Noncommutative Quantization

Agapitos Hatzinikitas
University of Crete, Department of Applied Mathematics,
L. Knosou-Ambelokipi, 71409 Iraklio Crete, Greece,
Email: ahatzini@tem.uoc.gr

and

Ioannis Smyrnakis
University of Crete, Department of Applied Mathematics,
L. Knosou-Ambelokipi, 71409 Iraklio Crete, Greece,
Email: smyrnaki@tem.uoc.gr

1 Introduction

One way of quantizing the free closed bosonic string is by splitting it into modes that satisfy the harmonic oscillator equation. These harmonic oscillator modes are then quantized and when substituted back into the bosonic string give the quantized version of the field [1]. In this work we explore the possibility of using a noncommutative version of the harmonic oscillator to perform the string quantization. This demands that we have at least two target space dimensions. For a particular noncommutative harmonic oscillator we will see that good geometric properties are preserved by the partition function as well as by certain 2n-point functions on the sphere.

2 The Noncommutative Harmonic Oscillator

Consider the classical harmonic oscillator Hamiltonian

$$H(q_i, p_i) = \sum_i \frac{1}{2}\left(q_i^2 + p_i^2\right). \tag{1}$$

This can be quantized through the canonical commutation relations $[q_i, p_j] = i\hbar\delta_{ij}$. As is well known excitations of the harmonic oscillator admit a particle interpretation.

Suppose now that these commutation relations are deformed. For simplicity assume $i, j = 1, 2$ and

$$[q_i, p_j] = i\hbar\delta_{ij} \quad ; \quad [q_1, q_2] = i\theta \quad ; \quad [p_1, p_2] = -i\theta. \tag{2}$$

The question that arises is how it is possible to quantize under these more general commutation relations.

One way of doing this is through deformation quantization [2], [3]. It is possible to define the noncommutative, associative star product [4], [5]

$$\star = \exp\left[\frac{i}{2}\begin{pmatrix} \overleftarrow{\partial}_{q_1}, & \overleftarrow{\partial}_{p_1}, & \overleftarrow{\partial}_{q_2}, & \overleftarrow{\partial}_{p_2} \end{pmatrix}\begin{pmatrix} 0 & \hbar & \theta & 0 \\ -\hbar & 0 & 0 & -\theta \\ -\theta & 0 & 0 & \hbar \\ 0 & \theta & -\hbar & 0 \end{pmatrix}\begin{pmatrix} \overrightarrow{\partial}_{q_1} \\ \overrightarrow{\partial}_{p_1} \\ \overrightarrow{\partial}_{q_2} \\ \overrightarrow{\partial}_{p_2} \end{pmatrix}\right]. \tag{3}$$

This gives

$$
\begin{aligned}
q_i \star p_j - p_j \star q_i &= i\hbar\delta_{ij}, \\
q_1 \star q_2 - q_2 \star q_1 &= i\theta, \\
p_1 \star p_2 - p_2 \star p_1 &= -i\theta.
\end{aligned} \tag{4}
$$

The Schrödinger equation is replaced by the $\star$-genvalue equation

$$
H(q_i, p_i) \star f(q_i, p_i) = E f(q_i, p_i). \tag{5}
$$

This splits into an equation for the imaginary part [6]

$$
\Bigg[\hbar(p_1\partial_{q_1} - q_1\partial_{p_1}) + \hbar(p_2\partial_{q_2} - q_2\partial_{p_2}) + \theta\,(q_2\partial_{q_1} - q_1\partial_{q_2}) + \theta\,(p_1\partial_{p_2} - p_2\partial_{p_1})\Bigg] f = 0 \tag{6}
$$

and an equation for the real part

$$
\left[\left(q_1^2 + p_1^2 + q_2^2 + p_2^2\right) - \frac{(\hbar^2+\theta^2)}{4}\left(\partial_{q_1}^2 + \partial_{p_1}^2 + \partial_{q_2}^2 + \partial_{p_2}^2\right)\right] f = 2Ef. \tag{7}
$$

