

On Classification of Non-Abelian Toda Systems

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A simple procedure to enumerate all Toda systems associated with complex classical Lie groups is given.

1 Introduction

By a Toda system we mean a system of nonlinear partial differential equations for functions of two real variables or one complex variable having a special form. A concrete Toda system is specified by the choice of a Lie group and by the choice of a \mathbb{Z} -gradation of the corresponding Lie algebra, see, for example, [1, 2]. Any Toda system is exactly or completely integrable, and this is actually enough to justify the necessity to investigate them. As a matter of fact, they arise in many mathematical and physical problems having as fundamental as application significance.

In this talk we discuss the classification of non-abelian Toda systems associated with classical Lie groups. Such a classification was performed earlier with the help of the root decomposition of the corresponding Lie algebras in papers [3, 4]. Here we use the approach which is not based on root techniques and appeals only to general properties of simple Lie algebras.

2 Toda systems

Let M be either the real manifold \mathbb{R}^2 or the complex manifold \mathbb{C} . Denote the standard coordinates on \mathbb{R}^2 by z^- and z^+ . In the case of the manifold \mathbb{C} we denote by z^- the standard complex coordinate z and by z^+ its complex conjugate \bar{z} ¹.

Recall that a Lie algebra \mathfrak{g} is said to be \mathbb{Z} -graded if there is given a representation

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m,$$

where

$$[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$$

for all $m, n \in \mathbb{Z}$. The subspace \mathfrak{g}_0 is a subalgebra of \mathfrak{g} .

Consider a real or complex matrix Lie group² G whose Lie algebra \mathfrak{g} is endowed with a \mathbb{Z} -gradation. Let for some positive integer l the subspaces \mathfrak{g}_{-m} and \mathfrak{g}_{+m} for $0 < m < l$ be trivial.

¹Actually one can assume that M is an arbitrary two-dimensional real manifold or one-dimensional complex manifold. Here z^- and z^+ are some local coordinates.

²The case of a general, not necessarily matrix, Lie group is considered in the paper [5], and in the book [2].

Denote by G_0 the connected Lie subgroup of G corresponding to the subalgebra \mathfrak{g}_0 . The Toda equations are the matrix equation for a mapping γ from M to G_0 which have the following form

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma].$$

Here c_- and c_+ are some fixed mappings from M to \mathfrak{g}_{-l} and \mathfrak{g}_{+l} respectively, satisfying the conditions

$$\partial_+c_- = 0, \quad \partial_-c_+ = 0.$$

When the Lie group G_0 is abelian the corresponding Toda system is said to be abelian, otherwise we deal with a nonabelian Toda system.

There exist the so-called higher grading [6, 7, 8, 9] and multi-dimensional [10, 11] generalisations of the Toda systems.

3 \mathbb{Z} -gradations

It is clear that to classify Toda systems one has to classify \mathbb{Z} -gradations of Lie algebras. Let us restrict ourselves to the case of Lie algebras corresponding to classical Lie groups. These are complex special linear, orthogonal and symplectic groups. All these groups and the corresponding Lie algebras are simple. This fact allows one to describe all \mathbb{Z} -gradations. Here the following facts are used.

Let us have some \mathbb{Z} -gradation of a Lie algebra \mathfrak{g} . Define a linear operator D acting on an element $x = \sum_{m \in \mathbb{Z}} x_m$ as

$$Dx = \sum_{m \in \mathbb{Z}} mx_m.$$

The Jacobi identity implies that the operator D satisfies the relation

$$D([x, y]) = [Dx, y] + [x, Dy].$$

Hence, D is a derivation of \mathfrak{g} . For any element of $x \in \mathfrak{g}$ the linear operator $\text{ad}(x)$ defined by

$$\text{ad}(x)y = [x, y]$$

is a derivation of \mathfrak{g} . Such a derivation is said to be internal. It is important for our consideration that any derivation of a semisimple Lie algebra is internal. Therefore, in the case when \mathfrak{g} is semisimple, for any \mathbb{Z} -gradation there exists an element q of \mathfrak{g} such that

$$Dx = \text{ad}(q)x = [q, x].$$

The operator $\text{ad}(q)$, or the element q itself, is called the grading operator, corresponding to the \mathbb{Z} -gradation under consideration. If the grading operator exists then the subspaces \mathfrak{g}_m can be described as

$$\mathfrak{g}_m = \{x \in \mathfrak{g} \mid [q, x] = mx\}.$$

Note now that if q is the grading operator generating some \mathbb{Z} -gradation, then the operator $\text{ad}(q)$ is diagonalisable. Let us recall the following fact from the theory of semisimple Lie algebras.

