Standard-Model Particles and Interactions from Field Equations on Spin 9+1 Dimensional Space

J. Besprosvany

Instituto de Física, Universidad Nacional Autónoma de México, México

We consider field equations set on an extended spin space that contain fermion and boson solutions. At given dimension, they determine the scalar symmetries. The standard field equations can be equivalently written in terms of such degrees of freedom, and are similarly constrained. At 9+1 dimensions, the $SU(3) \times SU(2)_L \times U(1)$ gauge groups emerge, as well as solution representations with quantum numbers of related gauge bosons, leptons, quarks, Higgs-like particles and others as lepto-quarks.

The current theory of elementary particles, the standard model (SM) is successful in describing their behavior, but it is phenomenological. The origin of the interaction groups, the particles' spectrum and representations, and parameters has remained largely unexplained. Still, the generalization of features of the model into larger structures with a unifying principle has suggested connections among the observables. Thus, additional dimensions in Kaluza-Klein theories are associated with gauge interactions, and larger groups in grand-unified theories [1] put some restrictions on them. Spin is a physical manifestation of the fundamental representation of the Lorentz group and it is more so in relation to space, which uses the vector representation. By setting Poincaré-invariant field equations on an extended spin space [2], described by a Clifford algebra, in this contribution we report on the restrictions and the classification of the symmetries and the representation solutions, at given dimension. They contain spin-1/2 and boson solutions with a fixed scalar representation. A field theory can be formulated in terms of such degrees of freedom. By constraining only the dimension of the spin space on which the fields are formulated, we derive the interactions and representations of the SM at 9+1 dimensions, the minimal space that contains them.

The Dirac equation

$$\gamma_0(i\partial_\mu\gamma^\mu - M)\Psi = 0, \tag{1}$$

uses an extended spin space when Ψ represents a matrix instead of, as traditionally, a four-entries (column) spinor. Eq. 1 contains four conditions over four spinors in a 4×4 matrix. There are, then, additional possible transformations and symmetry operations that further classify Ψ. The Dirac-operator transformation $(i\partial_\mu\gamma^\mu - M) \to U(i\partial_\mu\gamma^\mu - M)U^{-1}$ induces the left-hand side of the transformation

$$\Psi \to U\Psi U^\dagger, \tag{2}$$

and Ψ is postulated to transform as indicated on the right-hand side.

That the equation, the transformation and symmetry operators U, and the solutions Ψ occupy the same space is not only economical but it befittingly implements quantum mechanics, for it ultimately implies measuring apparatuses are not constituted differently in principle from the objects they measure.

U and Ψ can be classified in terms of Clifford algebras. In four dimensions (4-d) U is conventionally a 4×4 matrix containing symmetry operators as the Poincaré generators, but it can contain others, although, e. g., in the chiral massless case it can only carry an additional $U(2)$ scalar symmetry [2]. More symmetry operators appear if Eq. 1, $\mu = 0, ..., 3$, is assumed within the larger Clifford algebra $\mathcal{C}_N$, $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\mu, \nu = 0, ..., N-1$, where N is the (assumed even) dimension, whose structure is helpful in classifying the available symmetries, and which is represented by $2^{N/2} \times 2^{N/2}$ matrices. The usual 4-d Lorentz symmetry, generated in terms of $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, $\mu, \nu = 0, ..., 3$, is maintained and U contains also γ_a, $a = 4, ..., N-1$, and their products as possible symmetry generators. Indeed, these elements are scalars for they commute with the Poincaré generators, which contain $\sigma_{\mu\nu}$, and they are also symmetry operators of the massless Eq. 1, bilinear in the γ_μ matrices, which is not necessarily the case for mass terms (containing γ_0). In addition, their products with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ are Lorentz pseudoscalars. As $[\gamma_5, \gamma_a] = 0$, we can classify the (unitary) symmetry algebra as $\mathcal{S}_{N-4} = \mathcal{S}_{(N-4)R} \times \mathcal{S}_{(N-4)L}$, consisting of the projected right-handed $\mathcal{S}_{(N-4)R} = \frac{1}{2}(1 + \gamma_5)U(2^{(N-4)/2})$ and left-handed $\mathcal{S}_{(N-4)L} = \frac{1}{2}(1 - \gamma_5)U(2^{(N-4)/2})$ components.

