Classification of Singularities on the World Sheets of Relativistic Strings

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We show that relativistic strings of open and closed types in Minkowski space-time of dimension 3 and 4 have topologically stable singular points. This paper describes the structure of singularities, derives their normal forms, and introduces a local characteristic of singularity (topological charge), possessing a global law of conservation. Two other types of solutions (breaking and exotic strings) are also considered, which have singularities at arbitrary value of dimension.

Keywords: classical string theory, singularities of differentiable mappings, scientific visualization.

1. Introduction

Classical string theory considers time-like surfaces of extreme area in d-dimensional Minkowski space-time, called *the world sheets of strings.* These surfaces are spanned in motion through the space-time of 1-dimensional object, called relativistic string. The string theory is used in high energy physics to model the inner structure of elementary particles. For this purpose the world sheets (WS), having microscopic spatial sizes and infinitely extended in temporal direction, are considered as structured world lines of elementary particles. As a result, the internal characteristics of particles, such as mass and spin, are expressed in terms of string dynamics and can be derived from a small set of fundamental constants. The construction of string theory encountered several hard problems, concerning to quantum mechanical representation of infinite-dimensional groups of symmetries, which nowadays are solved completely only in high-dimensional space-time ($d = 26$) and in certain topologically non-trivial space-time manifolds [1, 2]. Also, certain subsets in the phase space of string theory were found in [3-5], which admit anomaly-free quantization at arbitrary dimension of the space-time. On the other hand, in several works [6-12] authors noticed singular properties of the classical string theory in low-dimensional ($d = 3, 4$) flat Minkowski space-time. In this paper we will discuss this topic in more detail.

Let WS is represented parametrically as $x_\mu(\sigma_1, \sigma_2)$. Time-likeness means that the WS admits a parametrization with $(\partial_1 x)^2 \geq 0$, $(\partial_2 x)^2 \leq 0$, i.e. one tangent vector should be time-like or light-like, while another one should be space-like or light-like. The area of WS in this case can be written as

$$A = \int\int d\sigma_1 d\sigma_2 \sqrt{(\partial_1 x\ \partial_2 x)^2 - (\partial_1 x)^2 (\partial_2 x)^2} = \text{extremum.}$$

Theorem 1 (type of extremum): a regular point of WS is a saddle point of area functional. (Proof of the theorem is placed in Appendix.)

The area of WS is minimal with respect to local variations, extended in temporal direction, changing mostly the length of strings in equal-time slices, see fig.1; and maximal for variations, extended in spatial direction, when mostly the interval of world lines of points on the string is changed.

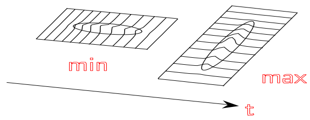


Fig.1. *Type of extremum is saddle point.*

String theory considers WS of various topological types, see fig.2: open strings – surfaces, homeomorphic to bands $I \times \mathbf{R}^1$, closed strings – cylinders $S^1 \times \mathbf{R}^1$, Y-shaped strings – 3 bands, glued together along one edge, and also the surfaces of more complex topology, corresponding to transitions between the described types (decays and transmutations of particles).

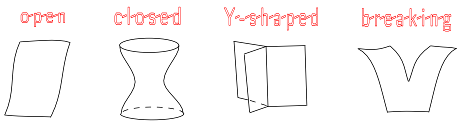


Fig.2. *Main topological types of WS.*

Extremum of area for each topological type leads to Lagrange-Euler equations, satisfied in internal points of WS, which have a form of local conservation of energy-momentum:

$$p_i^\mu = \delta A/\delta(\partial_i x_\mu), \quad \partial_i p_i^\mu = 0,$$

and boundary conditions, implying that total flow of momentum through the boundary vanishes. For example, open string satisfies the equation $p_i^\mu \epsilon_{ij} d\sigma_j = 0$ on the boundary (here $d\sigma_j$ is tangent element of WS boundary on the parameters plane, $\epsilon_{ij} d\sigma_j$ is normal element); for Y-shaped string $\sum p_i^\mu \epsilon_{ij} d\sigma_j = 0$ on the world line of the node, where the sum is taken for three surfaces attached to this line.

Construction of string theory usually proceeds in Hamiltonian approach. The coordinates on WS are distinguished: σ — compact coordinate, $\sigma \in I$ for open string, $\sigma \in S^1$ for closed string; $\tau \in \mathbf{R}^1$ — non-compact coordinate, called evolution parameter. The following denotations are introduced: $\dot{x} = \partial x/\partial\tau$, $x' = \partial x/\partial\sigma$, $p^\mu = p_\tau^\mu$, so that the action of string theory (area) can be written as $A = \int d\tau \int d\sigma\ \mathcal{L}$, where $\mathcal{L} = \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}$ is density of Lagrangian. Poisson brackets are introduced as $\{x_\mu(\sigma,\tau), p_\nu(\tilde{\sigma},\tau)\} = g_{\mu\nu}\delta(\sigma - \tilde{\sigma})$, where $g_{\mu\nu} = \mathrm{diag}(+1, -1, ..., -1)$ is metric tensor and $\delta()$ is Dirac's function. The problem is stated to find a solution $x(\sigma,\tau), p(\sigma,\tau)$ starting from the initial data $x(\sigma,0), p(\sigma,0)$. This evolution is described by a system of autonomous differential equations, simultaneous by τ (so that τ-dependence is usually omitted, implying that all variables are estimated at the same value of evolution parameter).

String theory is Hamiltonian theory with the 1st class constraints [13, 14], this means the following. The density of canonical Hamiltonian is defined by Legendre transformation as $\mathcal{H}_c = \dot{x}p - \mathcal{L}$. Substitution of p-definition in terms of $x', \dot{x}$ vanishes canonical Hamiltonian $\mathcal{H}_c = 0$, and additionally creates the following identities: $\Phi_1 = x'p = 0,\ \Phi_2 = x'^2 + p^2 = 0$. Appearance of these identities, called Dirac's constraints, is related with the symmetry of the action under the group of reparametrizations (right diffeomorphisms) of WS, i.e. smooth invertible mappings $(\sigma,\tau) \to (\tilde{\sigma},\tilde{\tau})$. Usually consideration of theories with constraints starts in extended phase space (x,p), where Hamiltonian is defined as linear combination of constraints, in our case $H = \int d\sigma(V_1\Phi_1 + V_2\Phi_2)$. Here the coefficients $V_{1,2}(\sigma)$ are arbitrary and called Lagrangian multipliers. On the surface $\Phi_i = 0$ the Hamiltonian vanishes[1], however its derivatives do not vanish and create Hamiltonian vector field:

$$\dot{x}_\mu(\sigma) = \delta H/\delta p_\mu(\sigma),\ \dot{p}_\mu(\sigma) = -\delta H/\delta x_\mu(\sigma).$$

[1]This scalar Hamiltonian does not coincide with the energy, which in relativistic theories is given by a component of momentum p_0.

This field is tangent to the surface $\Phi_i = 0$ due to the fact that the Poisson brackets $\{\Phi_i(\sigma), \Phi_j(\tilde{\sigma})\}$ vanish on the the surface $\Phi_i = 0$, in this case the constraints are said to be of the 1st class. Phase trajectory, integrated from such field, belongs to the surface $\Phi_i = 0$, and its projection to coordinate space $\{x\}$ gives a solution of Lagrange-Euler equations. In string theory Φ_i-terms of Hamiltonian generate infinitesimal shifts of points in tangent directions to the WS: Φ_1 generates the shifts $\delta x \sim x'$, while Φ_2 generates $\delta x \sim p \perp x'$, see fig.3. Together the constraints generate all possible reparametrizations of WS (connected component of right diffeomorphisms group).