It is possible to bring the imaginary part equation in standard symplectic form, while preserving a nice form for the real part equation. Let

$$
\begin{aligned}
\bar{q_1} &= q_1, \qquad \bar{p_1} = \frac{1}{\sqrt{\hbar^2+\theta^2}}(\hbar p_1 + \theta q_2), \\
\bar{q_2} &= \frac{1}{\sqrt{\hbar^2+\theta^2}}(\hbar q_2 - \theta p_1) \qquad \bar{p_2} = p_2.
\end{aligned} \tag{8}
$$

Using these new variables the imaginary part equation becomes

$$
\Bigg[(\bar{p}_1\partial_{\bar{q}_1} - \bar{q}_1\partial_{\bar{p}_1}) + (\bar{p}_2\partial_{\bar{q}_2} - \bar{q}_2\partial_{\bar{p}_2})\Bigg] f = 0. \tag{9}
$$

Note that the new variables satisfy canonical $\star$-commutation relations with $\hbar$ replaced by $\sqrt{\hbar^2+\theta^2}$. This equation implies that $f = f(z_1, z_2)$ where $z_i = 2(\bar{q}_i^2 + \bar{p}_i^2)$.

The real part equation can be expressed in terms of z_1, z_2 and takes the form

$$
\left[z_1\partial_{z_1}^2 + \partial_{z_1} + z_2\partial_{z_2}^2 + \partial_{z_2} + \frac{1}{(\hbar^2+\theta^2)}\left(E - \frac{(z_1+z_2)}{4}\right)\right] f(z_1, z_2) = 0. \tag{10}
$$

The solution to this equation is

$$
f_{nm}(z_1, z_2) = e^{-\frac{1}{2\sqrt{\hbar^2+\theta^2}}(z_1+z_2)} L_n\left(\frac{z_1}{\sqrt{\hbar^2+\theta^2}}\right) L_m\left(\frac{z_2}{\sqrt{\hbar^2+\theta^2}}\right), \tag{11}
$$

where

$$
L_m(\tilde{z}_i) = \frac{1}{m!} e^{\tilde{z}_i} \frac{d^m}{d^m \tilde{z}_i}\left(e^{-\tilde{z}_i} \tilde{z}_i^m\right) \tag{12}
$$

are the Laguerre polynomials. The energies corresponding to these $\star$-genfunctions are

$$
E_{nm} = \sqrt{\hbar^2+\theta^2}\,(n + m + 1)\,. \tag{13}
$$

It is again possible to define annihilation and creation operators

$$a_i = \frac{1}{\sqrt{2}}(\bar{q}_i + i\bar{p}_i) \qquad a_i^\dagger = \frac{1}{\sqrt{2}}(\bar{q}_i - i\bar{p}_i). \tag{14}$$

They satisfy the following $\star$-commutation relations

$$a_i \star a_j^\dagger - a_j^\dagger \star a_i = \delta_{ij}\sqrt{\hbar^2 + \theta^2}. \tag{15}$$

So the particle interpretation of excitations is not lost by the introduced noncommutativity.

More generally the commutation relations are governed by a general antisymmetric matrix M compatible with the Jacobi identity. The $\star$-product giving rise to the right $\star$-commutation relations is

$$\star = \exp\frac{i}{2}\left[\overleftarrow{\partial}_I^T M_{IJ} \overrightarrow{\partial}_J\right], \tag{16}$$

where $\partial_I^T = (\partial_{q_I}, \partial_{p_I})$, and $M_{IJ} = -M_{JI}^T$ are 2×2 matrix blocks. The imaginary part equation now becomes

$$X_I^T M_{IJ}\partial_J f = 0, \tag{17}$$

while the real part equation becomes

$$\left[X_I^T X_I - \frac{1}{4}(M_{IK_1}\partial_{K_1})^T(M_{IK_2}\partial_{K_2})\right] f = 2Ef. \tag{18}$$

Now we can use the following lemma [7]:

Lemma: Let (V, ω) be a symplectic vector space and $g : V \times V \to R$ be an inner product. Then there exists a basis $u_1, \cdots, u_n$, $v_1, \cdots, v_n$ of V which is both g-orthogonal and ω-standard. Moreover, this basis can be chosen such that $g(u_j, u_j) = g(v_j, v_j)$ for all j.