The solutions of Eq. 1 do not span all the matrix complex space, but this is achieved by considering also solutions of

$$\Psi\gamma_0(-i \overleftarrow{\partial}_\mu \gamma^\mu - M) = 0, \tag{3}$$

consistent with the transformation in Eq. 2, (the Dirac operator transforming accordingly).

It is not possible to find always solutions that simultaneously satisfy equations of the type 1 and 3 (except trivially), which means they are not simultaneously on-shell, but they satisfy at least one and therefore the Klein-Gordon equation. Indeed, the solutions of eqs. 1 and 3, can be generally characterized as bosonic since Ψ can be understood to be formed of spinors as $\sum_{i,j} a_{ij}|w_i\rangle\langle w_j|$.

Generalized operators acting on this tensor-product space (spinor $\times$ spinor $\times$ configuration or momentum space) further characterize the solutions. Positive-energy solutions, according to Eq. 1 are interpreted as negative-energy solutions from the right-hand side. This problem is overcome if we assume the hole interpretation for the $\langle w_j|$ components, which amounts to the requirement that operators generally acting from the right-hand side acquire a minus, and that the commutator be used for operator evaluation. Thus, the 4-d plane-wave solution combination $\frac{1}{4}[(1-\gamma_5)\gamma_0(\gamma_1-i\gamma_2)]e^{-ikx}$, with $k^\mu = (k, 0, 0, k)$, is a massless vector$-$axial ($V-A$) state propagating along $\hat{\mathbf{z}}$ with left-handed circular polarization, normalized covariantly according to $\langle\Psi_A|\Psi_B\rangle = tr\Psi_A^\dagger\Psi_B$, the generalized inner product for the solution space. In fact, combinations of solutions of Eqs. 1 and 3 can be formed with a well-defined Lorentz index: vector $\gamma_0\gamma_\mu$, pseudo-vector $\gamma_5\gamma_0\gamma_\mu$, scalar γ_0, pseudoscalar $\gamma_0\gamma_5$, and antisymmetric tensor $\gamma_0[\gamma_\mu, \gamma_\nu]$. For example, $A_\mu^\mathcal{C}(x) = \frac{i}{2}\gamma_0\gamma_\mu e^{-ikx}$ is a combination that transforms under parity into $A^{\mathcal{C}\mu}(\tilde{x})$, $\tilde{x}_\mu = x^\mu$, that is, as a vector. We may also view $\frac{1}{2}\gamma_0\gamma_\mu$ as an orthonormal polarization basis, $A_\mu = tr\frac{1}{2}\gamma_\mu A^\nu \frac{1}{2}\gamma_\nu 1$; just as n_μ in $A_\mu = g_{\mu\nu}A^\nu = n_\mu \cdot A^\nu n_\nu$. In fact, the sum of Eqs. of 1 and 3 implies [3] for a Ψ containing $\gamma_0 A\!\!\!/ = A^\mu\gamma_0\gamma_\mu$ that A^μ satisfies the free Maxwell's equations.

Solutions contain also products of γ_a matrices that define their scalar-group representation. For given N, there are variations of the symmetry algebra depending on the chosen Poincaré generators and Dirac equation, respectively, through the projection operators $\mathcal{P}_P$, $\mathcal{P}_D \in \mathcal{S}_{N-4}$, $[\mathcal{P}_P, \mathcal{P}_D] = 0$. $\mathcal{P}_P$ acts as in, e.g., $\mathcal{P}_P\sigma_{\mu\nu}$, and $\mathcal{P}_D$ modifies Eqs. 1 and 3 through $\mathcal{P}_D\gamma_0(i\partial_\mu\gamma^\mu - M)$. Together, they characterize the Lorentz and scalar-group solution representations. We require rank$\mathcal{P}_D \leq$ rank$\mathcal{P}_P$, for otherwise pieces of the solution space exist that do not transform properly. For $\mathcal{P}_D \neq 1$ Lorentz operators act trivially on one side of the solutions containing $1 - \mathcal{P}_P$, since $(1 - \mathcal{P}_P)\mathcal{P}_P = 0$, so we