Let \mathfrak{g} be a complex semisimple Lie algebra, and φ be its linear representation. If for some $x \in \mathfrak{g}$ the operator $\text{ad}(x)$ is diagonalisable, then the linear operator $\varphi(x)$ is also diagonalisable.

In the case of classical Lie algebras we always have a special representation, it is the defining representation. The above facts say in this case that any \mathbb{Z} -gradation is generated by the corresponding grading operator, and that this operator, up to an automorphism of the Lie algebra under consideration, is a diagonal matrix.

Now, let us proceed to the description of concrete \mathbb{Z} -gradations and corresponding Toda systems.

4 Special linear groups

We start with the Lie groups $\mathrm{SL}_n(\mathbb{C})$ and the corresponding Lie algebras $\mathfrak{sl}_n(\mathbb{C})$.

The grading operator q corresponding to a \mathbb{Z} -gradation of $\mathfrak{sl}_n(\mathbb{C})$ is a diagonalisable matrix. Hence, up to an internal automorphism any grading operator q has the block matrix form

$$q = \begin{pmatrix} \rho_1 I_{k_1} & 0 & \cdots & 0 \\ 0 & \rho_2 I_{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_p I_{k_p} \end{pmatrix}, \quad (1)$$

where I_k is the $k \times k$ identity matrix, and we assume that $\mathrm{Re} \rho_1 \geq \mathrm{Re} \rho_2 \geq \dots \geq \mathrm{Re} \rho_p$. The grading operator q of the above form belongs to the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ if $\sum_{a=1}^p k_a = n$ and

$$\sum_{a=1}^p \rho_a k_a = 0. \quad (2)$$

Represent a general element x of $\mathfrak{sl}_n(\mathbb{C})$ in the block matrix form

$$x = \begin{pmatrix} x_{11} & \cdots & x_{1p} \\ \vdots & \ddots & \vdots \\ x_{p1} & \cdots & x_{pp} \end{pmatrix}, \quad (3)$$

where x_{ab} is a $k_a \times k_b$ matrix. It is clear that

$$[q, x]_{ab} = (\rho_a - \rho_b)x_{ab}.$$

Hence, since q generates a \mathbb{Z} -gradation we should have $m_a = \rho_a - \rho_{a+1} \in \mathbb{Z}$. It follows that all numbers ρ_a have the same imaginary part, and in reality the equality (2) implies that they are real and we will assume that $\rho_1 > \rho_2 > \dots > \rho_p$. The numbers ρ_a can be expressed via the integers k_a and m_a :

$$\rho_a = \frac{1}{n} \left(-\sum_{b=1}^{a-1} m_b \sum_{c=1}^b k_c + \sum_{b=a}^{p-1} m_b \sum_{c=b+1}^p k_c \right). \quad (4)$$

Thus a \mathbb{Z} -gradation of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ is uniquely specified by the choice of p integers k_a such that $\sum_{a=1}^p k_a = n$ and by the choice of $p-1$ positive integers m_a . The grading operator q has the form (1), where the numbers ρ_a are given by the relation (4).

The grading structure of the Lie algebras $\mathfrak{sl}_n(\mathbb{C})$ can be depicted by the following scheme

$$\begin{pmatrix} 0 & m_1 & m_1 + m_2 & \cdots & \sum_{a=1}^{p-1} m_a \\ -m_1 & 0 & m_2 & \cdots & \sum_{a=2}^{p-1} m_a \\ -(m_1 + m_2) & -m_2 & 0 & \cdots & \sum_{a=3}^{p-1} m_a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sum_{a=1}^{p-1} m_a & -\sum_{a=2}^{p-1} m_a & -\sum_{a=3}^{p-1} m_a & \cdots & 0 \end{pmatrix}.$$

Here the numbers in the boxes correspond to the grading indices of the corresponding blocks in the block matrix representation (3) of a general element of $\mathfrak{sl}_n(\mathbb{C})$. Note, in particular, that the subalgebra \mathfrak{g}_0 is formed by all block diagonal matrices. The group G_0 is also formed by block diagonal matrices and is isomorphic to $\mathrm{GL}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_p}(\mathbb{C})$.

