superoperator in the Friedrichs model. We show that the expectation value of the time superoperator in the unstable state gives the lifetime of the particle and we calculate the fluctuations of T. In Section IV, we construct the entropy superoperator M, which manifests the irreversible aspect of the behavior of unstable systems at the microscopic level, and show the non-local character of M as it maps local states into non-local states. Finally, in Section V we present our conclusions.

2 Construction of the time super operator

We shall construct the time-super operator based on the analogy with the usual position operator of quantum mechanics. In our construction we use the Hamiltonian with continuous spectrum on the positive part of the real line. The diagonal representation of the Hamiltonian is

$$H = \sum_\alpha \int_0^\infty d\omega |\phi_\omega, \alpha\rangle \omega \langle \phi_\omega, \alpha|, \tag{4}$$

where ω is the energy, α is a degeneracy index and the energy eigenstates $|\phi_\omega, \alpha\rangle$ satisfy the orthonormality and completeness relations.

In the Liouville space there is an additional degeneracy of the eigenstates of L_H as the eigenvalues are energy differences $\omega - \omega'$. Introducing new variables

$$\nu \equiv \omega - \omega', \qquad \bar{\omega} \equiv \frac{\omega + \omega'}{2} \tag{5}$$

we can write the eigenstates of the Liouville operator, which are density operators given by the projectors on the eigenstates of the Hamiltonian, in the form

$$|\Phi_\nu, \bar{\omega}, \alpha, \alpha'\rangle\rangle \equiv |\phi_\omega, \alpha\rangle\langle\phi_{\omega'}, \alpha'|. \tag{6}$$

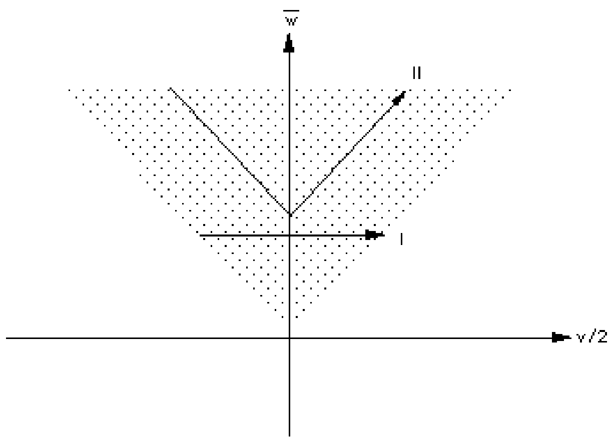

Fig. 1. Spectrum of L_H and the integration paths for the Fourier transform.

Then the eigenvalue equation of L_H is rewritten as

$$L_H|\Phi_\nu, \bar{\omega}, \alpha, \alpha'\rangle\rangle = \nu|\Phi_\nu, \bar{\omega}, \alpha, \alpha'\rangle\rangle. \tag{7}$$

The spectrum of ν runs from $-\infty$ to $+\infty$. As the eigenstates of L_H are density operators, we need to define their inner product in order to introduce the linear space structure. The inner product of two operators A and B is defined trough the trace $\langle\langle A, B\rangle\rangle \equiv Tr(AB)$. Now, similar to $\hat{x}$ and $\hat{p}$, the solution of the commutation relation (3) is constructed for the time superoperator T by taking the Fourier transform of the states $|\Phi_\nu, \bar{\omega}, \alpha, \alpha'\rangle\rangle$ over the variable ν. However, some care is necessary because the variable ν cannot vary independently of $\bar{\omega}$. Indeed, we have $\omega = \bar{\omega} + \nu/2 \geq 0$ and $w' = \bar{\omega} - \nu/2 \geq 0$, so we have $\bar{\omega} \geq |\nu|/2$. In the $(\nu/2, \bar{\omega})$ plane the allowed region is then only the shaded region shown in Fig. 1 (see [18] for a discussion on the spectrum of L_H). For given $\bar{\omega}$, the variable ν is restricted between the values $-2\bar{\omega}$ and $2\bar{\omega}$, as indicated by the path I of Fig. 1. In order to remove this restriction we choose integration paths such as II in Fig. 1. Along this path the vertical distance $E = \bar{\omega} - |\nu|/2$ to the lower edge of the shaded region remains constant (see also [3]).

