Energy-Momentum Conservation Laws in Gauge Theory with Broken Gauge Invariance

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If a Lagrangian of gauge theory of internal symmetries is not gauge-invariant, the energy-momentum fails to be conserved in general.

1. Introduction

We follow the geometric formulation of classical field theory where fields are represented by sections of a fibre bundle $Y \to X$, coordinated by (x^λ, y^i) (see [7, 12, 16] for a survey). Then gauge transformations are defined as automorphisms of $Y \to X$. A gauge transformation is called internal if it is a vertical automorphism of $Y \to X$, i.e., is projected onto the identity morphism of the base X. To study the invariance conditions and conservation laws, it suffices to consider one-parameter groups of gauge transformations. Their infinitesimal generators are projectable vector fields

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i \tag{1.1}$$

on a fibre bundle $Y \to X$. In particular, generators of internal gauge transformations are vertical vector fields

$$u = u^i(x^\mu, y^j)\partial_i. \tag{1.2}$$

We are concerned with a first order Lagrangian field theory. Its configuration space is the first order jet manifold J^1Y of $Y \to X$, coordinated by $(x^\lambda, y^i, y^i_\lambda)$. A first order Lagrangian is defined as a density

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega, \qquad \omega = dx^1 \wedge \cdots \wedge dx^n, \qquad n = \dim X, \tag{1.3}$$

on J^1Y. A Lagrangian L is invariant under a one-parameter group of gauge transformations generated by a vector field u (1.1) iff its Lie derivative

$$\mathbf{L}_{J^1u}L = J^1u\rfloor dL + d(J^1u\rfloor L) \tag{1.4}$$

along the jet prolongation J^1u of u vanishes. In this case, the first variational formula of the calculus of variations leads on-shell to the weak conservation law

$$d_\lambda \mathfrak{T}^\lambda_u \approx 0 \tag{1.5}$$

of the current

$$\begin{aligned} \mathfrak{T}_u &= \mathfrak{T}^\lambda_u \omega_\lambda, \qquad \omega_\lambda = \partial_\lambda \rfloor \omega, \\ \mathfrak{T}^\lambda_u &= (u^\mu y^i_\mu - u^i)\partial^\lambda_i \mathcal{L} - u^\lambda \mathcal{L}, \end{aligned} \tag{1.6}$$

along u. In particular, the current $\mathfrak{T}_u$ (1.6) along a vertical vector field u (1.2) reads

$$\mathfrak{T}_u^\lambda = -u^i \partial_i^\lambda \mathcal{L}. \tag{1.7}$$

It is called the Noether current.

It is readily observed that

$$\mathfrak{T}_{u+u'} = \mathfrak{T}_u + \mathfrak{T}_{u'}. \tag{1.8}$$

Note that any projectable vector field u (1.1), projected onto the vector field $\tau = u^\lambda \partial_\lambda$ on X, can be written as the sum

$$u = \widetilde{\tau} + (u - \widetilde{\tau}) \tag{1.9}$$

of some lift $\widetilde{\tau} = u^\lambda \partial_\lambda + \widetilde{\tau}^i \partial_i$ of τ onto Y and the vertical vector field $u - \widetilde{\tau}$ on Y. The current $\mathfrak{T}_{\widetilde{\tau}}$ (1.6) along a lift $\widetilde{\tau}$ onto Y of a vector field $\tau = \tau^\lambda \partial_\lambda$ on X is said to be the energy-momentum current [4, 7, 9, 13]. Then the decompositions (1.8) and (1.9) show that any current $\mathfrak{T}_u$ (1.6) along a projectable vector field u on a fibre bundle $Y \to X$ can be represented by a sum of an energy-momentum current and a Noether one.

Different lifts $\widetilde{\tau}$ and $\widetilde{\tau}'$ onto Y of a vector field τ on X lead to distinct energy-momentum currents $\mathfrak{T}_{\widetilde{\tau}}$ and $\mathfrak{T}_{\widetilde{\tau}'}$, whose difference $\mathfrak{T}_{\widetilde{\tau}} - \mathfrak{T}_{\widetilde{\tau}'}$ is the Noether current along the vertical vector field $\widetilde{\tau} - \widetilde{\tau}'$ on Y. The problem is that, in general, there is no canonical lift onto Y of vector fields on X, and one can not take the Noether part away from an energy-momentum current. Therefore, if a Lagrangian is not invariant under vertical gauge transformations, there is an obstruction for energy-momentum currents to be conserved [15].