Consider now Toda systems associated with the Lie group $\mathrm{SL}_n(\mathbb{C})$. Actually, it is more convenient to deal with the Lie group $\mathrm{GL}_n(\mathbb{C})$. Any grading operator for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ can be considered as a grading operator for the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.

It can be easily understood that to exhaust all Toda systems it suffices to consider only gradations with all numbers m_a equal to 1 [4]. In this case the mappings c_- and c_+ should take values in the subspaces \mathfrak{g}_{-1} and \mathfrak{g}_{+1} respectively. The general forms of such elements are

$$c_- = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ C_{-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & C_{-(p-1)} & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_{+1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{+(p-1)} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (5)$$

where for each $a = 1, \dots, p-1$ the mapping C_{-a} takes values in the space of $k_{a+1} \times k_a$ complex matrices, and the mapping C_{+a} takes values in the space of $k_a \times k_{a+1}$ complex matrices. Besides, these mappings must satisfy the relations

$$\partial_+ C_{-a} = 0, \quad \partial_- C_{+a} = 0.$$

Parametrise the mapping γ as

$$\gamma = \begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_p \end{pmatrix}, \quad (6)$$

where the mappings Γ_a take values in the Lie groups $\mathrm{GL}_{k_a}(\mathbb{C})$. In this parametrisation Toda equations take the form

$$\begin{aligned} \partial_+ (\Gamma_1^{-1} \partial_- \Gamma_1) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}, \\ \partial_+ (\Gamma_a^{-1} \partial_- \Gamma_a) &= -\Gamma_a^{-1} C_{+a} \Gamma_{a+1} C_{-a} + C_{-(a-1)} \Gamma_{a-1}^{-1} C_{+(a-1)} \Gamma_a, \quad 1 < a < p, \\ \partial_+ (\Gamma_p^{-1} \partial_- \Gamma_p) &= C_{-(p-1)} \Gamma_{p-1}^{-1} C_{+(p-1)} \Gamma_p. \end{aligned}$$

The simplest case is when one chooses $k_a = k$ and $C_{-a} = C_{+a} = I_k$:

$$\begin{aligned} \partial_+ (\Gamma_1^{-1} \partial_- \Gamma_1) &= -\Gamma_1^{-1} \Gamma_2, \\ \partial_+ (\Gamma_a^{-1} \partial_- \Gamma_a) &= -\Gamma_a^{-1} \Gamma_{a+1} + \Gamma_{a-1}^{-1} \Gamma_a, \quad 1 < a < p, \\ \partial_+ (\Gamma_p^{-1} \partial_- \Gamma_p) &= \Gamma_{p-1}^{-1} \Gamma_p. \end{aligned}$$

5 Orthogonal Lie groups

It is convenient for our purposes to define the complex orthogonal group $\mathrm{O}_n(\mathbb{C})$ as the Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ formed by the elements $a \in \mathrm{GL}_n(\mathbb{C})$ satisfying the condition

$$a^t J_n a = J_n, \quad (7)$$

where J_n is the skew-diagonal $n \times n$ unit matrix. The Lie algebra $\mathfrak{o}_n(\mathbb{C})$ of the Lie group $O_n(\mathbb{C})$ consists of $n \times n$ complex matrices x satisfying the condition

$$x^t J_n + J_n x = 0. \quad (8)$$

For a $k_1 \times k_2$ matrix a we will denote

$$a^T = J_{k_2} a^t J_{k_1}.$$

Actually, a^T is the transpose of a with respect to the skew diagonal. The conditions (7) and (8) can be written now as $a^T = a^{-1}$ and $x^T = -x$, respectively.