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[1]As for $\bar{\psi} = \psi^\dagger\gamma_0$, a unitary transformation can be applied to the fields and operators to convert them to a covariant form.

also get fermions. Table 1 depicts the distribution of Lorentz-representation solutions according to the matrix space they occupy in $\mathcal{S}_6$, when $\mathcal{P}_P = \mathcal{P}_D \neq 1$. Although it refers to $N = 10$ case, it is general for any N.

An interactive field theory can be constructed in terms of the above degrees of freedom. We consider a vector and fermion non-abelian gauge-invariant theory. The expression for the kinetic component of the Lagrangian density $\mathcal{L}_V = -\frac{1}{4}F_{\mu\lambda}^a g^{\lambda\eta}\delta_{ab}F^{b\mu}_{\eta} = -\frac{1}{4N_o}tr\mathcal{P}_D F_{\mu\lambda}^a \gamma_0\gamma^\lambda G_a F^{b\mu}_{\eta}\gamma_0\gamma_\eta G_b$ shows $\mathcal{L}_V$ is equivalent to a trace over combinations over normalized components $\frac{1}{\sqrt{N_o}}\gamma_0\gamma_\mu G_a$ with coefficients $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu^a A_\mu^a + gA_\mu^b A_\nu^c C_{bc}^a$, g the coupling constant, $\gamma_\mu \in \mathcal{C}_N$, $G_a \in \mathcal{S}_{N-4}$ the group generators, C_{bc}^a the structure constants, and $N_o = trG_a G_a$, where for non-abelian irreducible representations we use $trG_i G_j = 2\delta_{ij}$.

Similarly, the interactive part of the fermion gauge-invariant Lagrangian $\mathcal{L}_f = \frac{1}{2}\psi^{\alpha\dagger}\gamma_0(i\overset{\leftrightarrow}{\partial}_\mu -gA_\mu^a G_a)\gamma^\mu\psi^\alpha$, with ψ^α a massless spinor with flavor α, can be written $\mathcal{L}_{int} = -g\frac{1}{2N_o}tr\mathcal{P}_D A_\mu^a\gamma_0\gamma^\mu G_a$ $j_\lambda^{b\alpha}\gamma_0\gamma_\lambda G_b$, with $j_\mu^{a\alpha} = tr\Psi^{\alpha\dagger}\gamma_0\gamma_\mu G_a\Psi^\alpha$ containing $\Psi^\alpha = \psi^\alpha\langle\alpha|$, and $\langle\alpha|$ is a row state accounting for the flavor. $\mathcal{L}_{int}$ is written in terms of $\gamma_0\slashed{A}$, and $\gamma_0 j^{a\alpha}$, that is, the vector field and the current occupy the same spin space. This connection and the quantum field theory (QFT) understanding of this vertex as the transition operator between fermion states, exerted by a vector particle, with the coupling constant as a measure of the transition probability, justifies the interpretation for it $\frac{1}{2}gA^{a\mu}j_\mu^{a\alpha} = A^{a\mu}\frac{1}{\sqrt{N_o}}tr\Psi^{\alpha\dagger}\gamma_0\gamma_\mu G_a\Psi^\alpha$, leading to the identification $g \rightarrow 2\sqrt{\frac{K}{N_o}}$, K correcting for over-counted reducible representations, which is further clarified below. The theoretical assignment of g complements QFT, in which the coupling constant is set experimentally. It should be also understood as tree-level information, while the values are modified by the presence of a virtual cloud of fields, at given energy. Although in QFT the coupling constant is obtained perturbatively in terms of powers of the bare, which takes infinite values absorbed through renormalization, we may take the view that renormalization is a calculational device and that its physical value is a manifestation of the bare one; this is feasible for small coupling constants, which can give small corrections. Energy corrections are also necessary for a more detailed calculation.

