

Topological Interpretation of Dirac Equation and Geometrization of Physical Interactions

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Earlier we have shown that interacting electron-positron and electromagnetic fields can be considered as a certain microscopic distortion of pseudo-Euclidean properties of the Minkovsky 4-space-time. The known Dirac and Maxwell equations prove to be group-theoretical relations describing this distortion (nonmetrized closed 4-manifold). Here we apply the above geometrical approach to obtain equations for a neutrino interacting with its weak field. These equations contain some new terms and demonstrate geometrical mechanisms of gauge-invariance and P-T violation. Equations are also proposed for gravitational field and its microscopic quantum sources.

Introduction

There were many attempts to find out new interpretation of the mathematical formalism of quantum mechanics and new geometrical representation of all physical fields. Both these problems are usually considered separately. In our investigation, these problems are combined within a unique hypothesis. It was suggested that the relativistic quantum equations for the microscopic field sources and equations for these fields themselves describe some unified specific geometrical object—microscopic nonmetrized closed space-time 4-manifold.

In our previous publications, we proposed this topological interpretation for the Dirac and Maxwell equations which describe interacting electron-positron and electromagnetic fields. These equations can be written in the form [1]

$$i\gamma_1 \left(\frac{\partial}{\partial x_1} - ieA_1 \right) \psi - \sum_{\alpha=2}^4 i\gamma_\alpha \left(\frac{\partial}{\partial x_\alpha} - ieA_\alpha \right) \psi = m_e \psi, \quad (1)$$

$$F_{kl} = \frac{\partial A_k}{\partial x_l} - \frac{\partial A_l}{\partial x_k}, \quad (2)$$

$$\sum_{i=1}^4 \frac{\partial F_{ik}}{\partial x_i} = j_k^e, \quad j_k^e = e\psi * \gamma_1 \gamma_k \psi. \quad (3)$$

Here $\hbar = c = 1$, $x_1 = t$, $x_2 = x$, $x_3 = y$, $x_4 = z$, F_{kl} is the electromagnetic field tensor, A_k is the 4-potential, γ_k are the Dirac matrices, ψ is the Dirac bispinor, and m_e and e are electron mass and charge, respectively.

It was shown that Eqs.(1)–(3) can be interpreted as group-theoretical relations that account for the topological and metric properties of some unified microscopic nonmetrized closed space-time 4-manifold [2-5]. Equations (1)–(3) describe the above properties in this manifold-covering space and, this space proves to be a specific space — a space with so-called “semimetrical parallel translation” [6]. In this space,

$$\nabla_l G_{mn} = -\Gamma_l G_{mn},$$

where ∇_l is a covariant derivative, G_{mn} is a metric and Γ_l is a 4-vector (note that for the Riemann space in general relativity this derivative is zero). The Dirac spinors serve in (1) as basis vectors of the representation of the manifold fundamental group, while the electromagnetic field components prove to be components of a curvature tensor of the manifold-covering space, and 4-potentials serve as connections in this space. Energy, momentum components, mass, spin and particle-antiparticle states appear to be geometrical characteristics of the above manifold.

Geometrization of Weak Interaction

Does the above topological interpretation reflect any physical reality? It is impossible to give an answer within the framework of the considered one-particle low-energy approximation alone. New approach did not lead to predictions of new electrodynamical effects and did not explain anything that had not been explained before (as, e.g., deviations from the Newtonian gravitational law that was predicted by the theory of general relativity). So, we have to apply new concept to such problems where we encounter some difficulties and where there is a possibility of obtaining new physical results. Within the one-particle approximation, these may be some difficulties of the standard model of weak and strong interactions: gauge invariance and P-T violation mechanisms, nucleon-nucleon interaction at low energies, and so on. For many-particle quantum physics these may be problems of quantum chemistry and problems of quantum field theory. So, we have to choose the next step, and we will, at first, attempt to apply topological approach to consider weak interaction within one-particle approximation, where we need not solve the problem of geometrization of the second quantization procedure. In this section, we show that the geometrical concept leads to some new results, as compared with standard model.

As for the electromagnetic interaction, we suppose that the neutrino field together with its weak field can be considered as another kind of space-time 4-manifold. We have shown in our previous publications (see [1-3]) that topological properties of the manifold were represented by geometrical and symmetry properties of this manifold-covering space (the space with semimetrical translation in the case of interacting electron-positron and electromagnetic fields). Therefore, we have to find out the type of new manifold-covering space that could give an opportunity to reflect the known features of weak interaction.