The Lie algebras $\mathfrak{o}_n(\mathbb{C})$ are simple and it is clear that any \mathbb{Z} -gradation of $\mathfrak{o}_n(\mathbb{C})$ is generated by the corresponding grading operator, which has the form (1) and belongs to $\mathfrak{o}_n(\mathbb{C})$. Hence, a \mathbb{Z} -gradation of the Lie algebra $\mathfrak{o}_n(\mathbb{C})$ is uniquely specified by the choice of p integers k_a such that $\sum_{a=1}^p k_a = n$, $k_a = k_{p-a+1}$, and by the choice of $p-1$ positive integers m_a such that $m_a = m_{p-a}$.

Consider now the Toda equations associated with the Lie groups $O_n(\mathbb{C})$. The general form of the mappings c_- and c_+ is again given by (5), but here the mappings $C_{\pm a}$ should obey the relations

$$C_{-a}^T = -C_{-(p-a)}, \quad C_{+a}^T = -C_{+(p-a)}. \quad (9)$$

The Lie group G_0 in the case $p = 2s - 1$ is isomorphic to $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_{s-1}}(\mathbb{C}) \times SO_{k_s}(\mathbb{C})$ while in the case $p = 2s$ it is isomorphic to $GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_s}(\mathbb{C})$. We can use the same parametrisation (6) for γ as for the case of the Lie group $GL_n(\mathbb{C})$. Here one has

$$\Gamma_a^T = \Gamma_{p-a+1}^{-1}. \quad (10)$$

The Toda equations have the same form as for the case of the Lie groups $GL_n(\mathbb{C})$. One only has to take into account the relations (9) and (10). In the case of $p = 2s - 1$ we have s independent mappings Γ_a and the equations for them are

$$\begin{aligned} \partial_+ (\Gamma_1^{-1} \partial_- \Gamma_1) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}, \\ \partial_+ (\Gamma_a^{-1} \partial_- \Gamma_a) &= -\Gamma_a^{-1} C_{+a} \Gamma_{a+1} C_{-a} + C_{-(a-1)} \Gamma_{a-1}^{-1} C_{+(a-1)} \Gamma_a, \quad 1 < a < s, \\ \partial_+ (\Gamma_s^{-1} \partial_- \Gamma_s) &= -\Gamma_s^T C_{+(s-1)}^T \Gamma_{s-1}^{-1T} C_{-(s-1)}^T + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_s. \end{aligned}$$

Stress on that in this case $\Gamma_s^T = \Gamma_s^{-1}$. In the case $p = 2s$ we have the equations

$$\begin{aligned} \partial_+ (\Gamma_1^{-1} \partial_- \Gamma_1) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}, \\ \partial_+ (\Gamma_a^{-1} \partial_- \Gamma_a) &= -\Gamma_a^{-1} C_{+a} \Gamma_{a+1} C_{-a} + C_{-(a-1)} \Gamma_{a-1}^{-1} C_{+(a-1)} \Gamma_a, \quad 1 < a < s, \\ \partial_+ (\Gamma_s^{-1} \partial_- \Gamma_s) &= -\Gamma_s^{-1} C_{+s} \Gamma_s^{-1T} C_{-s} + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_s, \end{aligned}$$

where the mappings $C_{\pm s}$ satisfy the relations

$$C_{-s}^T = -C_{-s}, \quad C_{+s}^T = -C_{+s}. \quad (11)$$

6 Symplectic Lie groups

Define the Lie group $Sp_{2n}(\mathbb{C})$ as the Lie subgroup of the Lie group $GL_{2n}(\mathbb{C})$ formed by the elements $a \in GL_{2n}(\mathbb{C})$ which satisfy the condition

$$a^t K_{2n} a = K_{2n},$$

where

$$K_{2n} = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

Then the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ of $\mathrm{Sp}_{2n}(\mathbb{C})$ is formed by all $2n \times 2n$ complex matrices x satisfying the condition

$$x^t K_{2n} + K_{2n} x = 0.$$

The Lie algebras $\mathfrak{sp}_{2n}(\mathbb{C})$ are simple and any \mathbb{Z} -gradation of $\mathfrak{sp}_{2n}(\mathbb{C})$ is generated by the grading operator, which has the form (1) and belongs to $\mathfrak{sp}_{2n}(\mathbb{C})$. One can get convinced that a \mathbb{Z} -gradation of the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is uniquely specified by the same data as for a \mathbb{Z} -gradation of the Lie algebra $\mathfrak{o}_{2n}(\mathbb{C})$.