This means that for nondegenerate M, it is possible to find an orthogonal transformation R so that $R^T M R = J(M)$. Here

$$J(M)_{IJ} = \alpha_I(M)\delta_{IJ}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{19}$$

Making now the transformation $\bar{X} = R^T X$ the $\star$-genvalue equation becomes

$$\bar{H}\bar{\star}\bar{f} = E\bar{f}, \tag{20}$$

where $\bar{\star}$ is

$$\bar{\star} = \exp\left[\frac{i}{2}\overleftarrow{\partial}_I^T J(M)_{IK}\overrightarrow{\partial}_K\right]. \tag{21}$$

The imaginary part equation w.r.t. the new variables becomes

$$\sum_i \alpha_i(\bar{q}_i\partial_{\bar{p}_i} - \bar{p}_i\partial_{\bar{q}_i})\bar{f}(\bar{q}_i, \bar{p}_i) = 0 \tag{22}$$

while the real part equation takes the form

$$\sum_i \left[z_i\partial_{z_i}^2 + \partial_{z_i} - \frac{1}{\alpha_i^2}(\frac{z_i}{4} - E_i)\right] f(z_i) = 0. \tag{23}$$

This means that we really have a set of harmonic oscillators with $\hbar$ replaced by α_i.
So no matter what the original phase space commutation relations are, as long as they are non-degenerate, it is possible to choose new phase space variables that satisfy canonical commutation relations, with the value of $\hbar$ possibly modified, so that the Hamiltonian w.r.t the new variables is the harmonic oscillator Hamiltonian.

It is possible also to formulate the noncommutative harmonic oscillator in terms of path integrals, since we know the Hamiltonian of the system. It is natural to use the states $|\bar{q}_1, \bar{q}_2, \cdots >$ to define the partition function since the operators $\hat{\bar{q}}_i$, $\hat{\bar{q}}_j$ commute. Carrying out a calculation similar to the classical one for the partition function we get

$$< \bar{q}_i^F | e^{-i\frac{H\tau}{\hbar}} | \bar{q}_i^I >= (\prod_i \frac{1}{2\pi\alpha_i}) \int \prod_i Dp_i Dq_i e^{i\int_{t_I}^{t_F} dt [\frac{1}{2} X_i^T M_{ij}^{-1} \dot{X}_j - \frac{1}{\hbar} H(q_i, p_i)]}, \tag{24}$$

where $X_i^T = (q_i, p_i)$.

3 Field Theory

We will consider the free massless bosonic field in two target space dimensions, $X^i(\sigma, t)$, $i = 1, 2$

$$X^i(\sigma, t) = X_0^i + \frac{P_0^i}{2} t + \frac{N_0^i}{2} \sigma + \sum_{n>0} q_n^i(t) \cos n\sigma + \tilde{q}_n^i(t) \sin n\sigma. \tag{25}$$

The classical equation of motion, which is the wave equation, implies that the coefficients $q_n^i(t)$, $\tilde{q}_n^i(t)$ satisfy the classical harmonic oscillator equation.

We will quantize these harmonic oscillators following the noncommutative harmonic oscillator prescription with the commutation relations [8]

$$\begin{aligned} &[q_n^i, p_n^j] = i\hbar\delta_{ij}; \quad [q_n^1, q_n^2] = i\theta/n; \quad [p_n^1, p_n^2] = -in\theta, \\ &[\tilde{q}_n^i, \tilde{p}_n^j] = i\hbar\delta_{ij}; \quad [\tilde{q}_n^1, \tilde{q}_n^2] = i\theta/n; \quad [\tilde{p}_n^1, \tilde{p}_n^2] = -in\theta. \end{aligned} \tag{26}$$

All other commutation relations are taken to be 0. Here $\theta << \hbar$ is taken to be a possible correction to the usual commutation relations.

It is now possible to represent the above commutation relations by

$$\begin{aligned} q_n^1 &= \sqrt{\frac{\alpha}{2n}} (a_n^1 + a_n^{1\dagger}), \\ p_n^1 &= -i\sqrt{\frac{n}{2\alpha}} [(\hbar a_n^1 - i\theta a_n^2) - (\hbar a_n^{1\dagger} + i\theta a_n^{2\dagger})], \\ q_n^2 &= \sqrt{\frac{1}{2n\alpha}} [(\hbar a_n^2 - i\theta a_n^1) + (\hbar a_n^{2\dagger} + i\theta a_n^{1\dagger})] \equiv \sqrt{\frac{\alpha}{2n}} (A_n + A_n^\dagger), \\ p_n^2 &= -i\sqrt{\frac{n\alpha}{2}} (a_n^2 - a_n^{2\dagger}), \end{aligned} \tag{27}$$

where a_n^i, $a_n^{i\dagger}$ are annihilation-creation operators.