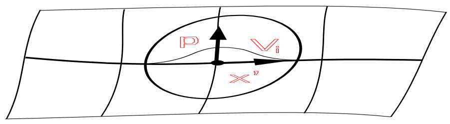


Fig.3. *Constraints generate reparametrizations.*

Coefficients $V_{1,2}$ influence only the parametrization of WS, a choice $V_1 = 0,\ V_2 = 1$ corresponds to conformal parametrization $\dot{x}x' = 0,\ \dot{x}^2 + x'^2 = 0$. This choice linearizes Hamiltonian equations. The equations can be solved by different methods, see [14, 15, 1, 11]. The result has the following geometric representation [14].

2. Structure of solutions: open and closed strings

Reconstruction of WS is based on the concept of *supporting curves*. Let's consider a curve $Q(\sigma)$ in Minkowski space with the following properties: (1) periodicity: $Q(\sigma + 2\pi) = Q(\sigma) + 2P$; (2) light-likeness: $Q'^2 = 0$.

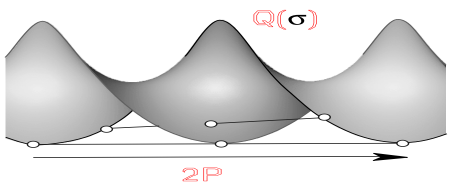


Fig.4. *WS of open string.*

WS of open string can be reconstructed by this curve as follows: $x(\sigma_1, \sigma_2) = (Q(\sigma_1)+Q(\sigma_2))/2,\ \sigma_1 \leq \sigma_2 \leq \sigma_1 + 2\pi$, see fig.4. The obtained surface has two edges, one coincident with $Q(\sigma)$, another one is $Q(\sigma) + P$. Translation by vector P transforms the WS to itself with the edges interchanged.

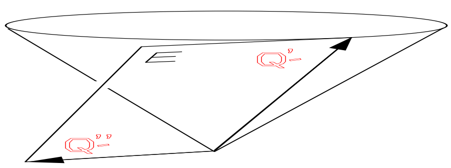


Fig.5. *Structure of WS near the edge.*

Theorem 2 (edge of WS): any vector from the tangent plane to open WS on its edge is orthogonal to Q'. This plane is tangent to the light cone in the direction of Q' (such planes are called *isotropic*). As a consequence, end points of open string move at light velocity perpendicularly to the direction of the string in these points.

To construct the WS of closed string, one should consider two supporting curves $Q_{1,2}(\sigma)$. Both should be light-like: $Q_{1,2}'^2 = 0$ and periodical with the same period: $Q_{1,2}(\sigma + 2\pi) = Q_{1,2}(\sigma) + P$. The WS is given by the formula: $x(\sigma_1, \sigma_2) = (Q_1(\sigma_1) + Q_2(\sigma_2))/2,\ \sigma_1 \leq \sigma_2 \leq \sigma_1 + 4\pi$.

Variables σ_i introduce light-like coordinates on WS: $(\partial_i x)^2 = 0$, related with Hamiltonian coordinates as $\sigma_{1,2} = \tau \pm \sigma$. The supporting curves can be reconstructed from initial data: coordinate x and density of momentum p on the string at value $\tau = 0$ by the formulae

$$
\begin{aligned}
&a) \qquad Q(\sigma) = x(\sigma) + \int_0^{\sigma} d\tilde{\sigma}\; p(\tilde{\sigma}), \text{ for open string;} \\
&b) \qquad Q_{1,2}(\sigma) = x(\pm\sigma) \pm \int_0^{\pm\sigma} d\tilde{\sigma}\; p(\tilde{\sigma}), \text{ for closed string.}
\end{aligned} \tag{1}
$$

In the last formula the curve Q_1 is given by upper choice of all signs, while Q_2 is given by lower signs. The functions $x(\sigma), p(\sigma)$ are continued to the whole axis of σ, see [1]: as 2π-periodical functions in (1b) and as even 2π-periodical functions in (1a).

Remark: Transition to complex variables $\tau \to i\tau$, $p \to ip$ in the above formulae gives a solution of Björling's problem [16] for minimal surfaces in Euclidean space (find a minimal surface, passing through a given curve and tangent to a given vector field on the curve).

The inverse formulae are:

a) $x(\sigma) = (Q(\sigma) + Q(-\sigma))/2,\ p(\sigma) = (Q'(\sigma) + Q'(-\sigma))/2,$
b) $x(\sigma) = (Q_1(\sigma) + Q_2(-\sigma))/2,\ p(\sigma) = (Q_1'(\sigma) + Q_2'(-\sigma))/2.$

From (1ab) we see that the vector P coincides with the total energy-momentum of the string. In particular case $Q_1 = Q_2$ the WS of closed string degenerates to 2-folded WS of open string. In this case the periods of supporting curves are $P_{closed} = 2P_{open}$, as a result, each of two coincident sheets of resulting open string receives a half of energy-momentum of original closed string.

Subspace, orthogonal to P, forms the center-of-mass frame (CMF). In projection to CMF supporting curves become closed. The curves, whose temporal component $Q^0(\sigma)$ *is not* monotonous function, will be considered in the next section. Here we consider supporting curves, satisfying at each point the condition $Q'^0 > 0$. Such curves can be restored by their projection to CMF:

$$
Q_0' = |\vec{Q}'| \;\Rightarrow\; Q_0(\sigma_1) - Q_0(0) = \int_0^{\sigma_1} |\vec{Q}'(\sigma)| d\sigma = L(\sigma_1),
$$

$L(\sigma)$ is the arc length of the curve $\vec{Q}(\sigma)$ between the points $\vec{Q}(0)$ and $\vec{Q}(\sigma)$. The total length of the curve $\vec{Q}(\sigma)$ is equal to $2\sqrt{P^2}$ for open string and $\sqrt{P^2}$ for closed string. One can parameterize the curve $\vec{Q}(\sigma)$ by its length: $\sigma = 2\pi L/L_{tot}$, then $Q_0' = |\vec{Q}'| = L_{tot}/2\pi$. This parametrization is called Rohrlich gauge [17].

At the value of space-time dimension $d = 3, 4$ the strings have stable singular points. Their appearance has the following reason.

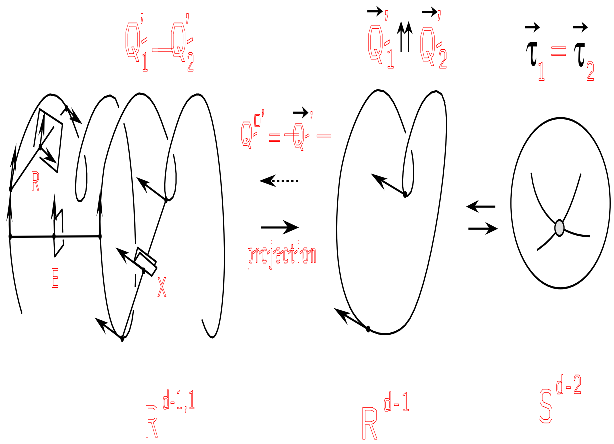


Fig.6. *Singularities of WS correspond to parallel tangent vectors on supporting curve. R – regular point, E – edge, X – internal singular point.*

Tangent vectors to WS in a point $x(\sigma_1, \sigma_2)$ are $Q'(\sigma_1)$ and $Q'(\sigma_2)$, i.e. tangent vectors to the supporting curve in corresponding points. If these two vectors are linearly independent, they define tangent plane to WS, and this point is regular (point R on fig.6). Otherwise, if these vectors are parallel, the point is singular. The point E with $\sigma_2 - \sigma_1 = 0$ or 2π belongs to the edge of WS. The point X with $0 < \sigma_2 - \sigma_1 < 2\pi$ is internal singular point.

Fig.7. *Self-intersection of hodograph $\vec{\tau}(\sigma)$.*

For supporting curves with $Q'^0 > 0$ the linear dependence of vectors Q'_1 and Q'_2 in the space-time is equivalent to the coincidence of unit tangent vectors $\vec{\tau} = \vec{Q}'/|\vec{Q}'|$ to *the projection* of supporting curve to CMF. Therefore, singularities of WS correspond to the points of self-intersection of hodograph $\vec{\tau}(\sigma_1) = \vec{\tau}(\sigma_2)$ (for closed string – intersection of two hodographs $\vec{\tau}_1(\sigma_1) = \vec{\tau}_1(\sigma_2)$). Hodographs belong to a sphere S^{d-2}: to a circle for $d = 3$ and to 2-dimensional sphere for $d = 4$. As a result, for $d = 3$ the intersection of hodographs is 1-dimensional set, and for $d = 4$ the transversal intersections are located in isolated points, see fig. 7. In the case $d > 4$ the intersections can be removed by small variations of hodographs on the sphere S^{d-2}.