Denoting as $\xi \equiv (E, \alpha, \alpha')$ the set of variables that are constant along the path II, we relabel

$$|\Phi_\nu, E, \alpha, \alpha'\rangle\!\rangle = |\Phi_\nu, \xi\rangle\!\rangle \, .$$

Then the eigenstates of T, which we call "age eigenstates" may be obtained by the Fourier transform:

$$|\Phi(t), \xi\rangle\!\rangle = \int_{-\infty}^{\infty} d\nu e^{-i\nu t} |\Phi_\nu, \xi\rangle\!\rangle \, . \tag{8}$$

The time superoperator is then

$$T = \int_{-\infty}^{\infty} dt \sum_{\xi} |\Phi(t), \xi\rangle\!\rangle t \langle\!\langle \Phi(t), \xi| ., \tag{9}$$

where $\sum_\xi$ means summation over the discrete variables and integration over E. Similar to the position operator we may also represent the time superoperator in terms of its conjugate variable by replacing $t \Rightarrow i\partial/\partial\nu$ as

$$T = \int_{-\infty}^{\infty} d\nu \sum_{\xi} |\Phi_\nu, \xi\rangle\!\rangle i \frac{\partial}{\partial \nu} \langle\!\langle \Phi_\nu, \xi| .. \tag{10}$$

We introduce the average value $\langle T \rangle_\rho$ associated with a state ρ as

$\langle T \rangle_\rho \equiv$ h F7 1 Tffl 7.9706 0 0 7.9706 0 0 Tmfl 30.851 81.985 9dfl 3(1)]TJfl F6

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where $|1\rangle$ and $|\omega,\alpha\rangle$ form an orthonormal basis and $V_\alpha(\omega)$ is real. The state $|\omega,\alpha\rangle$ represents the bare particle in its ground state together with a field mode (or "photon") with the energy ω and a degeneracy index α, while the state $|1\rangle$ represents the bare particle in its excited state with no photons present (for the one dimensional Friedrichs model we have $\omega = |k|$ and $\alpha = \pm 1$ depending on the sign of the momentum k.)

The exact eigenstates of H corresponding to the continuous spectrum are known [13]. The eigenstates

$$|\phi_{\omega,\alpha}\rangle \equiv |\omega,\alpha\rangle + \frac{\lambda V_\alpha(\omega)}{\eta^+(\omega)}\left[|1\rangle + \sum_{\alpha'}\int_0^\infty d\omega' \frac{\lambda V_\alpha(\omega')}{\omega-\omega'+i0}|\omega',\alpha'\rangle\right], \tag{15}$$

where $\eta^+(\omega) \equiv \eta(\omega + i\varepsilon)$ with

$$\eta(z) \equiv z - \omega_1 - \sum_\alpha \int_0^\infty d\omega \frac{\lambda^2 V_\alpha^2(\omega)}{z-\omega}. \tag{16}$$

correspond to the "in" states of scattering theory, i.e., the asymptotic time-evolved free-states for $t \to +\infty$. We may obtain "out" states by changing $\epsilon \to -\epsilon$. Changing the sign of ϵ leads to $-T$ instead of T.

For the interaction satisfying

$$\omega_1 > \lambda^2 \sum_\alpha \int_0^\infty d\omega \frac{V_\alpha^2(\omega)}{\omega} \tag{17}$$

the bare excited state of the particle $|1\rangle$ becomes unstable due to resonances [19] and disappears from the spectrum of the total Hamiltonian which contains now only continuous part. We can thus apply our construction from the previous section. This is so because the instability condition (17) guarantees that $[\eta(z)]^{-1}$ is analytic at the first Riemann sheet except for the cut $[0,+\infty)$ and therefore, there is no discrete eigenvalues. In the second Riemann sheet the Green's function $[\eta(z)]^{-1}$ may have complex poles. We assume that $[\eta^+(\omega)]^{-1}$ continued to the lower half plane has a pole at $z_1 \equiv \tilde\omega_1 - i\gamma/2$, where $\tilde\omega_1$ is the energy of the particle shifted due to the interaction, and $\gamma > 0$ is the decay rate of the excited unstable state.