Note that there exists the category of so called natural fibre bundles $T \to X$ which admit the canonical lift $\widetilde{\tau}$ of any vector field τ on X [11]. This lift is the infinitesimal generator of a one-parameter group of general covariant transformations of T. For instance, any tensor bundle

$$T = (\overset{m}{\otimes} TX) \otimes (\overset{k}{\otimes} T^*X) \tag{1.10}$$

over X is of this type. The canonical lift onto T (1.10) of a vector field τ on X is

$$\widetilde{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^{\alpha_1} \dot{x}^{\nu\alpha_2\cdots\alpha_m}_{\beta_1\cdots\beta_k} + \ldots - \partial_{\beta_1} \tau^\nu \dot{x}^{\alpha_1\cdots\alpha_m}_{\nu\beta_2\cdots\beta_k} - \ldots] \frac{\partial}{\partial \dot{x}^{\alpha_1\cdots\alpha_m}_{\beta_1\cdots\beta_k}}. \tag{1.11}$$

For instance, gravitation theory is a gauge field theory on natural bundles. Its Lagrangians are invariant under general covariant transformations. The corresponding conserved energy-momentum current on-shell takes the form

$$\mathfrak{T}_{\widetilde{\tau}}^\lambda \approx d_\mu U^{\mu\lambda}, \tag{1.12}$$

where $U^{\mu\lambda} = -U^{\lambda\mu}$ is the generalized Komar superpotential [1, 6, 7, 14]. Other energy-momentum currents differ from $\mathfrak{T}_{\widetilde{\tau}}$ (1.12) in Noether currents, but they fail to be conserved because almost all gravitation Lagrangians are not invariant under vertical (non-holonomic) gauge transformations.

Here, we focus on energy-momentum conservation laws in gauge theory of principal connections on a principal bundle $P \to X$ with a structure Lie group G. These connections are sections of the fibre bundle

$$C = J^1P/G \to X, \tag{1.13}$$

and are identified to gauge potentials [7, 12, 16]. The well-known result claims that, if L is a gauge-invariant Lagrangian on J^1C in the presence of a background metric g, we have the familiar covariant conservation law

$$\nabla_\lambda (t_\mu^\lambda \sqrt{\mid g \mid}) \approx 0 \tag{1.14}$$

of the metric energy-momentum tensor

$$t^\mu_\beta\sqrt{\mid g\mid} = 2g^{\mu\alpha}\partial_{\alpha\beta}\mathcal{L}, \tag{1.15}$$

where ∇ is the covariant derivative with respect to the Levi–Civita connection of the background metric g [9]. Moreover, other energy-momentum conservation laws differ from (1.14) in superpotentials terms $d_\mu d_\lambda U^{\mu\lambda}$. Here, we show that the conservation law (1.14) locally holds without fail. However, no energy-momentum current is conserved if a principal bundle P is not trivial and a Lagrangian of gauge theory on P is not gauge-invariant.

Two examples of non-invariant Lagrangians are examined. The first one is the Chern–Simons Lagrangian whose Euler–Lagrange operator is gauge-invariant. In this case, we have a conserved quantity, but it differs from an energy-momentum current. Another example is the Yang–Mills Lagrangian in the presence of a background field, e.g., a Higgs field.

2. Lagrangian conservation laws

The first variational formula provides the following universal procedure for the study of Lagrangian conservation laws in classical field theory.

Remark 2.1. Let J^2Y be the second order jet manifold coordinated by $(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu})$. Recall the following standard notation: of the contact form $\theta^i = dy^i - y^i_\lambda dx^\lambda$, the horizontal projection

$$h_0(dx^\lambda) = dx^\lambda, \qquad h_0(dy^i) = y^i_\lambda dx^\lambda \qquad h_0(dy^i_\mu) = y^i_{\lambda\mu}dx^\lambda,$$

the total derivative

$$d_\lambda = \partial_\lambda + y^i_\lambda\partial_i + y^i_{\lambda\mu}\partial^\mu_i,$$

and the horizontal differential $d_H = dx^\lambda \wedge d_\lambda$ such that $d_H \circ h_0 = h_0 \circ d$.