Let us now show that, in the one-particle approximation adopted in this work (low energies), weak interaction can be represented as a manifestation of the torsion in the covering space of a 4-manifold representing weak field and its sources. Note that in due time, Einstein attempted at including electromagnetic field into a unified geometrical description of physical fields by “adding” torsion to the Riemannian space-time curvature, which reflects the presence of a gravitational field in general relativity [7]. Since the curvature of covering space corresponds, within our approach, to the electromagnetic field, we will attempt to include weak interaction into the topological approach by including torsion in this space.

It is known that weak interaction breaks the mirror space-time symmetry. On the other hand this symmetry can be violated in space with torsion (at least, left screw looks like a right one in the mirror). So, it is natural to assume that, within our topological approach, torsion may be due to weak interaction. Let us consider the case where the electromagnetic field is absent, i.e., the curvature of covering space is zero. A space with torsion but without curvature is so called “the space with absolute parallelism” [7, 8]. Thus, at first the challenge is to determine how does Eq. (1) change if the interparticle interaction is due only to torsion, which transforms the pseudo-Euclidean covering space into a space with absolute parallelism.

Let us denote the torsion tensor by S_{lm}^k ; then the problem can be formulated as follows. It is necessary to “insert” the tensor S_{lm}^k or some of its components into Eq. (1) (where now $A_l = 0$) so that the resulting equation remains invariant about the Lorentz transformations and adequately

describes the experimental data (e.g., violation of spatial and time symmetry by weak interaction). Among the spaces with torsion, there are so-called “spaces with semisymmetric parallel translation” [6,9]. The torsion tensor S_{lm}^k for such spaces is defined by the antisymmetric part of connection and can be represented as [6]

$$S_{lm}^k = K_l I_m^k - S_m A_l^k. \quad (4)$$

Here, K_l is a 4-vector and I_l^k is the identity tensor. We will investigate the above-mentioned kind of spaces as a candidate for the covering space because this space has some preferential direction (vector K_l) and, so, may in a way, be “responsible” for the P and T violation.

The vector K_l has the property that an infinitesimal parallelogram remains closed upon the parallel translation in the hyperplanes perpendicular to this vector [6, 9]. This indicates that, in the considered space, the sliding symmetry of Eq. (1) (the product of reflection operators γ_l and translation operators $\partial/\partial x_l$) can only be retained in the 3-hyperplanes perpendicular to K_l , while the translation symmetry can be retained along K_l . This is the first thing we have to take into account when looking for the analogy of Eq. (1) to describe a neutrino and its weak field.

The second important point is that, unlike the case of electromagnetic interaction, we can now fix the invariant orientation in every point of the covering space. It follows from the fact that, in addition to vector K_l , we may introduce into consideration one more vector B_l

$$B_l = S_{lp}^p = K_l A_p^p - K_p A_l^p. \quad (5)$$

Having two nonparallel vectors K_l and B_l we can define the antisymmetric second-rank tensor (bivector) T_{lm}

$$T_{lm} = \frac{1}{2}(K_l B_m - B_l K_m), \quad (6)$$

and it is known that every bivector defines orientation in the space [8,9].

After taking into account the above symmetry properties, we obtain the following analogy of Dirac Eq. (1) for a neutrino interacting with its weak field

$$i\sigma_1 \nabla_1 \varphi - \sum_{\alpha=2}^4 i\sigma_\alpha \nabla_\alpha \varphi + \frac{g^2}{2} T_{lm} \sigma_m \nabla_l \varphi = m_N \varphi, \quad (7)$$

where m_N is the neutrino mass, σ_1 is a two-row unit matrix, $\sigma_{2,3,4}$ are the Pauli matrices ($\sigma_{x,y,z}$), φ is a two-component spinor, and ∇_l are the covariant derivatives

$$\nabla_l = \frac{\partial}{\partial x_l} - igD_l. \quad (8)$$

Here, igD_l is a connection in the considered covering space. Note that we introduce the coupling constant g by inserting $B \rightarrow igB$, $K \rightarrow igK$, $D \rightarrow igD$.