Substituting this representation into the expansion for $X^i(\sigma, \tau)$ one gets

$$\begin{aligned} X^1(\sigma, t) &= X_0^1 + \frac{P_0^1}{2} t + \frac{N_0^1}{2} \sigma + \frac{i}{2} \sum_{n>0} \frac{1}{n} \left(C_n e^{-in(\frac{a}{\hbar} t - \sigma)} + \tilde{C}_n e^{-in(\frac{a}{\hbar} t + \sigma)} \right), \\ X^2(\sigma, t) &= X_0^2 + \frac{P_0^2}{2} t + \frac{N_0^2}{2} \sigma + \frac{i}{2} \sum_{n>0} \frac{1}{n} \left(D_n e^{-in(\frac{a}{\hbar} t - \sigma)} + \tilde{D}_n e^{-in(\frac{a}{\hbar} t + \sigma)} \right), \end{aligned} \tag{28}$$

where C_n, D_n, $\tilde{C}_n$, $\tilde{D}_n$ depend on the annihilation-creation operators in the following manner

$$C_n = -\sqrt{\frac{\alpha}{2}}\left(b_n^1 + ia_n^1\right); \qquad \tilde{C}_n = \sqrt{\frac{\alpha}{2}}\left(b_n^1 - ia_n^1\right); \tag{29}$$

$$D_n = -\sqrt{\frac{\alpha}{2}}\,\mathrm{sign}(n)\left(B_n + iA_n\right); \qquad \tilde{D}_n = \sqrt{\frac{\alpha}{2}}\,\mathrm{sign}(n)\left(B_n - iA_n\right); \tag{30}$$

$$A_n = \frac{\hbar a_n^2 - i\theta a_n^1}{\alpha}; \qquad B_n = \frac{\hbar b_n^2 - i\theta b_n^1}{\alpha} \tag{31}$$

and satisfy

$$[C_n, C_{-n}] = n\alpha; \quad [C_n, D_{-n}] = in\theta sign(n); \quad [D_n, D_{-n}] = n\alpha. \tag{32}$$

Similar commutation relations hold for $\tilde{C}_n$, $\tilde{D}_n$. The operators b_n^i, ${b_n^i}^\dagger$ are the annihilation-creation operators associated with $\tilde{q}_n^i(t)$.

For the zero modes it is reasonable to take the commutation relations

$$[p_L^i, x_L^i] = [p_R^i, x_R^i] = -\alpha i, \tag{33}$$

where $\frac{x_L^i + x_R^i}{2} = X_0^i$ and

$$\begin{aligned} p_R^i &= \frac{\hbar}{\alpha}\frac{P_0^i}{2} - \frac{N_0^i}{2}, \\ p_L^i &= \frac{\hbar}{\alpha}\frac{P_0^i}{2} - \frac{N_0^i}{2}, \end{aligned} \tag{34}$$

It is possible to define a generator of time translations, the Hamiltonian of the system

$$\begin{aligned} H &= H_0 + \sum_{n>0}\left[\frac{\alpha^2}{\hbar^2}(C_{-n}C_n + \tilde{C}_{-n}\tilde{C}_n + D_{-n}D_n + \tilde{D}_{-n}\tilde{D}_n)\right. \\ &+ \left. i\frac{\alpha\theta}{\hbar^2}(D_{-n}C_n - C_{-n}D_n + \tilde{D}_{-n}\tilde{C}_n - \tilde{C}_{-n}\tilde{D}_n)\right], \end{aligned} \tag{35}$$

where

$$H_0 = \frac{1}{2}\sum_{i=1}^{2}\left((p_R^i)^2 + (p_L^i)^2\right). \tag{36}$$

It is also possible to define a generator of spatial translations, the momentum of the system

$$\begin{aligned} P &= P_0 + \sum_{n>0}\left[\frac{\alpha}{\hbar}(C_{-n}C_n - \tilde{C}_{-n}\tilde{C}_n + D_{-n}D_n - \tilde{D}_{-n}\tilde{D}_n)\right. \\ &+ \left. i\frac{\theta}{\hbar}(D_{-n}C_n - C_{-n}D_n - \tilde{D}_{-n}\tilde{C}_n + \tilde{C}_{-n}\tilde{D}_n)\right], \end{aligned} \tag{37}$$

where

$$P_0 = \frac{\hbar}{2\alpha}\sum_{i=1}^{2}\left((p_R^i)^2 - (p_L^i)^2\right). \tag{38}$$

Note that the momentum and the Hamiltonian commute.