Let u be a projectable vector field on a fibre bundle $Y \to X$ and

$$J^1u = u + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i \tag{2.1}$$

its jet prolongation onto J^1Y. The Lie derivative (1.4) of a Lagrangian L along J^1u reads

$$\mathbf{L}_{J^1u}L = [\partial_\lambda u^\lambda\mathcal{L} + (u^\lambda\partial_\lambda + u^i\partial_i + (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i)\mathcal{L}]\omega. \tag{2.2}$$

The first variational formula states its canonical decomposition over J^2Y:

$$\begin{aligned}\mathbf{L}_{J^1u}L = u_V\rfloor\mathcal{E}_L + d_H h_0(u\rfloor H_L) = \\ (u^i - y^i_\mu u^\mu)(\partial_i - d_\lambda\partial^\lambda_i)\mathcal{L}\omega - d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}]\omega,\end{aligned} \tag{2.3}$$

where $u_V = (u\rfloor\theta^i)\partial_i$,

$$\mathcal{E}_L = (\partial_i\mathcal{L} - d_\lambda\partial^\lambda_i\mathcal{L})\theta^i \wedge \omega, \tag{2.4}$$

is the Euler–Lagrange operator, and

$$H_L = L + \partial^\lambda_i\mathcal{L}\theta^i \wedge \omega_\lambda = \partial^\lambda_i\mathcal{L}dy^i \wedge \omega_\lambda + (\mathcal{L} - y^i_\lambda\partial^\lambda_i\mathcal{L})\omega \tag{2.5}$$

is the Poincaré–Cartan form.

The kernel of the Euler-Lagrange operator $\mathcal{E}_L$ (2.4) is given by the coordinate relations

$$\delta_i\mathcal{L} = (\partial_i - d_\lambda\partial^\lambda_i)\mathcal{L} = 0, \tag{2.6}$$

and defines the Euler–Lagrange equations. Their classical solution is a section s of the fibre bundle $X \to Y$ whose second order jet prolongation $J^2 s$ lives in (2.6).

Remark 2.2. Note that different Lagrangians L and L' lead to the same Euler–Lagrange operator if their difference $L_0 = L - L'$ is a variationally trivial Lagrangian whose Euler–Lagrange operator vanishes. Such a Lagrangian takes the form

$$L_0 = h_0(\epsilon) \tag{2.7}$$

where ϵ is a closed n-form on Y [8, 16]. We have locally $\epsilon = d\sigma$ and

$$L_0 = h_0(d\sigma) = d_H(h_0(\sigma)).$$

On the shell (2.6), the first variational formula (2.3) leads to the weak identity

$$\partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y^i_\mu \partial_\lambda u^\mu)\partial^\lambda_i]\mathcal{L} \approx -d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i \mathcal{L} - u^\lambda \mathcal{L}]. \tag{2.8}$$

If the Lie derivative $\mathbf{L}_{J^1 u} L$ (2.2) vanishes, we obtain the weak conservation law $0 \approx -d_H \mathfrak{T}_u$ (1.5) of the current $\mathfrak{T}_u$ (1.6). It takes the coordinate form

$$0 \approx -d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i \mathcal{L} - u^\lambda \mathcal{L}]. \tag{2.9}$$

Remark 2.3. It should be emphasized that, from the first variational formula, the symmetry current (1.6) is defined modulo the terms $d_\mu(c^{\mu\lambda}_i(y^i_\nu u^\nu - u^i))$, where $c^{\mu\lambda}_i$ are arbitrary skew-symmetric functions on Y. Here we leave aside these boundary terms which are independent of a Lagrangian.

The weak conservation law (2.9) leads to the differential conservation law

$$\partial_\lambda(\mathfrak{T}^\lambda \circ s) = 0$$

on a solution s of the Euler–Lagrange equations. This differential conservation law implies the integral law

$$\int\limits_{\partial N} s^* \mathfrak{T} = 0, \tag{2.10}$$

where N is a compact n-dimensional submanifold of X and ∂N denotes its boundary.