We see that the last term on the left-hand side of Eq.(7) is of the second order in the small coupling constant g , but this term plays the main role in the space-time symmetry violation. The above term corresponds within the Lagrangian formalism to the new additional pseudoscalar term L_{ad} (in the standard Lagrangian)

$$L_{ad} = g^2 T_{lm} \varphi^+ \sigma_m \nabla_l \varphi. \quad (9)$$

The fields D_l in Eqs.(7),8) can be obtained using the symmetry properties of the covering space with semisymmetric translation. This symmetry must be the same after rotations in the 3-hyperplane perpendicular to the vector K_l . Let us assume that OX_1 axis is aligned with the vector

K_l after a certain rotation expressed by the $U(1)$ representation (note that this rotation reflects the gauge invariance of electromagnetic interaction which we did not include into consideration). Then the above hyperplane is a usual 3-space and for the two-component spinors the above rotations in this space are represented by the matrices of the $SU(2)$ representation. So, this symmetry proves to be just the $SU(2)$ -gauge invariance used in the standard model of weak interaction for calculating weak fields (D_l fields in our case). This means that the fields D_l in Eqs. (7), (8) can be expressed in terms of twelve Yang-Mills fields D_l^α ($\alpha = x, y, z$) [10]

$$D_l = \frac{1}{2}(D_l^x \sigma_x + D_l^y \sigma_y + D_l^z \sigma_z). \quad (10)$$

Then, the weak field strength has the form

$$G_{lm} = \frac{\partial D_l}{\partial x_m} - \frac{\partial D_m}{\partial x_l} - ig(D_l D_m - D_m D_l). \quad (11)$$

Finally, the equations for weak interactions read

$$i\sigma_1 \nabla_1 \varphi - \sum_{\alpha=2}^4 i\sigma_\alpha \nabla_\alpha \varphi + \frac{g^2}{2} T_{lm} \sigma_m \nabla_l \varphi = m_N \varphi, \quad (12)$$

$$G_{lm} = \frac{\partial D_l}{\partial x_m} - \frac{\partial D_m}{\partial x_l} - ig(D_l D_m - D_m D_l), \quad (13)$$

$$\sum_{i=1}^4 \frac{\partial G_{ik}}{\partial x_i} = j_k^N, \quad j_k^N = g\varphi * \sigma_k \varphi, \quad (14)$$

$$D_l = \frac{1}{2}(D_l^x \sigma_x + D_l^y \sigma_y + D_l^z \sigma_z). \quad (15)$$

In the following publications we will represent the interdependence between T_{lm} and D_l (it is a purely geometrical problem).

Gravitational interaction

Gravitational field in general relativity is geometrized by representing classical trajectories of macroscopic bodies as geodesics in the curved Riemannian space-time. The field sources are also supposed to be macroscopic objects. In this paper we propose the hypothesis for the geometrisation of gravitational field generated by microscopic sources of gravitational field (elementary gravitational charges).

The main idea is as follows. Electrons are microscopic elementary sources of electromagnetic field and both of them (field and sources) are represented, within our topological approach, by the special 4-manifold. The topological properties of this manifold are described by the Dirac-Maxwell equations (1)-(3). Neutrinos are microscopic elementary sources of weak field, and both of them (field and sources) are represented by a different special 4-manifold and the topological properties of this manifold are described by equations (7)-(11). It is natural to assume that elementary sources of gravitational field are also certain microscopic particles and that both of them (field and these sources) can be considered as a special 4-manifold and can be described by group-theoretical relations similar to Eqs. (1)-(3) and Eqs. (12)-(15).

We obtained above relations for gravitational interaction assuming that the covering space for corresponding 4-manifold is the Riemann space (the same as in general relativity). Instead of the spinor fields describing electromagnetic and weak interactions, we use vector field ϕ_l . Then the

analogy of the Dirac equation for the free electron-positron field will be the known Lorentz-invariant Proca equation for vector field ϕ_l describing free particles with mass m and spin 1 [11]

$$\frac{\partial\phi_{lm}}{\partial x_m} = m^2\phi_l, \quad \phi_{lm} = \frac{\partial\phi_l}{\partial x_m} - \frac{\partial\phi_m}{\partial x_l}. \quad (16)$$