From (1) we derive a formula for linear density of energy-momentum on the string:

$$dp_0/dl = 2\gamma,\ d\vec{p}/dl = (\vec{\tau}_1 + \vec{\tau}_2)\gamma,\ \gamma = (2(1 - \vec{\tau}_1\vec{\tau}_2))^{-1/2}.$$

In the singular points and on the edge of open WS $\gamma \to \infty$, therefore in these points the linear density of energy-momentum tends to infinity.

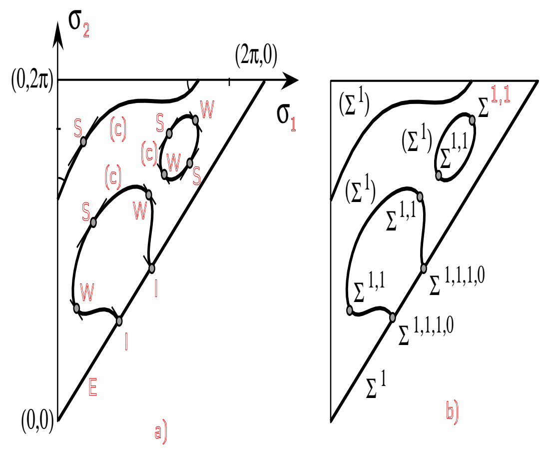


Fig.8. *a) singular lines on parametric plane (scheme); b) classification of singularities according to degeneration of the 1st differential [18].*

Case $\mathbf{d = 3}$: representing $\vec{\tau}(\sigma) = (\cos\varphi(\sigma), \sin\varphi(\sigma))$, singularities are defined by the equation $\varphi(\sigma_1) = \varphi(\sigma_2) \bmod 2\pi$.

For open string this equation should be solved inside a triangle on (σ_1, σ_2)-plane, shown on fig.8a, and then continued to the whole WS using the trivial symmetries $\sigma_1 \leftrightarrow \sigma_2$, $\sigma_{1,2} \to \sigma_{1,2} + 2\pi k$, $k \in \mathbf{Z}$. For closed string the equation $\varphi_1(\sigma_1) = \varphi_2(\sigma_2) \bmod 2\pi$ should be solved in a square $0 < \sigma_{1,2} < 2\pi$, and continued by $\sigma_{1,2} \to \sigma_{1,2} + 2\pi k$. Typically, for supporting curves in general position, solutions of these equations form smooth curves on parametric plane, denoted on fig.8a as (c), for which the following behavior is possible:

- curves extend on the WS not reaching its edges;
- curves can form closed loops;
- for open strings the curve (c) can terminate on the edge.

In the last case the curve (c) enters to the edge at right angle, due to the symmetry $\sigma_1 \leftrightarrow \sigma_2$. Additionally, the points are marked on fig.8a, where the curve (c) is tangent to directions $d\sigma_2 = \pm d\sigma_1$. The evolution is performed by equal-time slices $x_0 = Const$, which in Rohrlich gauge correspond to $\sigma_1 + \sigma_2 = Const$. A structure of WS near singular points is shown on fig.9:

$$\begin{aligned}
E: &\quad (v, u^2, 0) \\
c: &\quad (v, u^2, u^3) \\
S: &\quad (v, u^2, vu^3) \\
W: &\quad (v^2 + 2u, v^3 + 3vu, v^4 + 4v^2u) \\
I: &\quad (v^2 + u, v^4 + 2v^2u, v^6 + 3v^4u)
\end{aligned}$$

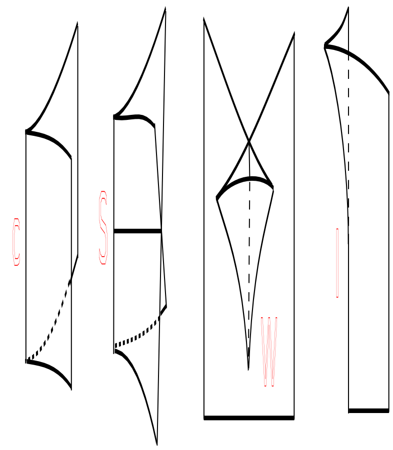


Fig.9. *Normal forms of singularities, $d = 3$.*

The normal forms are given in light coordinates $(t, y, x - t)$, so that the direction $(1, 0, 0)$ is light-like, and together with the direction $(0, 1, 0)$ it defines the isotropic plane.

E: edge of WS;

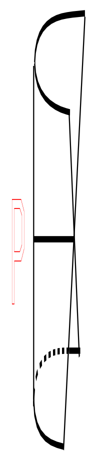
c: equal-time slices of WS represent a cusp $(0, u^2, u^3)$, moving in the space-time in light-like direction $(1, 0, 0)$, i.e. cusp moves at light velocity perpendicularly to its own direction $(0, 1, 0)$.

S: line of self-intersection of the surface terminate on the cusp line, in this point "the wings" of cusp pass through each other.

W: swallowtail [18], a point of creation or annihilation of two cusps.

I: for open string the cusp can appear/disappear alone at the edge of WS, in the point where this curve has inflection: $\varphi'(\sigma_I) = 0$. In other words, a cusp absorbed by the end of the string inflects its trajectory.

Case $\mathbf{d = 4}$



$$\begin{aligned}
E: &\quad (v, u^2, 0, 0) \\
P: &\quad (v, u^2, vu, u^3)
\end{aligned}$$

Fig.10. *Normal forms of singularities, $d = 4$.*

Here we use light coordinates $(t, y, z, x - t)$. The direction $(1, 0, 0, 0)$ is light-like, and together with $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ it defines an isotropic 3-dimensional hyperplane, tangent to light cone in 4 dimensions.

Equal-time slices of WS near point (P) give a curve, which is smooth at $t \neq 0$ and has a cusp $(0, u^2, 0, u^3)$ at the instant of time $t = 0$. Projections to almost any[2] 3-dimensional subspace transform this singularity to *pinch point*, where the surface has a form of *Whitney umbrella*: (v, u^2, vu).

The following theorems describe the transformation of WS to normal forms. Let $\vec{Q} : S^1 \to \mathbf{R}^n$ be C^∞-smooth analytical function with $\vec{Q}'(\sigma) \neq 0$, i.e. smooth closed curve in $\mathbf{R}^n$. Let σ be a natural parameter (length) on the curve.

Definition ($n = 2$): Let $\varphi(\sigma)$ be polar angle of the vector $\vec{Q}'(\sigma)$. Let's denote $\varphi_i = \varphi(\sigma_i)$. We say that the curve is *in general position*, if in every pair of points where the equation $\varphi_1 = \varphi_2 \bmod 2\pi$ is satisfied, one of the following properties is valid:

(c) $\varphi_1' \neq \pm\varphi_2'$;
(S) $\varphi_1' = \varphi_2' \neq 0,\ \varphi_1'' \neq \varphi_2''$;
(W) $\varphi_1' = -\varphi_2' \neq 0,\ \varphi_1'' \neq \varphi_2''$;
(E) $\sigma_1 = \sigma_2,\ \varphi_1' \neq 0$;
(I) $\sigma_1 = \sigma_2,\ \varphi_1' = 0,\ \varphi_1'' \neq 0,\ \varphi_1''' \neq 0$.

For two smooth closed curves $\vec{Q}_{1,2}$ in $\mathbf{R}^2$ let's denote $\varphi_i = \varphi_i(\sigma_i)$. We say that two curves are *in general position to each other* if in every pair of points where the equation $\varphi_1 = \varphi_2 \bmod 2\pi$ is satisfied, one of the properties (c),(S),(W) is valid.