Consider now the corresponding Toda equations. The mappings c_- and c_+ have the forms (5) with the mappings $C_{\pm a}$ satisfying the relations

$$\begin{aligned} C_{-a}^T &= -C_{-(p-a)}, & C_{+a}^T &= -C_{+(p-a)}, & a &\neq s-1, s, \\ J_{k_{s-1}} C_{-(s-1)}^t K_{k_s} &= -C_{-s}, & K_{k_s} C_{+(s-1)}^t J_{k_{s-1}} &= -C_{+s} \end{aligned}$$

when $p = 2s - 1$, and

$$\begin{aligned} C_{-a}^T &= -C_{-(p-a)}, & C_{+a}^T &= -C_{+(p-a)}, & a &\neq s, \\ C_{-s}^T &= C_{-s}, & C_{+s}^T &= C_{+s}, \end{aligned}$$

when $p = 2s$.

The Lie group G_0 in the case of $p = 2s - 1$ is isomorphic to $\mathrm{GL}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_{s-1}}(\mathbb{C}) \times \mathrm{Sp}_{k_s}(\mathbb{C})$, while in the case of $p = 2s$ it is isomorphic to $\mathrm{GL}_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_s}(\mathbb{C})$. We can use the parametrisation (6) for the mapping γ . Here, in the case of $p = 2s - 1$, one has

$$\Gamma_a^T = \Gamma_{p-a+1}^{-1}, \quad a \neq s+1, \quad \Gamma_s^t K_{k_s} \Gamma_s = K_{k_s},$$

whereas in the case of $p = 2s$

$$\Gamma_a^T = \Gamma_{p-a+1}^{-1}$$

for any a . The independent Toda equations in the case of $p = 2s - 1$ have the form

$$\begin{aligned} \partial_+ (\Gamma_1^{-1} \partial_- \Gamma_1) &= -\Gamma_1^{-1} C_{+1} \Gamma_2 C_{-1}, \\ \partial_+ (\Gamma_a^{-1} \partial_- \Gamma_a) &= -\Gamma_a^{-1} C_{+a} \Gamma_{a+1} C_{-a} + C_{-(a-1)} \Gamma_{a-1}^{-1} C_{+(a-1)} \Gamma_a, \quad 1 < a < s, \\ \partial_+ (\Gamma_s^{-1} \partial_- \Gamma_s) &= \Gamma_s^{-1} K_{k_s} C_{+(s-1)}^t \Gamma_{s-1}^{-1t} C_{-(s-1)}^t K_{k_s} + C_{-(s-1)} \Gamma_{s-1}^{-1} C_{+(s-1)} \Gamma_s. \end{aligned}$$

In the case of $p = 2s$ one has the same equations as for the Lie groups $\mathrm{O}_{2n}(\mathbb{C})$, but with the mappings $C_{\pm s}$ satisfying instead of (11) the relations

$$C_{-s}^T = C_{-s}, \quad C_{+s}^T = C_{+s}.$$

7 Simplest example

A first non-abelian Toda system which was integrated explicitly was a system associated with the Lie group $\mathrm{O}_5(\mathbb{C})$. We start this section with the description of this system along lines used in early papers [12, 13, 14]. The Lie algebra $\mathfrak{o}_5(\mathbb{C})$ of the Lie group $\mathrm{O}_5(\mathbb{C})$ is of type B_2 . Let h_1, h_2 be Cartan generators, and $x_{\pm 1}, x_{\pm 2}$ be the Chevalley generators corresponding to the simple roots