We take the gravitational interaction into account by replacing usual derivatives in Eq.(16) by the covariant derivatives with the standard connection in Riemann space [8, 9]. The resulting equations for the field sources are

$$\frac{\partial\phi_{lm}}{\partial x_m} - \Gamma_{ml}^p\phi_{pm} - \Gamma_{mm}^p\phi_{lp} = m_G^2\phi_l, \quad (17)$$

where the connection Γ_{ml}^p has the standard form [8, 9]

$$\Gamma_{mn}^l = \frac{1}{2}G^{lp} \left(\frac{\partial G_{pm}}{\partial x_n} + \frac{\partial G_{pn}}{\partial x_m} - \frac{\partial G_{mn}}{\partial x_p} \right). \quad (18)$$

Here G_{mn} is the metrics of Riemann space, m_G is the elementary gravitational charge.

The connection Γ_{mn}^l is not a tensor with respect to the arbitrary coordinate transformation [8, 9]. This means that the physical observable variables should be connected directly not with Γ_{mn}^l but with some tensor that is expressed through Γ_{mn}^l . As in general relativity, we use the Ricci tensor R_{ik} [8, 9]

$$R_{ik} = R_{pi,k}^p,$$

where $R_{si,k}^p$ is the Riemannian space curvature tensor. The Ricci tensor may be expressed through Γ_{mn}^l as [8, 9]

$$R_{ik} = \frac{\partial\Gamma_{il}^l}{\partial x_k} - \frac{\partial\Gamma_{ik}^l}{\partial x_l} - \Gamma_{ik}^l\Gamma_{lp}^p + \Gamma_{ip}^l\Gamma_{kl}^p. \quad (19)$$

These equations can be considered as an analogy of the first pair of the Maxwell equations.

We suppose that the microscopic source of gravitational field is a gravitational current j_l^G of our vector field ϕ_l which has the standard form [11]

$$j_l^G = -im_G \left(\phi^p * \frac{\partial\phi_p}{\partial x_l} - \phi^p \frac{\partial\phi_p^*}{\partial x_l} \right), \quad (20)$$

Then the analog of the second pair of Maxwell equations will be equation for the Ricci tensor

$$\frac{\partial R_{ik}}{\partial x_i} = j_k^G. \quad (21)$$

Finally, we have the following system of equations describing, within the framework of topological approach, a gravitational field generated by its microscopic sources,

$$\frac{\partial\phi_{lm}}{\partial x_m} - \Gamma_{ml}^p\phi_{pm} - \Gamma_{mm}^p\phi_{lp} = m_G^2\phi_l, \quad (22)$$

$$\phi_{lm} = \frac{\partial\phi_l}{\partial x_m} - \frac{\partial\phi_m}{\partial x_l}, \quad (23)$$

$$R_{ik} = \frac{\partial\Gamma_{il}^l}{\partial x_k} - \frac{\partial\Gamma_{ik}^l}{\partial x_l} - \Gamma_{ik}^l\Gamma_{lp}^p + \Gamma_{ip}^l\Gamma_{kl}^p, \quad (24)$$

$$\frac{\partial R_{ik}}{\partial x_i} = -im_G \left(\phi^p * \frac{\partial\phi_p}{\partial x_k} - \phi^p \frac{\partial\phi_p^*}{\partial x_k} \right). \quad (25)$$

The proposed elementary gravitational charge m_G (mass m_G is not yet defined), represents one more candidate for a rather numerous family of so-called WINP particles (Weakly Interacting

Neutral Particle [12]). The possibilities of experimental confirmation will be considered in detail elsewhere. Note in conclusion that, contrary to general relativity, the Riemann space appears here not as a real curved space-time but as an “effective” Riemann space (covering space used for the formal description of a more complicated geometrical object—microscopic nonmetrized closed topological space-time 4-manifold).

Conclusion

We showed that, as in the case of electromagnetic interaction, the topological concept of quantum mechanics also does not contradict the known equations for weak interaction, although leads to some new small terms that are responsible for the P-T violation.

Within the new approach the $U(1)SU(2)$ gauge invariance seems to be the standard Lorentz rotation to new frame position, where electromagnetic and weak fields are most symmetric.

The hypothesis is also proposed about new elementary particle that can be considered as an elementary microscopic gravitational charge, and the equations describing these particles and their gravitational fields are suggested.

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