Definition ($n = 3$): For the curve in $\mathbf{R}^3$ let's denote $\vec{Q}_i' = \vec{Q}'(\sigma_i)$. The curve is in general position, if in every pair of points where the equation $\vec{Q}_1' = \vec{Q}_2'$ is satisfied, one of the following properties is valid:

(E) $\sigma_1 = \sigma_2,\ \vec{Q}_1'' \neq 0$;
(P) $\vec{Q}_1'' \nparallel \vec{Q}_2'',\ |\vec{Q}_1''| \neq |\vec{Q}_2''|$.

For two curves in $\mathbf{R}^3$ let's denote $\vec{Q}_i' = \vec{Q}_i'(\sigma_i)$. Two curves are in general position to each other if the property (P) is valid in every pair of points where the equation $\vec{Q}_1' = \vec{Q}_2'$ is satisfied.

Theorem 3 (general position): curves in general position form an open everywhere dense set in C^∞-topology.

Theorem 4 (normal forms): for supporting curves in general position the singularities of the WS in their neighborhood can be transformed to one of the above written normal forms by LR-diffeomorphisms, i.e. by smooth invertible transformations of parameters plane $(\sigma_1, \sigma_2) \to (\tilde{\sigma}_1, \tilde{\sigma}_2)$ and Minkowski space $x \to \tilde{x}$. The linear part of the transformation of Minkowski space, defined by Jacoby matrix $\partial\tilde{x}/\partial x$, maps light-like direction $Q_1' \| Q_2'$ to light-like direction $(1, 0, .., 0)$ and also maps to each other the isotropic planes, related with these directions.

Theorem 3 means that curves in general position represent a general case in the set of all smooth curves. Refinement in the theorem 4 relates the isotropic planes in the image and pre-image spaces and allows to write normal forms in light coordinates.

Σ-classification [18]. Σ-class (Thom's symbol) of singularity is defined by a kernel of the 1st differential of considered mapping, in our case: solution of an equation $Q_1' d\sigma_1 + Q_2' d\sigma_2 = 0$, which in Rohrlich gauge follows $d\sigma_1 + d\sigma_2 = 0$. This kernel is 1-dimensional, so that the curves (c), edges (E) and points (P) are singularities of class Σ^1. The lines $d\sigma_1 + d\sigma_2 = 0$ are shown by hatching on fig.8b. The points (W), where the hatching is tangent to the curves (c), correspond to singularities of class $\Sigma^{1,1}$ (cusp on the line of cusps). For points (I) Thom's symbol is not defined, however in Boardman's refining scheme these points are classified as $\Sigma^{1,1,1,0}$. Points (S) are of class Σ^1, i.e. are not distinguished from surrounding points of cusp line (c) by Σ-classification.

[2]Exceptional directions of projection belong to a plane $t = x$, they create higher order singularities.

Stability. Let's call the singularity of WS *weak stable* (W-stable), if all WS in ϵ-vicinity of the given WS_0 have a singularity. The singularity is called LR-stable [18], if all WS in ϵ-vicinity of WS_0 can be transformed to WS_0 by LR-diffeomorphism (in this case the singularity will be certainly W-stable). The described singularities are LR-stable if ϵ-vicinity of WS is defined in C^∞-topology: $|\delta\vec{Q}^{(n)}| < \epsilon_n,\ n = 0, 1, 2...$ The singularities are W-stable in C^1-topology: $|\delta\vec{Q}'| < \epsilon$ (in this case they already are not LR-stable, example is shown on fig.11d: near the point of transversal self-intersection of hodograph $\vec{Q}'(\sigma)$ in its $|\delta\vec{Q}'| < \epsilon$ -vicinity C^1-small but C^2-large variations exist, changing the structure of self-intersection). C^1-small variations of supporting curves are equivalent to variations of initial data, small in the sense $|\delta x'| < \epsilon,\ |\delta p| < \epsilon$. From physical point of view it is also interesting to consider C^0-small variations: $|\delta\vec{Q}| < \epsilon$, equivalent to small variation of coordinate $|\delta x| < \epsilon$ and small change of integrals $|\int d\sigma\delta p| < \epsilon$, taken over finite segments of the string. The variations of supporting curve, which are C^0-small but C^1-large, correspond to vanishing loops, displayed in fig.11a. These loops can eliminate the coincidence of tangent vectors, throwing the hodograph to an opposite side of a circle, fig.11b, therefore removing the singularity. More detailed consideration shows that in the case $d = 3$ this process generates new cusps (c') and swallowtails (W), see fig.11c, so that the singularity can be removed in small region of deformation, but not completely from the whole WS. The pinch points can be completely eliminated by C^0-small deformations, see fig.11e. This process is followed by a creation of a small loop on the string, fig.11f, which propagates at light velocity and passes through the point (P), preventing the appearance of the instantaneous cusp there. In this region of the string the linear density of energy-momentum is large but is not infinite.

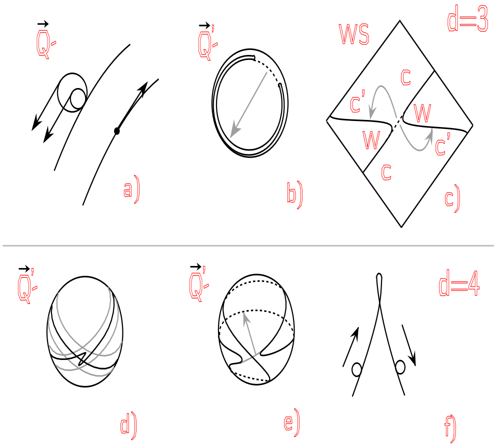


Fig.11. *Deformations, changing structure of singularity.*

It's also interesting to study the properties of singularities in Fourier representation [1], where the coefficients of expansion $Q'(\sigma) = \sum a_n e^{in\sigma}$ are used as coordinates in the phase space of the string. In finite dimensional subspaces $\{a_n, n \leq N;\ a_n = 0, n > N\}$ subsets with $\vec{Q}' \neq 0$ and only transversal intersections of hodograph $\vec{\tau} = \vec{Q}'/|\vec{Q}'|$ are open, therefore the singularities are W-stable; and because in such sets $\vec{\tau}$ are analytical, the singularities are LR-stable. For creation of the vanishing loops the infinite number of Fourier coefficients is necessary. In the space of all $\{a_n\}$ the definitions of C^0- and C^1-small variations can be reproduced by certain requirements for the rate of descent of Fourier coefficients.

This table presents several settings of topology and the resulting stability type of singularities on WS:

Topology	$d = 3$	$d = 4$	$d > 4$
$\lvert\delta\vec{Q}^{(n)}\rvert < \epsilon_n\ (C^\infty)$	LR	LR	—
$\lvert\delta\vec{Q}'\rvert < \epsilon\ (C^1)$	W	W	—
$\lvert\delta\vec{Q}\rvert < \epsilon\ (C^0)$	W	—	—
$\lvert\delta a_n\rvert < \epsilon;\ a_n = 0, n > N$	LR	LR	—
$\lvert\delta a_n\rvert < \epsilon/n^p, p > 1\ (C^1)$	W	W	—
$\lvert\delta a_n\rvert < \epsilon/n^p, p > 0\ (C^0)$	W	—	—

WS can also have stable self-intersections, whose properties are the same as for generic surfaces. The following tables summarize all stable singularities for WS and generic surfaces (in C^∞-topology).

Self-intersections

Surface	$d = 3$	$d = 4$	$d > 4$
generic, WS	lines	points	—

Others

Surface	$d = 3$	$d = 4$	$d > 4$
generic	pinch points	—	—
WS	cusp lines and their singularities (S),(W),(I)	pinch points	—

Global structure of singularities: *($d = 3$, closed string).*
Further consideration will be done in CMF.

Theorem 5 (presence of singularities) [7]: all WS of closed string in 3-dimensional Minkowski space necessarily have singularities.