Remark 2.4. It may happen that a current $\mathfrak{T}$ (1.6) takes the form

$$\mathfrak{T} = W + d_H U = (W^\lambda + d_\mu U^{\mu\lambda})\omega_\lambda, \tag{2.11}$$

where the term W vanishes on-shell ($W \approx 0$) and

$$U = U^{\mu\lambda}\omega_{\mu\lambda}, \qquad \omega_{\mu\lambda} = \partial_\mu \rfloor \omega_\lambda, \tag{2.12}$$

is a horizontal $(n-2)$-form on $J^1 Y$. Then one says that $\mathfrak{T}$ reduces to a superpotential U (2.12) [3, 7, 13]. In this case, the integral conservation law (2.10) becomes tautological. At the same time, the superpotential form (2.11) of $\mathfrak{T}$ implies the following integral relation

$$\int\limits_{N^{n-1}} s^* \mathfrak{T} = \int\limits_{\partial N^{n-1}} s^* U, \tag{2.13}$$

where N^{n-1} is a compact oriented $(n-1)$-dimensional submanifold of X with the boundary ∂N^{n-1}. One can think of this relation as being a part of the Euler–Lagrange equations written in an integral form.

Remark 2.5. Let us consider conservation laws in the case of gauge transformations preserving the Euler-Lagrange operator $\mathcal{E}_L$, but not necessarily a Lagrangian L. Let u be a generator of these transformations. Then we have

$$\mathbf{L}_{J^2u}\mathcal{E}_L = 0,$$

where J^2u is the second order jet prolongation of the vector field u. There is the relation

$$\mathbf{L}_{J^2u}\mathcal{E}_L = \mathcal{E}_{\mathbf{L}_{J^1u}L}$$

[5, 7], and we obtain $\mathcal{E}_{\mathbf{L}_{J^1u}L} = 0$. It follows that the Lie derivative $\mathbf{L}_{J^1u}L$ is a variationally trivial Lagrangian. Hence, it takes the form $h_0(\epsilon)$ (2.7). Then the weak identity (2.8) comes to the weak equality

$$h_0(\epsilon) \approx -d_H\mathfrak{T}_u. \tag{2.14}$$

A glance at this expression shows that

$$h_0(e) = W + d_H\phi, \tag{2.15}$$

where $W \approx 0$. Then the equality (2.14) leads to the weak conservation law

$$0 \approx d_H(\phi + \mathfrak{T}_u), \tag{2.16}$$

but the conserved quantity $\phi + \mathfrak{T}_u$ is not globally defined, unless ϵ is an exact form. For instance, let $Y \to X$ be an affine bundle. In this case, $\epsilon = \varepsilon + d\sigma$ where ε is an n-form on X [8]. Since the weak equality $\varepsilon \approx 0$ implies the strong one $\varepsilon = 0$, we obtain from the expression (2.15) that ε is also an exact form. Thus, the conserved quantity in the conservation law (2.16) is well defined.

Remark 2.6. Background fields do not live in the dynamic shell (2.6) and, therefore, break Lagrangian conservation laws as follows. Let us consider the product

$$Y_{\mathrm{tot}} = Y \underset{X}{\times} Y' \tag{2.17}$$

of a fibre bundle Y, coordinated by (x^λ, y^i), whose sections are dynamic fields and a fibre bundle Y', coordinated by (x^λ, y^A), whose sections are background fields which take the background values

$$y^B = \phi^B(x), \qquad y^B_\lambda = \partial_\lambda\phi^B(x). \tag{2.18}$$

A Lagrangian L is defined on the total configuration space J^1Y_{tot}. Let u be a projectable vector field on Y_{tot} which also projects onto Y' because gauge transformations of background fields do not depend on the dynamic ones. This vector field takes the coordinate form

$$u = u^\lambda(x)\partial_\lambda + u^A(x^\mu, y^B)\partial_A + u^i(x^\mu, y^B, y^j)\partial_i. \tag{2.19}$$

Substitution of (2.19) in (2.3) leads to the first variational formula in the presence of background fields

$$\begin{aligned}&\partial_\lambda u^\lambda\mathcal{L} + [u^\lambda\partial_\lambda + u^A\partial_A + u^i\partial_i + (d_\lambda u^A - y^A_\mu\partial_\lambda u^\mu)\partial^\lambda_A + \\ &\quad (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i]\mathcal{L} = (u^A - y^A_\lambda u^\lambda)\partial_A\mathcal{L} + \partial^\lambda_A\mathcal{L}d_\lambda(u^A - y^A_\mu u^\mu) + \\ &\quad (u^i - y^i_\lambda u^\lambda)\delta_i\mathcal{L} - d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}].\end{aligned} \tag{2.20}$$