α_1, α_2 . Consider the \mathbb{Z} -gradation generated by the grading operator $q = 2h_1 + h_2$. The grading subspaces have the forms

$$\begin{aligned}\mathfrak{g}_{-1} &= \mathbb{C} \mathfrak{g}^{-\alpha_1} \oplus \mathbb{C} \mathfrak{g}^{-\alpha_1 - \alpha_2} \oplus \mathbb{C} \mathfrak{g}^{-2\alpha_1 - \alpha_2} \\ \mathfrak{g}_0 &= \mathbb{C} \mathfrak{g}^{-\alpha_2} \oplus \mathfrak{h} \oplus \mathbb{C} \mathfrak{g}^{\alpha_2} \\ \mathfrak{g}_{+1} &= \mathbb{C} \mathfrak{g}^{\alpha_1} \oplus \mathbb{C} \mathfrak{g}^{\alpha_1 + \alpha_2} \oplus \mathbb{C} \mathfrak{g}^{2\alpha_1 + 2\alpha_2}.\end{aligned}$$

Choose the fixed mappings c_- and c_+ as

$$c_+ = [x_{+1}, x_{+2}], \quad c_- = [x_{-2}, x_{-1}],$$

and parametrise the mapping γ as

$$\gamma = \exp(a_+ x_{+2}) \exp(a_- x_{-2}) \exp(a_1 h_1 + a_2 h_2)$$

Then the Toda equations take the form

$$\partial_+ \partial_- a_1 = -2e^{-a_1} (1 + a_- a_+), \quad (12)$$

$$\partial_+ (\partial_- a_1 - \partial_- a_2 - a_- \partial_- a_+) = -e^{-a_2} (1 + 2a_- a_+), \quad (13)$$

$$\partial_+ (e^{a_1 - 2a_2} \partial_- a_+) = 2e^{-2a_2} a_+, \quad (14)$$

$$\partial_+ [e^{-a_1 + 2a_2} (\partial_- a_- - a_-^2 \partial_- a_+)] = 2e^{-2a_1 + 2a_2} a_- (1 + a_- a_+). \quad (15)$$

Now, let us show how the system described above enters into our classification of non-abelian Toda systems. First of all notice that the Lie group $O_5(\mathbb{C})$ is locally isomorphic to the Lie group $Sp_4(\mathbb{C})$. Moreover, the formulation of the system was based only on local properties of the Lie group $O_5(\mathbb{C})$. Therefore, we will come to the same system starting from the Lie group $Sp_4(\mathbb{C})$. This also allows one to include the system under consideration in a series of non-abelian Toda systems associated with the Lie groups $Sp_{2n}(\mathbb{C})$ which have an extremely simple form.

Endow the Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ with a \mathbb{Z} -gradation generated by the grading operator

$$q = \frac{1}{2} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

The parametrisation of the mapping γ corresponding with this \mathbb{Z} -gradation has the form

$$\gamma = \begin{pmatrix} \Gamma & 0 \\ 0 & (\Gamma^T)^{-1} \end{pmatrix}.$$

Choose the mappings c_- and c_+ as

$$c_- = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}.$$

The Toda equations have in our case the form

$$\partial_+ (\Gamma^{-1} \partial_- \Gamma) = -(\Gamma^T \Gamma)^{-1}. \quad (16)$$

In the case of $n = 2$, if one parametrises Γ as

$$\Gamma = \begin{pmatrix} e^{a_1} (1 + a_- a_+) & e^{-a_1 + a_2} a_+ \\ e^{a_1} a_- & e^{-a_1 + a_2} \end{pmatrix},$$

then the Toda equations (16) take the form (12)–(15). Certainly, the equation (16) is more attractive than the equations (12)–(15).

In the case of $n = 1$ the mapping Γ is just a function, and we have the equation

$$\partial_+(\Gamma^{-1}\partial_-\Gamma) = -\Gamma^{-2}.$$

Introducing a function F via the relation

$$\Gamma = e^F$$

one comes to the famous Liouville equation

$$\partial_+\partial_-\Gamma = -e^{-2F}.$$

Therefore, it is quite natural for general n to call the equation

$$\partial_+(\Gamma^{-1}\partial_-\Gamma) = -(\Gamma^T\Gamma)^{-1}$$

the non-abelian Liouville equation.

8 Conclusions

We have shown that the classification of the non-abelian Toda systems associated with complex classical Lie groups can be performed using only some general properties of the semisimple Lie algebras. The arising block matrix structure appears to be very convenient. For example, it allows one to find explicit forms for W -algebras corresponding to non-abelian Toda systems [15, 16, 17].

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