Remark: at certain conditions (central symmetry of supporting curves) the WS has singularity of type "collapse", where the string for an instance of time shrinks to a point. Example: $Q_{1,2}(\sigma) = (\sigma, \cos\sigma, \pm\sin\sigma)$, the supporting curves, whose projection to CMF are two oppositely oriented circles. This singularity is unstable: small variations of the curves unfold it to a small closed cusp line.

The velocity of cusp $\vec{v}$ is orthogonal to its direction $\vec{k}$.
Definition: *topological charge of cusp* is a number c, equal to $+1$, if a rotation from $\vec{v}$ to $\vec{k}$ is *counterclockwise*; and equal to -1 if this rotation is clockwise.

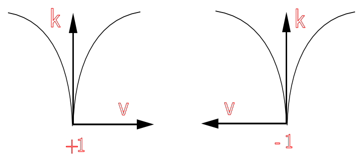


Fig.12. *Topological charge.*

Theorem 6 (conservation of topological charge):
Total topological charge of the string, equal to the sum of topological charges of all cusps, is constant in time and equals $n_1 + n_2$. Here n_i are the numbers of revolutions of vectors $\vec{Q}_i'(\sigma)$ in complete passage around supporting curves ($n_i > 0$, if revolutions are counterclockwise; $n_i < 0$, if they are clockwise).

Theorem 7 (permanent regime): Let supporting curves $\vec{Q}_i(\sigma)$ have no inflection points. Let $\operatorname{sign} n_1 = \operatorname{sign} n_2$. In this case cusps do not collide and topological charges of all cusps have the same sign equal to $\operatorname{sign} n_{1,2}$. As a result, *the number of cusps* on the string is constant in time and equals $|n_1 + n_2|$.

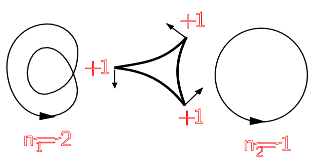


Fig.13. *Permanent regime.*

Remark: curves with $n_i = 0$ (e.g. figure "∞") necessarily have inflection points and violate permanent conditions. Thus, these conditions imply $|n_i| \geq 1$, and the strings under permanent conditions always have $N \geq 2$ cusps.

Theorem 8 (collision of cusps): intersection of cusp lines is unstable, i.e. small variations of supporting curves transform it either to scattering or to annihilation/creation mode (fig.14). Cusps are created/annihilated by pairs in singular points of type W. At the moment of creation cusps have equal velocities and opposite directions, so that conservation of topological charge is valid: $(+1, -1) \leftrightarrow 0$.

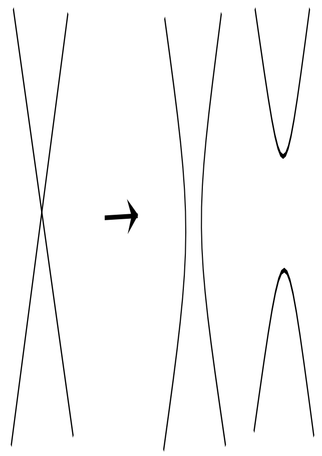


Fig.14. *Intersection of cusp lines is unstable.*

Open strings can be free of singularities. Example [3]: the supporting curve $\vec{Q}(\sigma)$ is a circle, WS is helicoid fig.4, the string in CMF is a straight line, rotating at constant angular velocity. Open string is a degenerate case of closed one, so that analogous theorems for open strings can be obtained in the limit $\vec{Q}_1 \to \vec{Q}_2$. This limit maps two cusps of closed string to the edges of WS, while other cusps become 2-folded, see fig.15a,b. From the obtained two coincident sheets only one represents the open string, so that a half of the topological charges of singularities should be taken, leading to the charge ± 1 for cusps and $\pm 1/2$ for the edges. Absorption of the cusp at the edge corresponds to the process fig.15c: $(+1, -1/2) \leftrightarrow +1/2$.

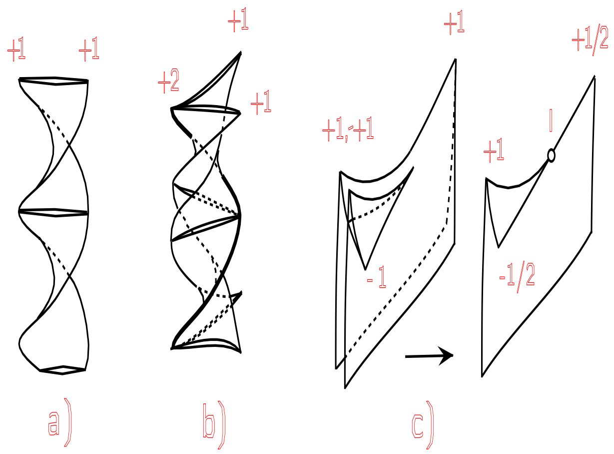


Fig.15. *Folding of WS: closed $\to$ open.*

Theorem 6': total topological charge of open string, equal to the sum of topological charges of cusps and edges, is constant in time and equals n, the number of revolutions of vector $\vec{Q}'(\sigma)$ in the passage around supporting curve.

Theorem 7': if the supporting curve $\vec{Q}(\sigma)$ has no inflection points, the cusps do not collide and do not reach the edges, and the topological charges of all cusps and edges have the same sign, equal to sign n. The total number of cusps in this case is constant in time and equals $|n|-1$.

Theorem 8': there exists stable intersection of cusp line with the edge of WS in the points of type I. This singularity can be obtained in folding of the swallowtail with the cusp line in the limit $\vec{Q}_1 \to \vec{Q}_2$, as shown on fig.15c.

Remark: the described local elements can be assembled to more complicated patterns, particularly, the following process can be found both on closed and open WS, see fig.16: a cusp (0) is initially present on the string, then the pair of cusps (1,2) appears, one cusp from the pair annihilates the cusp (0), while another one becomes the cusp (0) in the next period of evolution.

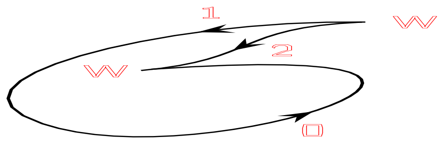


Fig.16. *Z-process.*

Several examples of singularities on strings at $d = 3$ are shown on fig.21a-d.

Global structure of singularities: $(d = 4)$
Stable singularities are pinch points, periodically located on WS. In evolution instantaneous cusps appear on the string in its passage through the pinch points, periodically at the same point in CMF. 3-dimensional projections of the WS of open string from 4-dimensional Minkowski space-time are shown on fig.21e,f. Two types of singular points can be found in this figure: $P, P', ...$ – singular points, existing on WS itself, which are projected to pinch points in projections to 3-dimensional space, such as (xyz), (xyt), shown in the figure; Q – pinch point, which appears only on a specific projection and therefore is not physically important.

For the pinch points the topological charge also can be introduced, characterizing the behavior of singularities in continuous deformations (homotopies) of the WS. For closed strings the transversal intersections of two closed oriented curves $\vec{Q}'_{1,2}(\sigma)$ on the sphere S^2 are characterized by indices [19], equal to $+1/-1$ if a pair of tangent vectors $(\vec{Q}''_1, \vec{Q}''_2)$ has coincident/opposite orientation with a frame defining global orientation on the sphere S^2. Due to the theorems [19], the sum of all indices is invariant under homotopies and for two closed curves on a sphere is always equal to zero. Therefore, the number of pinch points counted in one period of WS of closed string is even: $n = 0, 2, 4, 6...$ and in continuous deformations of WS the pinch points appear/disappear pairwise: $(+1, -1) \leftrightarrow 0$.