Then the following identity

$$\begin{aligned}&\partial_\lambda u^\lambda\mathcal{L} + [u^\lambda\partial_\lambda + u^A\partial_A + u^i\partial_i + (d_\lambda u^A - y^A_\mu\partial_\lambda u^\mu)\partial^\lambda_A + \\ &\quad (d_\lambda u^i - y^i_\mu\partial_\lambda u^\mu)\partial^\lambda_i]\mathcal{L} \approx (u^A - y^A_\lambda u^\lambda)\partial_A\mathcal{L} + \partial^\lambda_A\mathcal{L}d_\lambda(u^A - y^A_\mu u^\mu) - \\ &\quad d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}]\end{aligned}$$

holds on the shell (2.6). A total Lagrangian L usually is constructed to be invariant under gauge transformations of the product (2.17). In this case, we obtain the weak identity

$$(u^A - y^A_\mu u^\mu)\partial_A\mathcal{L} + \partial^\lambda_A\mathcal{L}d_\lambda(u^A - y^A_\mu u^\mu) \approx d_\lambda[(u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L}]. \tag{2.21}$$

in the presence of background fields on the shell (2.18). Given a background field ϕ (2.18), there always exists a vector field on a fibre bundle $Y \to X$ such that the left-hand side of the equality (2.21) vanishes. This is the horizontal lift

$$\widetilde{\tau} = \tau^\lambda(\partial_\lambda + \Gamma^A_\alpha \partial_A)$$

onto Y' of a vector field τ on X by means of a connection Γ on $Y' \to X$ whose integral section is ϕ, i.e., $\partial_\lambda \phi^A = \Gamma^A_\lambda \circ \phi$. However, the Lie derivative of a Lagrangian L along this vector field need not vanish.

3. Noether conservation laws in gauge theory

Let $\pi_P : P \to X$ be a principal bundle with a structure Lie group G which acts on P on the right

$$R_g : p \mapsto pg, \quad p \in P, \quad g \in G. \tag{3.1}$$

A principal bundle P is equipped with a bundle atlas $\Psi_P = \{(U_\alpha, \psi^P_\alpha\}$ whose trivialization morphisms ψ^P_α obey the equivariance condition

$$\psi^P_\alpha(pg) = \psi^P_\alpha(p)g, \qquad \forall g \in G, \qquad \forall p \in \pi_P^{-1}(U_\alpha). \tag{3.2}$$

A gauge transformation in gauge theory on a principal bundle $P \to X$ is defined as an automorphism Φ_P of $P \to X$ which is equivariant under the canonical action (3.1), i.e., $R_g \circ \Phi_P = \Phi_P \circ R_g$ for all $g \in G$. The infinitesimal generator of a one-parameter group of these gauge transformations is a G-invariant vector field ξ on P. It is naturally identified to a section of the quotient $T_G P = TP/G$ of the tangent bundle $TP \to P$ by the canonical action R_G (3.1). Due to the equivariance condition (3.2), any bundle atlas Ψ_P of P yields the associated bundle atlase $\{U_\alpha, T\psi^P_\alpha/G)\}$ of $T_G P$. Given a basis $\{e_p\}$ for the right Lie algebra $\mathfrak{g}_r$ of the group G, let $\{\partial_\lambda, e_p\}$ be the corresponding local fibre bases for the vector bundles $T_G P$. Then a section ξ of $T_G P \to X$ reads

$$\xi = \xi^\lambda \partial_\lambda + \xi^p e_p. \tag{3.3}$$

The infinitesimal generator of a one-parameter group of vertical gauge transformations is a G-invariant vertical vector field on P identified to a section $\xi = \xi^p e_p$ of the quotient

$$V_G P = VP/G \subset T_G P \tag{3.4}$$

of the vertical tangent bundle VP of P by the canonical action R_G (3.1).

The Lie bracket of two sections ξ and η of the vector bundle $T_G P \to X$ reads

$$[\xi, \eta] = (\xi^\mu \partial_\mu \eta^\lambda - \eta^\mu \partial_\mu \xi^\lambda)\partial_\lambda + (\xi^\lambda \partial_\lambda \eta^r - \eta^\lambda \partial_\lambda \xi^r + c^r_{pq} \xi^p \eta^q) e_r, \tag{3.5}$$

where c^r_{pq} are the structure constants of the Lie algebra $\mathfrak{g}_r$. Putting $\xi^\lambda = 0$ and $\eta^\mu = 0$, we obtain the Lie bracket

$$[\xi, \eta] = c^r_{pq} \xi^p \eta^q e_r \tag{3.6}$$

of sections of the vector bundle $V_G P \to X$. A glance at the expression (3.6) shows that the typical fibre of $V_G P \to X$ is the Lie algebra $\mathfrak{g}_r$. The structure group G acts on $\mathfrak{g}_r$ by the adjoint representation.