Using the exact eigenstates (15) of the total Hamiltonian we repeat our construction of the eigenstates of the time superoperator and consider

$$\langle T\rangle_{\rho_1} = \langle\!\langle \rho_1|T|\rho_1\rangle\!\rangle\,, \quad |\rho_1\rangle\!\rangle \equiv |1\rangle\langle 1|\,. \tag{18}$$

Thus we obtain

$$\langle T\rangle_{\rho_1} = \int_{-\infty}^{+\infty} \mathrm{d}t \sum_\xi \langle\!\langle \rho_1|\Phi(t),\xi\rangle\!\rangle t \langle\!\langle \Phi(t),\xi|\rho_1\rangle\!\rangle, \tag{19}$$

where

$$|\Phi_\nu,\xi\rangle\!\rangle = |\Phi_\nu,E,\alpha,\alpha'\rangle\!\rangle = \begin{cases} |\phi_{E+\nu,\alpha}\rangle\langle\phi_{E,\alpha'}| & \text{for}\, \nu>0 \\ |\phi_{E,\alpha}\rangle\langle\phi_{E-\nu,\alpha'}| & \text{for}\, \nu<0 \end{cases}. \tag{20}$$

In the weak coupling case determined by the conditions $\lambda \ll 1$ and $\tilde\omega_1 \gg \gamma$ we have

$$\begin{aligned} \langle T\rangle_{\rho_1} &= -i\lambda^4 \sum_{\alpha,\alpha'} \int_0^\infty d\nu \int_0^\infty d\bar\omega \frac{V_\alpha^2(\bar\omega+\nu)}{|\eta^+(\bar\omega+\nu)|^2} \\ &\times \frac{V_{\alpha'}^2(\bar\omega)}{|\eta^+(\bar\omega)|^2}\left(\frac{\eta^-(\bar\omega+\nu)'}{\eta^-(\bar\omega+\nu)} - \frac{\eta^+(\bar\omega+\nu)'}{\eta^+(\bar\omega+\nu)}\right), \end{aligned} \tag{21}$$

where $\eta^{\pm}(\omega)' \equiv \partial\eta^{\pm}(\omega)/\partial\omega$. Then using the exact relation

$$\frac{1}{2\pi i}\sum_{\alpha}\frac{\lambda^2 V_{\alpha}^2(\omega)}{|\eta^{+}(\omega)|^2} = \left[\frac{1}{\eta^{+}(\omega)} - \frac{1}{\eta^{-}(\omega)}\right] \tag{22}$$

and the approximations

$$\begin{aligned} \eta^{\pm}(\omega)' &= 1 + O(\lambda^2)\,, \\ 1/(\eta^{+}(\omega)) &= 1/(\omega - z_1) + O(\lambda^2)\,, \end{aligned} \tag{23}$$

we evaluate the mean lifetime of the particle

$$\langle T\rangle_{\rho_1} \approx \frac{1}{\gamma}\,. \tag{24}$$

The expectation value of the time superoperator in the unstable state $|1\rangle\langle 1|$ is then approximately the mean lifetime of the unstable state. The fluctuation of the value of the lifetime we call age fluctuation

$$(\Delta T)^2_{\rho} \equiv \langle T^2\rangle_{\rho} - (\langle T\rangle_{\rho})^2.$$

It can be obtained by evaluating $\langle T^2\rangle_{\rho_1}$ that gives

$$\langle T^2\rangle_{\rho_1} \approx \frac{2}{\gamma} \tag{25}$$

and then

$$\Delta T_{\rho_1} \approx \frac{1}{\gamma}\,. \tag{26}$$

The age fluctuation of the unstable state $|\rho_1\rangle\rangle$ is therefore, as large as its average age. This fluctuation specifies a new interpretation of the time-energy uncertainty relation. Indeed, the energy fluctuation in the bare excited state is

$$(\Delta H)^2_{\rho_1} = \sum_{\alpha}\int_0^{\infty} \mathrm{d}\omega\lambda^2 V_{\alpha}^2(\omega) \tag{27}$$

and we have

$$(\Delta T)_{\rho_1}(\Delta L_H)_{\rho_1} = \frac{1}{\gamma}\left(2\sum_{\alpha}\int_0^{\infty}\mathrm{d}\omega\lambda^2 V_{\alpha}^2(\omega)\right)^{1/2}\,, \quad (\Delta L_H)^2_{\rho} = 2(\Delta H)^2_{\rho}\,. \tag{28}$$

Then using the instability condition (17) we come to

$$(\Delta T)_{\rho}(\Delta L_H)_{\rho} \geq \frac{1}{2},$$

that must be satisfied due to the commutation relation (3).