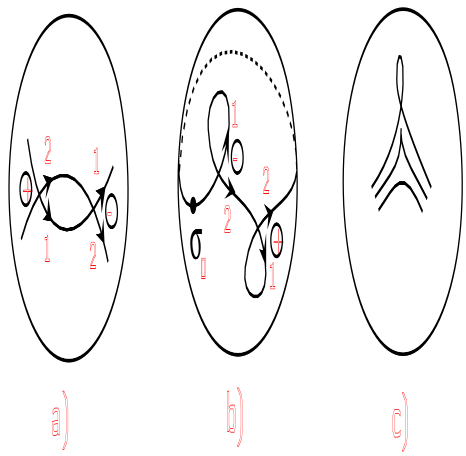


Fig.17. *Topological charge of pinch point.*

For open strings the points of self-intersection of the curve $\vec{Q}'(\sigma)$ can appear/disappear alone, followed by the creation of cusp $\vec{Q}''(\sigma) = 0$, see fig.17c. This situation corresponds to the creation of inflection point on the supporting curve $\vec{Q}(\sigma)$ in 3-dimensional space. On WS the pinch point moves to the edge ($\vec{Q}'_1 = \vec{Q}'_2$, $\sigma_1 \to \sigma_2$) and disappears in the inflection point of the edge. The pinch points can also appear/disappear pairwise in internal regions of WS during its continuous deformation, preserving a local characteristic, introduced as follows [19]. Let's fix on the curve $\vec{Q}'(\sigma)$ a point σ_0, not coincident with self-intersection. Let's pass a curve starting from σ_0 and mark the tangent vectors in intersection points: in the 1st passage through the intersection point write 1 on the corresponding tangent vector, and in the 2nd passage write 2, see fig.17b. Then assign to the intersection point a number ± 1 dependently on the orientation of the frame $(1, 2)$. The sum of these numbers is called index of self-intersection (Whitney number) of the closed curve. This number depends on a choice of σ_0 (in a passage of σ_0 through the point of self-intersection its index changes the sign), however *the parity* of Whitney number does not depend on σ_0 and is invariant under those homotopies which do not create the cusps $\vec{Q}''(\sigma) = 0$.

3. Exotic solutions

Solutions of this form correspond to the supporting curves, whose temporal component $Q_0(\sigma)$ is non-monotonous function, see fig.18. Such curves can be explicitly constructed, specifying tangent vector in the form $Q'(\sigma) = a_0(\sigma)(1, \vec{n}(\sigma))$, where $|\vec{n}(\sigma)| = 1$, $\vec{n}(\sigma)$ is 2π-periodic function, and $a_0(\sigma)$ is 2π-periodic function of variable sign. Corresponding WS is shown on fig.21g. On this figure (cABh) is a supporting curve, which has two cusps A,B. These cusps induce cusp lines on WS: (fRAd) and (gBRe), which separate the WS into a number of pieces. Here R=(A+B)/2.

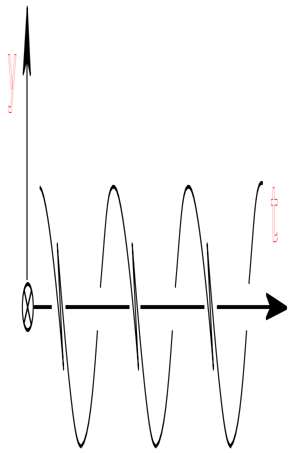


Fig.18. *Supporting curve, non-monotonous in temporal direction.*

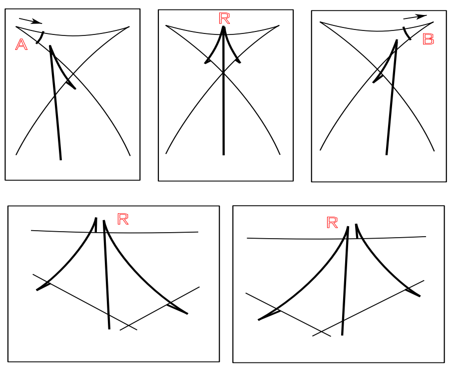


Fig.19. *Exotic solutions.*

Equal-time slices of this WS contain disconnected parts. There is a long string, which is permanently present in the system. Additionally, the following processes occur:
• in point A new short string appears from vacuum;
• in point R it recombinates with the long string: is attached to the long string while a part of the long string is detached;
• in point B short string disappears.

Therefore, these WS correspond to the processes of creation, annihilation and recombination of strings. Further analysis [11] shows that the density of energy for solutions of this kind is not everywhere positive: the regions, marked (+) on fig.21g, possess positive energy, while the regions (−) have negative energy (theoretical physics uses for such solutions term “exotic matter” [20]). For such solutions the components of strings, which appear/disappear in vacuum near points A,B, have zero total energy-momentum and orbital momentum, so that conservation laws do not prevent these processes. It was also shown in [11] that WS of this kind are time-like, so that the square root in the Lagrangian of the string is real-valued, however the cusp lines of WS correspond to branching points of this square root, therefore there are several choices of sign in Lagrangian. Exotic solutions correspond to such a choice, that the areas of the parts marked (±) on fig.21g give opposite contribution to the action. It was shown in [11] that exotic solutions are present at arbitrary dimension of the space-time and occupy regions, i.e. are not rare in the phase space of covariant Hamiltonian string theory. This phase space is formed by the coefficients of Fourier expansion of the function $a(\sigma)$. Rejection of exotic solutions from the theory, possible only by means of explicit requirement $a_0(\sigma) > 0$, or equivalent requirement in terms of Fourier coefficients, creates additional difficulties in quantum theory and actually is never done.

4. Break of the string

Considering string’s breaking or any other transmutation process, it is needed to fix the topological class of the process and find the extremum of action in this class. For instance, breaking of open string into two open strings corresponds to the diagram fig.2, right. The initial and final positions of the strings should be fixed, and the surface should be varied to achieve the extremum of area, assuming the boundaries and breaking point free. Such surfaces can be reconstructed using the following algorithm [6], see fig.21i:

1. Let’s represent the WS of open string by a supporting curve Q (which is as usual 2P-periodical and light-like). Q is the 1st edge of the WS; Q+P is the 2nd edge.

2. Take two arbitrary points A,C on the curve Q, lying inside one period. Take their middle B = (A+C)/2. This will be *breaking point* on the WS. Take also points A’=A-2P and B’=(A’+C)/2=B-P.

3. Consider the following curves: AB, obtained from a segment AC of supporting curve, by homothetic contraction to the point A with coefficient 1/2; BC, obtained from AC by homothetic contraction to the point C with the same coefficient (AB and BC are congruent); and analogously: A’B’ is A’C 1/2-contracted to A’; B’C is A’C 1/2-contracted to C (A’B’ and B’C are congruent). Let A”BC’ = A’B’C+P (parallel translation by P).

Remark: Curves ABC and A”BC’ belong to the WS. These curves mark a path of light signals emitted from the point B on the WS and are called *characteristics.*

4. Consider triangular parts of the WS:

1 = part, restricted by arcs AB, BC and AC;

2 = part, restricted by arcs A”B, BC’ and A”C’.

Shift these parts repeatedly by vectors BA and BC’:

1’ = 1+BA, 1” = 1’+BA, ...

2’ = 2+BC’, 2” = 2’+BC’, ...

The sequence of images $\{1, 1', 1'', ...\}$ forms a connected surface (parts match each other along the characteristic AB and its images; connection is continuous but generally not smooth – the WS has a fracture along characteristics). Construct $\{2, 2', 2'', ...\}$ analogously.

5. A part of initial WS, restricted by edges Q,Q+P and arcs A"B, BC (lying on the left of A"BC), unified with the surfaces $\{1,1',1'',...\},\{2,2',2'',...\}$, form a complete WS for decay "open $\rightarrow$ 2 open".

Remarks:

a) Products of decay $\{1',1'',...\},\{2',2'',...\}$ can be generated from the supporting curves by the common rule "locus of middles", fig.4. Supporting curves in this case are periodical continuations of arcs CA and A"C'. Semi-periods are equal to energy-momentum, and the property $P = P_1 + P_2$ (conservation of energy-momentum in the decay process) is evident on fig.21i.