As for the initial formulation, $\mathcal{P}_D$ restricts the possible gauge symmetries that can be constructed in the Lagrangian, for $\gamma_0\gamma_\mu G_i$ needs to be contained in the space it projects. Thus, $\mathcal{P}_P$ and $\mathcal{P}_D$ determine the symmetries, which are global, and in turn, determine the allowed gauge interactions. Furthermore, they fix the representations, assumed physical. The $N = 6$ case has been researched [2] and connections have been found to the $SU(2)_L \times U(1)$, electroweak sector of the SM. It is apparent that the minimal algebra that includes the SM groups requires $N = 9 + 1$, with $\mathcal{S}_6$, on which we will concentrate. There are limited ways in which we may represent the SM interactions in such a matrix space and only one giving the correct fermion and boson quantum numbers. In order to have fermions we need $\mathcal{P}_P \neq 1$. Account of the quark quantum numbers requires that their left-handed $SU(3) \times SU(2)_L$, and right-handed $SU(3) \times U(1)_Y$ symmetry generators be direct-product reducible representations occupying, respectively, 6×6 matrix pieces of $\mathcal{S}_{6L}$ and $\mathcal{S}_{6R}$. The remaining 2×2 matrix into which $\mathcal{S}_{6L}$ is broken is associated to the $SU(2)_L$ acting on the $SU(3)$-singlet leptons, and that of $\mathcal{S}_{6R}$ to a $U(1)$ describing the right-handed leptons hypercharge, and an inert $U(1)$ that gives rise to two fermion generations (all this applies also to antiparticles). There are additional $U(1)$ symmetries which can be assigned in correspondence to SM symmetries.

The algebraic expression for different generators as they appear in 9+1 d is given in Ref. [4]. For example, the choice $Y = Y_+ + Y_-$ with $V + A$ and $V - A$ components, respectively, that give the correct hypercharge fermion quantum numbers can be deduced from the condition that it not be axial, separately for quarks and leptons, namely, $tr\gamma_5 YL = 0$, and $tr\gamma_5 YB = 0$, where $L = P_{L(V+A)} + P_{L(V-A)}$, $B = P_{B(V+A)} + P_{B(V-A)}$, with $P_{L,B}$ the respective lepton and boson projectors, which leads to the correct ratios between right-handed and left-handed lepton and quark

hypercharges. The anomaly-cancellation condition sets the ratio between lepton and quark hypercharges. For $N = 6$, the electric charge $Q = \frac{1}{2}Y + I_3$ can also be deduced [2] as one of the operators commuting with the Hamiltonian when mass terms are added, which sets the form of Y.

Table 1. Arrangement of $\mathcal{S}_6$ scalar components of $N = 9 + 1$ solutions.   (a) $\mathcal{S}_6$ is divided into four 6-d 8×8 matrix blocks, with fermion (F), vector (and axial-) (V), and scalar (and pseudo-) and antisymmetric (S,A) terms.

	F	F
F	V (b)	S,A (c)
F	S,A (d)...	V (d)

$\mathcal{P}_P = \mathcal{P}_D = B + L \in \mathcal{S}_6$ is the only option describing SM fermions. In Table 1 are the solution representations resulting from such massless Hamiltonian, classified with the above generators. Each of the four types of solution in Table 1 is an 8×8 matrix, one of which pieces (b) is given explicitly in Table 2. The other components can be found in Ref. [4] with the solutions' quantum numbers. (b) and (d) contain vectors (and axial-) γ-bilinear solutions that are in the adjoint representation, with (b) also containing the flavor group, and fermion solutions. (c) has γ-linear solutions conformed of scalar (and pseudo-), antisymmetric tensors, and fermions. Fixing $\mathcal{P}_D$, the physically feasible ($trG_iG_j = 2\delta_{ij}$) gauge groups are comprised[5] of the $U(1)$ and compact simple algebras generated in it. With the conditions of anomaly-cancelling, renormalizability, B, L conservation, the allowed Lagrangian reduces basically to that of the SM, with particles with correct Lorentz and gauge group representations: gluons, weak, and hypercharge vectors. A Higgs-like particle can also be constructed from the neutral and charged scalar $SU(2)_L$ doublets in (c). There are two generations of $SU(2)_L$ lepton and quark doublets $(e, \nu)_L$, $(u, d)_L$ and e_R, u_R, d_R singlets (generation choice arbitrary). We also find particles beyond the SM as scalar leptoquarks in (b), and $V - A$ iso-gluons in (c).