To proceed, Wick rotate to Euclidean space. Let $\tau = -i\frac{\alpha}{\hbar}t$, $w = \tau + i\sigma$, $z = e^{-w}$, $\theta \to -i\theta$. If

$$X^i(z,\bar{z}) = \frac{X^i(z) + \tilde{X}^i(\bar{z})}{2}, \tag{39}$$

then

$$\begin{aligned}
X^1(z) &= x_R^1 - ip_R^1 \ln z + i\sum_{n\neq 0}\frac{1}{n}C_n z^{-n}, \\
\tilde{X}^1(\bar{z}) &= x_L^1 - ip_L^1 \ln \bar{z} + i\sum_{n\neq 0}\frac{1}{n}\tilde{C}_n \bar{z}^{-n}, \\
X^2(z) &= x_R^2 - ip_R^2 \ln z + i\sum_{n\neq 0}\frac{1}{n}D_n z^{-n} \\
\tilde{X}^2(\bar{z}) &= x_L^2 - ip_L^2 \ln \bar{z} + i\sum_{n\neq 0}\frac{1}{n}\tilde{D}_n \bar{z}^{-n}.
\end{aligned} \tag{40}$$

Using the known commutation relations of the modes it is possible to compute the propagators

$$\begin{aligned}
&< X^1(z)X^1(w) >=< (x_R^1)^2 > -\alpha\ln(z-w), \\
&< X^2(z)X^2(w) >=< (x_R^2)^2 > -\alpha\ln(z-w), \\
&< X^1(z)X^2(w) >=< x_R^1 x_R^2 > -\theta\ln(1-\frac{w}{z}), \\
&< X^2(z)X^1(w) >=< x_R^2 x_R^1 > +\theta\ln(1-\frac{w}{z}).
\end{aligned} \tag{41}$$

There are new singularities introduced at $z, w = 0$ but it is not possible to avoid them by changing the zero mode commutation relations because then it becomes impossible to find generators of time and space translations.

In this theory the generators of conformal transformations on the primary fields based on the action field X^1 are different from the generators of conformal transformations on the primary fields based on the action field X^2. The stress energy tensor for the first is

$$T^1(z) = -\frac{1}{2\alpha} : (\partial_z X^1(z))^2 : \tag{42}$$

while for the second is

$$T^2(z) = -\frac{1}{2\alpha} : (\partial_z X^2(z))^2 : . \tag{43}$$

The modes of these operators satisfy two copies of the Virasoro algebra, but they do not commute among themselves.

It is possible to use the generators of space and time translations to compute the Euclidean partition function

$$Z = Tr(e^{2\pi i\frac{\tau_1}{\hbar}P}e^{-2\pi\frac{\hbar}{\alpha}\frac{\tau_2}{\hbar}H}). \tag{44}$$

This turns out to be

$$Z = \frac{Z_0^2}{|\eta(\tau)|^4}. \tag{45}$$

Here Z_0 is the contribution from the zero modes which is

$$Z(\tau, \bar{\tau}) = \sum_{n,m=-\infty}^{\infty} q^{\frac{1}{2}(\sqrt{\alpha}\frac{m}{2R} - \frac{nR}{\sqrt{\alpha}})^2} \bar{q}^{\frac{1}{2}(\sqrt{\alpha}\frac{m}{2R} + \frac{nR}{\sqrt{\alpha}})^2}. \quad (46)$$

In computing the zero mode contribution we have taken the quantization condition to be

$$P_0^i = \frac{\alpha^2}{\hbar}\frac{m}{R}; \qquad N_0^i = 2\pi n R,$$

where R is the compactification radius.