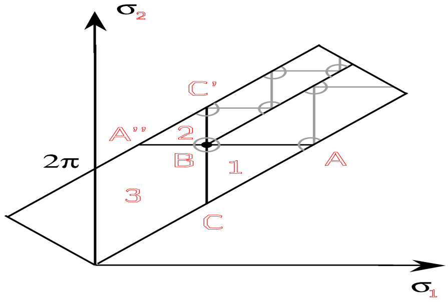


Fig.20. *Break "open $\rightarrow$ 2 open" on parameters plane.*

b) On the plane of parameters (σ_1, σ_2) the characteristics AB, BC, A"B, BC' correspond to straight lines $\sigma_i = Const$. The parts 1,2 are triangles, restricted by these lines. From here we have an equivalent algorithm of WS reconstruction: take triangles 1,2 on the parameters plane, map them into space-time using $x(\sigma_1,\sigma_2) = (Q(\sigma_1) + Q(\sigma_2))/2$, obtain their images in translations, described above, and take the union of surfaces $\{1,1',1'',...\} \cup \{2,2',2'',...\} \cup 3$ to represent the complete WS.

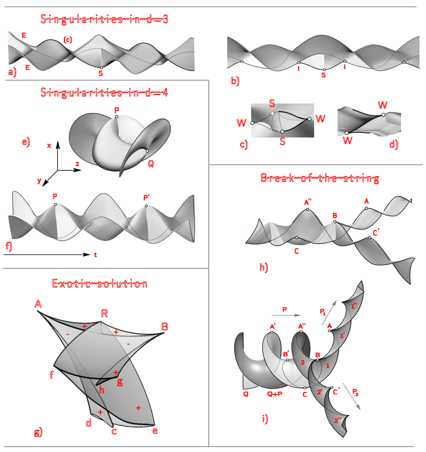


Fig.21. *Singular world sheets. Images are created by computer program [21], representing string dynamics in Avango*$^{\mathrm{TM}}$ *Virtual Environment* [22].

Theorem 9 (extremal property): WS, constructed by such algorithm, has extreme area.

Theorem 10 (break in a singular point): the products of string breaking have no fracture along the characteristics if and only if B is singular point of original WS (see fig.21h).

Conclusion

Using the geometrical method [14] for explicit representation of solutions in string theory, we classify the singular points on them. It is shown that the world sheets of open and closed strings in Minkowski space-time of dimension 3 and 4 have stable singularities, which cannot be eliminated by small deformations of the surface in considered class. In dimension 3 singularities are cusps, propagating along WS at light velocity, appearing/disappearing by pairs in inner regions of WS or alone on the boundary of WS. In dimension 4 singularities are isolated points, looking like instantaneous cusps periodically located on WS. Higher dimensional cases have no stable singularities.

We show that at definite conditions the string theory has solutions of type $I\times\mathbf{R}^1$ whose boundary is not immersed in Minkowski space-time, but has cusp-like singularities. These solutions possess not everywhere positive density of energy and correspond to spontaneous creation of strings from vacuum. Such processes lead to instability of vacuum state in string theory, mentioned earlier in work [8].

Considering the processes of string breaking by the scheme [6], we have found certain relations between singularities on strings and the breaking processes. Particularly, smooth WS can appear only as a result of breaking of singular WS in one of the singular points. This fact gives a possibility to construct models of decay of elementary particles, where smooth WS describe the particles with long lifetime, while singular WS decay finally to the smooth ones by a sequence of breaks in the singular points. The dimensions $d=3,4$ are naturally selected by these models as the only values of dimension where stable singularities of WS exist. Consideration of these models on quantum level is possible at least for subsets of phase space [3-5], admitting anomaly-free quantization at $d=3,4$.

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Appendix

Here we present the proofs for the theorems stated above.

T1: Let $x(\sigma,\tau)$ be an extremal surface, presented in a neighborhood of a regular point (σ_0,τ_0) in conformal parametrization: $\dot{x}^2 = -x'^2 > 0,\ \dot{x}x' = 0$. Consider a local variation $x \to x + \epsilon\delta x$, i.e. $\delta x \in C^\infty$ is vanishing outside of a small vicinity of the point (σ_0,τ_0). Note that parametrization on disturbed surface is not conformal any more. Computing variation of the area:

$$\begin{aligned}
&\delta((\dot{x}x')^2 - \dot{x}^2x'^2)^{1/2} = \epsilon\ \delta\mathcal{L}_1 + \epsilon^2\delta\mathcal{L}_2 + O(\epsilon^3),\\
&\delta\mathcal{L}_1 = \dot{x}\delta\dot{x} - x'\delta x',\ \delta\mathcal{L}_2 = (\dot{x}^2(\delta\dot{x}^2 - \delta x'^2) +\\
&((\dot{x} - x')(\delta x' - \delta\dot{x}))((\dot{x} + x')(\delta x' + \delta\dot{x})))/2\dot{x}^2.
\end{aligned}$$

The terms linear in δx give contribution $\delta A_1 = \epsilon\int\delta\mathcal{L}_1 d\tau d\sigma = \epsilon\int(-\ddot{x} + x'')\delta x d\tau d\sigma = 0$ (here boundary terms vanish because δx is local, $\ddot{x} = x''$ is the condition of extremum in conformal parametrization). Let's consider variations of a special form: $\delta x(\sigma,\tau) = (0, 0, F(\sigma,\tau), 0...)$ in a system of coordinates where $\dot{x}(\sigma_0,\tau_0) = (c, 0, 0...),\ x'(\sigma_0,\tau_0) = (0, c, 0...)$, i.e. the variation orthogonal to a tangent plane to WS in the point (σ_0,τ_0). For such variations $\delta\mathcal{L}_2 = (F'^2 - \dot{F}^2)R/2$, where $R = 1+(\dot{x}_2^2 - x_2'^2)/\dot{x}^2$. Due to $\dot{x}^2 = -x'^2 > 0,\ \dot{x}x' = 0$ we have inequality $R \geq 0$, and because $R_0 = 1$ in the point (σ_0,τ_0), we have $R > 0$ in the vicinity of this point. Now it's clear that $\delta\mathcal{L}_2$ is not positively defined, however there is a possibility that δA_2 will be positive after the integration. Let's consider an explicit example: $F(\sigma,\tau) = f(\sigma^2/a^2 + \tau^2/b^2)$, where $f(\rho) \in C^\infty$ is monotonous in $\rho \in [0, 1]$ and $f(\rho) = 0$ for $\rho > 1$. In the limit of small a, b we have $\delta A_2 = \epsilon^2\int\delta\mathcal{L}_2 d\tau d\sigma = 2\pi\epsilon^2 ab(a^{-2} - b^{-2})I$, where $I = \int_0^1 f'^2\rho^3 d\rho > 0$, so that $\delta A_2 < 0$ for $a > b > 0$ (maximum) and $\delta A_2 > 0$ for $0 < a < b$ (minimum), see fig.1.

T2: The tangent plane to open WS on the edge is spanned on two vectors (Q', Q'') and is isotropic due to $Q'^2 = Q'Q'' = 0$. The tangent vector to the string x', obtained as equal-time slice of WS $x_0' = 0$, is contained in the tangent plane to WS, and on the edge it is orthogonal to Q': $x'Q' = 0 \ \Rightarrow\ \vec{x}'\vec{Q}' = 0$, so that the direction of the string at the end point and the velocity of the end point are orthogonal.

T3: The curves, which *are not* in general position, define a closed stratified submanifold in the space of multijets ${}_2J^3(S^1, \mathbf{R}^n)$ [23], whose codimension is greater than the dimension of considered mapping. Due to criterions, given in [24], this case can be eliminated by a small variation of the mapping. The statement of the theorem follows from Thom's transversality theorem [25, 18], generalized to multijet space in [23].