The theory thus presented succeeds in reproducing many aspects of the SM. We obtain the groups $SU(2)_L \times SU(3) \times U(1)$, with corresponding vector bosons acting on quarks and leptons in two generations with correct quantum numbers and chiralities. A Higgs particle is also obtained. By using the allowed vertices with normalized-polarisation fields we are able to calculate coupling constants, around electroweak breaking. It is not obvious that SM features can be described and yet a special model configuration among few possible gives such predictions. An assortment of new particles beyond the SM are also obtained but not all need be stable or appear at low energies.

<u>Table 2.</u> (b) $\frac{1}{2}(1+\gamma_5)\gamma_0\gamma_\mu$ fields with (throughout) notation $X^{Y(\mathbf{i},\mathbf{c})}$, where Y is the hypercharge, omitted for $Y = 0$, except for fields $X^{(Y)}$, labelled according to the particle's Y they give, and $\mathbf{i}$, $\mathbf{c}$ label respectively the $SU(2)_L$ isospin, $SU(3)$ color representations, omitted for singlets $(\mathbf{1},\mathbf{1})$. Here and on, all empty places are occupied by antiparticles, except for the flavor-assigned upper-left box here. We get $V + A$ hypercharge carriers $B_\mu^{(-2)}$, $B_\mu^{(-4/3)}$, $B_\mu^{(2/3)}$, gluons $\tilde{A}_\mu^{(\mathbf{1},\mathbf{8})}$, hyper-gluons $\bar{A}_\mu^{(\mathbf{1},\mathbf{8})}$, $A_\mu^{2(\mathbf{1},\mathbf{8})}$; hyper-triplets $\tilde{A}_\mu^{-10/3(\mathbf{3},\bar{\mathbf{3}})}$, $A_\mu^{-4/3(\mathbf{1},\bar{\mathbf{3}})}$; and isospin-singlet leptons e_L, and quarks $\bar{u}_L$ $\bar{d}_L$.

	e_L^2	$\bar{u}_L^{-4/3(\mathbf{1},\bar{\mathbf{3}})}$	$\bar{d}_L^{\ 2/3(\mathbf{1},\bar{\mathbf{3}})}$
	$B_\mu^{(-2)}$	$A_\mu^{-10/3(\mathbf{1},\bar{\mathbf{3}})}$	$A_\mu^{-4/3(\mathbf{1},\bar{\mathbf{3}})}$
		$B_\mu^{(4/3)}$ $\tilde{A}_\mu^{(\mathbf{1},\mathbf{8})}$ $\bar{A}_\mu^{(\mathbf{1},\mathbf{8})}$	B_μ^2 $A_\mu^{2(\mathbf{1},\mathbf{8})}$
			$B_\mu^{(-2/3)}$ $\tilde{A}_\mu^{(\mathbf{1},\mathbf{8})}$ $\bar{A}_\mu^{(\mathbf{1},\mathbf{8})}$

(b)

With the Poincaré and SM-gauge symmetric Lagrangian presentation of the model, renormalization and quantization can be applied, leading to a QFT formulation. Its simplicity, with spin as its basic building block, should allow for a generalization and application in theories such as supersymmetry [6] and those accounting for gravity, such as supergravity [6] and strings.

There is a robustness to the predictions, for extension of the model into such structures, requiring additional reducible representations and quantum numbers, will not change the results. On the other hand, inclusion into other models with different irreducible representations should modify them, making the $N = 9 + 1$ case unique.

The close connection between the results hence derived and the physical particles' phenomenology makes plausible the idea that, as with the spin, the gauge vector and matter fields, and their interactions originate in the structure of space-time.

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