The conformal blocks of the form $<: e^{ik_1 X^1(z)} :: e^{ik_2 X^2(w)} :>$ are going to be zero because of the zero mode contribution $< e^{ik_1 x_R^1} e^{ik_2 x_R^2} >$. The first nonzero contribution that contains both $X^1(z)$ and $X^2(z)$ will occur at the level of the four point function. For this we get

$$<: e^{ikX^1(u)} :: e^{-ikX^1(v)} :: e^{i\lambda X^2(w)} :: e^{-i\lambda X^2(z)} :>= (u-v)^{-\alpha k^2}(w-z)^{-\alpha\lambda^2} \left[\frac{(u-w)(v-z)}{(u-z)(v-w)}\right]^{\theta k \lambda}. \quad (47)$$

So we see that the θ-noncommutativity introduces a cross-ratio contribution to the four point function. So the global conformal invariance is preserved at this level. It is possible to generalize the above formula to the case of the 2n-point functions of the form below:

$$< \prod_{i=1}^{n} : e^{ik_i X^{\sigma(i)}(z_i)} :: e^{-ik_i X^{\sigma(i)}(w_i)} :>= \prod_{i=1}^{n} (z_i - w_i)^{-\alpha k_i^2} \prod_{i<j} Cr(i,j)^{k_i k_j (\alpha \delta_{\sigma(i),\sigma(j)} + \theta \epsilon_{\sigma(i),\sigma(j)})}, \quad (48)$$

where $\sigma : \{1, \cdots, n\} \to \{1, 2\}$ and $Cr(i,j) = \frac{(z_j - z_i)(w_j - w_i)}{(w_j - z_i)(z_j - w_i)}$. Note that when a particular pair of insertions z_i, w_i is far from the rest of the insertions then the 2n-point function factorizes into a product of a two point function and a (2n-2)-point function as expected.

It is possible to understand the form of the partition function in terms of path integrals. Let

$$X^i(\sigma, \tau) = \frac{q_0^i(t)}{\sqrt{2}} + N_0^i \sigma + \sum_{n>0} q_n^i(t) \cos n\sigma + \tilde{q}_n^i(t) \sin n\sigma. \quad (49)$$

The classical Hamiltonian of the system becomes

$$H = \frac{1}{2\pi}\int_0^{2\pi} ((\dot{X}^i)^2 + (X^{i'})^2) d\sigma = \sum_{i=1}^{2}\left[H_0 + \sum_{n=1}^{\infty}(H_n^i + \tilde{H}_n^i) + (N_0^i)^2\right], \quad (50)$$

where $H_n^i = \frac{(\dot{q}_n^i)^2}{2} + n^2 \frac{(q_n^i)^2}{2}$, $\tilde{H}_n^i = \frac{(\dot{\tilde{q}}_n^i)^2}{2} + n^2 \frac{(\tilde{q}_n^i)^2}{2}$ are the harmonic oscillator Hamiltonians.

Using (24), the partition function becomes

$$\begin{aligned} Z &= \int \prod_{i=1,2} Dp_0^i Dq_0^i e^{\frac{i}{\hbar}\oint \frac{\hbar}{\alpha}(p_0^i \dot{q}_0^i + \tilde{p}_0^i \dot{\tilde{q}}_0^i)} \prod_{n=1}^{\infty} Dp_n^i Dq_n^i D\tilde{p}_n^i D\tilde{q}_n^i e^{\frac{i}{\hbar}\oint X_n^{iT}(M^{(n)})_{ij}^{-1}\dot{X}_n^j + \tilde{X}_n^{iT}(M^{(n)})_{ij}^{-1}\dot{\tilde{X}}_n^j} e^{-\frac{i}{\hbar}\oint H} \\ &= \int \prod_{i=1,2} D\bar{p}_0^i D\bar{q}_0^i e^{\frac{i}{\alpha}\oint(\bar{p}_0^i \dot{\bar{q}}_0^i + \bar{\tilde{p}}_0^i \dot{\bar{\tilde{q}}}_0^i)} \prod_{n=1}^{\infty} D\bar{p}_n^i D\bar{q}_n^i D\bar{\tilde{p}}_n^i D\bar{\tilde{q}}_n^i e^{\frac{i}{\alpha}\oint(\bar{p}_n^i \dot{\bar{q}}_n^i + \bar{\tilde{p}}_n^i \dot{\bar{\tilde{q}}}_n^i)} e^{-\frac{i}{\hbar}\oint H} \\ &= \int \prod_{i=1,2} D\bar{q}_0^i e^{\frac{i}{\alpha}\oint \frac{1}{2}(\dot{\bar{q}}_0^i)^2 - \frac{i}{\alpha}\oint (N_0^i)^2} \prod_{n=1}^{\infty} D\bar{q}_n^i D\bar{\tilde{q}}_n^i e^{\frac{i}{\alpha}\oint\left[\frac{1}{2}((\dot{\bar{q}}_n^i)^2 - n^2(\bar{q}_n^i)^2) + \frac{1}{2}((\dot{\bar{\tilde{q}}}_n^i)^2 - n^2(\bar{\tilde{q}}_n^i)^2)\right]}, \end{aligned} \quad (51)$$