T4: The normal forms are given by the lowest order linear independent terms in Taylor's expansion of WS near the singular point of each type. The central part of the theorem is a proof that the higher order terms do not change the structure of singularity and can be compensated by LR-diffeomorphisms. For instance, Taylor's expansion of WS written for the case (c) in Rohrlich gauge in light coordinates, related with the direction $Q'_1 = Q'_2$, contains together with the normal form $(x_0, x_2, x_-) = (v, u^2, u^3)$ the additions to x_2, x_--components of the form $v^k u^n$ with the following set of possible indices $(k, n) \in \{k \geq 2, n = 0\} \cup \{k = 0, n \geq 3\} \cup \{k \geq 1, n \geq 2\}$, where the term with $k = 0, n = 3$ enters to x_2-component only. These terms can be presented as $f_{kn} = x_0^k x_2^{n/2}$ for even n and $f_{kn} = x_0^k x_- x_2^{(n-3)/2}$ for odd n (i.e. they belong to an ideal, spanned on monomials v, u^2, u^3), so that their addition is equivalent to smooth mapping $x \to x + f_{kn}(x)$. All these mappings have unit Jacoby matrix at $x = 0$ except of the case $k = 0, n = 3$, which corresponds to invertible linear mapping: $(x_0, x_2, x_-) \to (x_0, x_2 + cx_-, x_-)$, preserving the directions $(1, 0, 0)$ and $(0, 1, 0)$. Convergent Taylor's series $\sum c_{kn} v^k u^n$ correspond to convergent $\sum c_{kn} f_{kn}(x)$, so that the constructed mapping and its inverse are analytical. Therefore, higher order corrections to the normal form (c) can be compensated by analytical L-diffeomorphism, preserving light-like direction and related isotropic plane. In the case $(S): \ (v, u^2, vu^3)$ the terms u^{2n+1} cannot be compensated by L-diffeomorphisms (they do not belong to the ideal). The obstacle here is a line of self-intersection, whose position on the parameters plane $v = 0$ is invariant under L-diffeomorphisms, while is changed by addition of u^{2n+1}. It's easy to prove that these terms can be removed by analytical R-diffeomorphisms (reparametrizations). In the same way the transformation to normal form is performed for other singularities. In each case it's needed to verify that the constructed mapping is analytical and its linear part is defined by non-degenerate Jacoby matrix of upper triangular form: $J = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$. The last property ensures that the linear part of constructed mapping preserves the light-like direction and isotropic plane, initially related with $Q'_1 = Q'_2$.

T5: If two hodographs on the circle S^1 have no intersection, one of them should cover an arc with angular size $\Delta\varphi < \pi$. This is impossible due to condition $\oint d\sigma e^{i\varphi} = 0$, equivalent to closeness of the curve $\vec{Q}(\sigma)$.

T6: In Rohrlich gauge the topological charge of cusp is defined by orientation of a pair $(\vec{Q}'_1, \vec{Q}''_1 + \vec{Q}''_2)$ and is equal to $\mathrm{sign}(\varphi'_1 + \varphi'_2)$. This number is a particular case of the following topological invariant. Let's consider a mapping $S^1 \times S^1 \to S^1$, defined by the function $f(\sigma_1, \sigma_2) = -\varphi_1(\sigma_1) + \varphi_2(\sigma_2)$. The cusp line (c) is zero level $f(\sigma_1, \sigma_2) = 0$. The vector $(-\varphi'_1, \varphi'_2)$ is a normal and (φ'_2, φ'_1) is tangent element to (c), which can be used to define the orientation of (c). Let's consider another oriented contour (k) on the torus, with tangent element $(d\sigma_1, d\sigma_2)$, intersecting (c) in a certain point. The number $\nu = \mathrm{sign}(-\varphi'_1 d\sigma_1 + \varphi'_2 d\sigma_2)$ in this point is called *index of intersection*. Let's also consider a mapping $S^1 \to S^1$, defined by a restriction of $f(\sigma_1, \sigma_2)$ to the contour (k). The same number $\nu = \mathrm{sign}(df)$, estimated in pre-images of point 0 for this mapping (i.e. in (c)∩(k)), is called *degree of mapping*. According to the theorems, proved in [19], the sum of such numbers estimated in all points of intersection (c)∩(k), is invariant under homotopies of mapping f and contour (k). In our case it equals $-n_1(k) + n_2(k)$, where $n_i(k)$ are numbers of revolutions of $\varphi_i(\sigma_i)$ corresponding to a complete passage of (k). The statement of the theorem corresponds to the case when (k) is equal-time slice: $(\sigma_1, \sigma_2) = (\tau_0 - \sigma, \tau_0 + \sigma), \ \sigma \in [0, 2\pi]$, so that $\nu = \mathrm{sign}(\varphi'_1 + \varphi'_2)$ and $\sum \nu_i = n_1 + n_2$, where are n_i are numbers of revolutions of φ_i for basis cycles on torus. Another invariant $n_1 - n_2$ equals to the intersection index of (c) with the trajectory of point σ_0 on the string: $(\sigma_1, \sigma_2) = (\tau - \sigma_0, \tau + \sigma_0), \ \tau \in [0, 2\pi]$, representing the total number of windings of cusp lines around the cylinder of closed string WS.

T7: If the functions φ_i are monotonous and $\mathrm{sign}\,\varphi'_1 = \mathrm{sign}\,\varphi'_2$, then on equal-time slice we have a function $f(\sigma) = -\varphi_1(\tau_0 - \sigma) + \varphi_2(\tau_0 + \sigma)$, which is also monotonous and has the interval of variation $f(2\pi) - f(0) = 2\pi(n_1 + n_2)$. In this case the equation $f(\sigma) = 2\pi k, \ k \in \mathbf{Z}$ has on the interval $\sigma \in [0, 2\pi)$ exactly $|n_1 + n_2|$ isolated solutions. Other statements of the theorem follow from T6.

T8: Intersection of cusp lines corresponds to a saddle point of the function $f(\sigma_1, \sigma_2) = -\varphi_1(\sigma_1) + \varphi_2(\sigma_2)$ on the level $f = 0$. Small variations transform the cusp lines similarly to recombination of hyperbolas $x^2 - y^2 = c$, when c passes through 0. Then, using conditions $\varphi_1' + \varphi_2' = 0,\ d(\varphi_1' + \varphi_2') = (\varphi_1'' - \varphi_2'')d\sigma_1 \neq 0,\ d\sigma_2 = -d\sigma_1$, valid in the point W, we see that the function $\varphi_1' + \varphi_2'$ changes sign in the passage through this point, so that created cusps have opposite topological charges, while the velocities $\vec{Q}'_{1,2}$ are equal.

T9: Internal regions of the patches $\{1, 1', 1'', ...\}$, $\{2, 2', 2'', ...\}, 3$ are constructed by the rule $x(\sigma_1, \sigma_2) = (Q(\sigma_1) + Q(\sigma_2))/2$, so that Lagrange-Euler equations $\partial_i p_i = 0$ are satisfied there. It is only needed to check that condition $\Delta I = \Delta p_i \epsilon_{ij} d\sigma_j = 0$ is satisfied on the lines of connection (characteristics), so that the flow of momentum, coming out one patch, is equal to the flow of momentum, coming into another patch. Here Δp_i is a discontinuity of momentum on the characteristics. Computing p_i: $p_1 = Q'(\sigma_2),\ p_2 = Q'(\sigma_1)$ (see [11]), we have $\Delta I = -\Delta Q'(\sigma_1)d\sigma_1 + \Delta Q'(\sigma_2)d\sigma_2$, where $\Delta Q'(\sigma_i)$ are discontinuities of tangent vector to supporting curve. Then, because the discontinuity $\Delta Q'(\sigma_i) \neq 0$ propagates along the characteristic $d\sigma_i = 0$ with the same i, we have $\Delta I = 0$. Actually, a discontinuity occurs only for components of momentum, tangential to characteristics. The flow of momentum through the free edge vanishes due to the identities $Q'(\sigma_1) = Q'(\sigma_2),\ d\sigma_1 = d\sigma_2$. Then, considering the contours, separating the break point and the points (A,C'...), where characteristics intersect the edges, fig.20, we see that the flow of momentum through these contours is not changed when the contour is continuously deformed to a point. Then, using the fact that p_i are bounded: $|p_i| < Const$ (even in the singular point of WS), we see that flow of momentum vanishes in this limit, therefore the momentum has no leakage in these points.

T10: String breaks in singular point, when the tangent vectors to supporting curve in points A,C are parallel. In this case the periodical continuation of curves CA and A"C' on fig.21h is C^1-smooth.