4 Entropy superoperator

Following Misra et. al. [4, 20] we define the entropy superoperator

$$M = \Lambda^{\dagger}\Lambda, \tag{29}$$

where non-unitary superoperator $\Lambda = F(T)$ is a function of T and the function $F(t)$ is an arbitrary non-increasing function of t. This non-unitary transformation was introduced in [4, 20] for unstable dynamical systems such as Baker transformation. Using our age eigenstates (20) we can write Λ operator for the Friedrichs model in the form

$$\Lambda = \int_{-\infty}^{\infty} dt \sum_{\xi} |\Phi(t), \xi\rangle\rangle F(t) \langle\langle \Phi(t), \xi| \,. \tag{30}$$

From the non-increasing property of $F(T)$ we see that the entropy superoperator satisfies

$$M \geq 0\,, \quad \frac{d}{dt}\langle M\rangle_{\rho(t)} \leq 0\,, \rho) \quad \langle M\rangle_{\rho} = \langle\langle \rho|M|\rho\rangle\rangle, \tag{31}$$

which makes it suitable for characterization of the irreversible aspect of the behavior of unstable systems and allows the introduction of the concept of microscopic dynamical entropy.

As shown in [20] there is a close relation between the microscopic entropy and non-locality. For the baker map, Λ transforms points in phase space into ensembles corresponding to non-local distributions. The basic objects, like $\tilde{\rho} \equiv \Lambda^{\dagger}\rho$ defining the entropy (29), (31), are therefore non-local [2]. In quantum mechanics Λ superoperator defines a non-unitary transformation that introduces delocalization in the space representation. As an example, we choose $F(t) = \theta(-t)$ in the Friedrichs model. The position states $|x\rangle$ of the photons are related to the momentum states as

$$\langle x|\omega, \alpha\rangle = (2\pi)^{-1/2} \exp(i\alpha\omega x), \quad (k = \alpha|\omega|)\,. \tag{32}$$

The corresponding density operator $|\rho_x\rangle\rangle = |x\rangle\langle x|$ has vanishing projection onto the initial state $|\rho_1\rangle\rangle = |1\rangle\langle 1|$

$$\langle x|\rho_1|x\rangle = \langle\langle \rho_x|\rho_1\rangle\rangle = 0\,. \tag{33}$$

For the transformed states $\tilde{\rho}_1 = \Lambda\rho_1$ and $\tilde{\rho}_x = \Lambda\rho_x$ we have a non-vanishing overlap

$$\langle\langle \tilde{\rho}_x|\tilde{\rho}_1\rangle\rangle = \langle\langle \rho_x|\Lambda^{\dagger}\Lambda|\rho_1\rangle\rangle \sim \gamma e^{-\gamma|x|} \tag{34}$$

demonstrating the delocalization of $|\rho_1\rangle\rangle$ induced by the Λ transformation. Therefore the entropy superoperator M (29) demonstrates the non-local correlations of the localized unstable state $|\rho\rangle\rangle$ with the other parts of the system as the expectation value $\langle\langle \rho_1|M|\rho_1\rangle\rangle$ is non-local in x due to (34).

Our final remark is that although the choice of the decreasing function $F(t)$ is not unique, the important qualitative conclusion is that the existence of the time superoperator implies the existence of the entropy superoperator. In order to have meaningful definition of the entropy in quantitative terms, the function $F(t)$ should be linked to the intrinsic decay rates of the system, i.e., to γ.

5 Concluding remarks

We presented a general construction of the time superoperator for quantum systems with bounded from below Hamiltonian that has only continuous spectrum. Like in classical mechanics where the time superoperator [4], [5] leads to the description in terms of probability and ensembles, beyond the trajectory description, in quantum mechanics the time superoperator leads to a formulation based on density matrices that goes beyond the wave function description.

We have constructed a time superoperator T in the Friedrichs model in which the interaction of an excitable particle with the radiation field leads to irreversible decay process due to Poincaré resonances. In this example we see that when the instability condition (17) is not satisfied, the time operator cannot be constructed because there is no resonances between the discrete state and the continuum and an isolated point appears in the spectrum of the total Hamiltonian H. This isolated point spoils the possibility of obtaining a complete set of age eigenstates by Fourier transformation of the eigenstates of the Liouvillian (8).

Although the Friedrichs model presents many simplifications, our method can still be applied to more complicated models, including models with virtual transitions [21] and field theoretical models [22].

We have shown that the expectation value of T with the bare unstable particle state gives the lifetime of the particle. The fluctuation of the lifetime, i.e., the age fluctuation, gives a meaning to the time-energy uncertainty relation. In analogy with the Misra-Prigogine-Courbage formulation, we have also introduced an entropy superoperator constructed from the age eigenstates, and shown that this operator is non-local in configuration space in accordance with the fact that non-locality is a common feature in theories that incorporate the increase of entropy as a fundamental principle [20].

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