where in the last step we have rescaled time by letting $\frac{\alpha}{\hbar}t \to t$. The integration is w.r.t. time and $X_n^i(t) = (q_n^i(t), p_n^i(t))$.

Since we compactify $X^i(\sigma, \tau)$ on a circle of radius R, we have

$$\frac{q_0^i(\pi T)}{\sqrt{2}} = \frac{q_0^i(-\pi T)}{\sqrt{2}} + 2\pi n' R, \tag{52}$$

where $2\pi T$ is the rescaled time period. To satisfy this boundary condition we write

$$q_0^i(t) = \sqrt{2} R n' \frac{t}{T} + q_0^{i'}(t), \tag{53}$$

where $q_0^{i'}(t)$ is periodic.

Introducing this into the partition function and computing the oscillator integrals we get

$$Z = (\frac{2\pi R}{\sqrt{\alpha}})^2 T det^{-1}\left(\frac{d^2}{dt^2}\right) \prod_{n=1}^{\infty} det^{-2}(\frac{d^2}{d\tau^2} + n^2) \cdot \left(\sum_{n,n'} e^{\frac{i}{\alpha}(\frac{2\pi R^2 n'^2}{T} - 2\pi n^2 R^2 T)}\right)^2. \tag{54}$$

It is easy to evaluate the above determinants under periodic boundary conditions by explicitly finding the eigenvalues of the corresponding operators. They turn out to be

$$\begin{aligned} det(\frac{d^2}{dt^2} + n^2) &= C sin^2 \pi n T, \\ det'\left(\frac{d^2}{dt^2}\right) &= C T^2. \end{aligned} \tag{55}$$

Finally continuing to imaginary time ($t \to -it$) and Poisson resumming in n' we get

$$Z = \frac{1}{|\eta(T)|^4}\left(\sum_{n,m} q^{\frac{n^2 R^2}{\alpha} - \frac{m^2}{4R^2}\alpha}\right)^2 = \frac{1}{|\eta(T)|^4}\left(\sum_{n,m} q^{\frac{1}{2}(\frac{m\sqrt{\alpha}}{2R} + \frac{nR}{\sqrt{\alpha}})^2} q^{\frac{1}{2}(\frac{m\sqrt{\alpha}}{2R} - \frac{nR}{\sqrt{\alpha}})^2}\right)^2. \tag{56}$$

This is precisely the partition function that we got from noncommutative quantization.

Note that, although there is little difference in the calculation of the partition function from the canonical quantization case, there is a big difference in the calculation once insertions are introduced because then it is no longer convenient to transform to barred variables, since the action fields are expanded in the unbarred variables, and certainly the integration results are expected to be different from the canonical case.

4 Conclusions

What we have seen is that it is possible to use a noncommutative harmonic oscillator to quantize bosonic strings in more than one flat target space dimensions. The particle interpretation of excitations is not lost in this quantization procedure. The stress energy tensors corresponding to each coordinate field do not commute to each other. Nevertheless the modes of each stress energy tensor independently satisfy the Virasoro algebra with central charge one. For the particular case of the noncommutative harmonic oscillator described in section two we get that the partition function of the two-dimensional bosonic string compactified on the torus preserves its modular invariance but with a modular parameter that depends on the noncommutativity parameter θ. The four point function as well as certain 2n-point functions acquire cross ratio dependent factors that go to unity as θ goes to zero. Finally it is shown how to formulate the theory in terms of path integrals. In fact the partition function is rederived using path integrals.

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