A principal connection on a principal bundle $P \to X$ is defined as a global section A of the affine jet bundle $J^1 P \to P$ which is equivariant under the right action (3.1), i.e.,

$$J^1 R_g \circ A = A \circ R_g, \qquad \forall g \in G. \tag{3.7}$$

Due to this equivariance condition, there is one-to-one correspondence between the principal connections on a principal bundle $P \to X$ and the global sections A of the quotient C (1.13) of the first order jet manifold J^1P of a principal bundle $P \to X$ by the jet prolongation of the canonical action R_G (3.1). The quotient C (1.13) is an affine bundle over X. Given a bundle atlas Ψ_P of P, it is provided with bundle coordinates (x^λ, a^q_λ) such that $A^q_\lambda = a^q_\lambda \circ A$ are coefficients of the familiar local connection form $A^q_\lambda dx^\lambda \otimes e_q$ on X, i.e., a^q_λ are coordinates of gauge potentials. Therefore C (1.13) is called the connection bundle. Gauge transformations of P generated by the vector field (3.3) induce gauge transformations of C whose generator is

$$\xi_C = \xi^\lambda \partial_\lambda + (\partial_\mu \xi^r + c^r_{pq} a^p_\mu \xi^q - a^r_\lambda \partial_\mu \xi^\lambda)\partial^\mu_r. \tag{3.8}$$

The configuration space of gauge theory is the first order jet manifold J^1C of C coordinated by $(x^\lambda, a^q_\lambda, a^q_{\lambda\mu})$. It admits the canonical splitting over C which takes the coordinate form

$$a^r_{\lambda\mu} = \frac{1}{2}(\mathcal{F}^r_{\lambda\mu} + \mathcal{S}^r_{\lambda\mu}) = \frac{1}{2}(a^r_{\lambda\mu} + a^r_{\mu\lambda} - c^r_{pq} a^p_\lambda a^q_\mu) + \frac{1}{2}(a^r_{\lambda\mu} - a^r_{\mu\lambda} + c^r_{pq} a^p_\lambda a^q_\mu). \tag{3.9}$$

Let L be a Lagrangian on J^1C. One usually requires of L to be invariant under vertical gauge transformations. It means that the Lie derivative $\mathbf{L}_{J^1\xi_{CY}} L$ of L along the jet prolongation (2.1) of any vertical vector field

$$\xi_C = (\partial_\lambda \xi^r + c^r_{qp} a^q_\lambda \xi^p)\partial^\lambda_r \tag{3.10}$$

on C vanishes. Coefficients ξ^q of this vector field play the role of gauge parameters. Then we come to the well-known Noether conservation law. The key point is that, since the vector fields (3.10) depends on derivatives of gauge parameters, the Noether current in gauge theory reduces to a superpotential as follows.

The first variational formula (2.3) leads to the strong equality

$$0 = (\partial_\mu \xi^r + c^r_{qp} a^q_\mu \xi^p)\delta^\mu_r \mathcal{L} + d_\lambda[(\partial_\mu \xi^r + c^r_{qp} a^q_\mu \xi^p)\partial^{\lambda\mu}_r \mathcal{L}]. \tag{3.11}$$

Due to the arbitrariness of gauge parameters ξ^p, this equality falls into the system of equalities

$$c^r_{pq}(a^p_\mu \partial^\mu_r \mathcal{L} + a^p_{\lambda\mu}\partial^{\lambda\mu}_r \mathcal{L}) = 0, \tag{3.12a}$$
$$\partial^\mu_q \mathcal{L} + c^r_{pq} a^p_\lambda \partial^{\mu\lambda}_r \mathcal{L} = 0, \tag{3.12b}$$
$$\partial^{\mu\lambda}_p \mathcal{L} + \partial^{\lambda\mu}_p \mathcal{L} = 0. \tag{3.12c}$$

One can think of them as being the equations for a gauge-invariant Lagrangian. As is well known, there is a unique solution of these equations in the class of quadratic Lagrangians. It is the conventional Yang-Mills Lagrangian L_{YM} of gauge potentials on the configuration space J^1C. In the presence of a background metric g on the base X, it reads

$$L_{\mathrm{YM}} = \frac{1}{4\varepsilon^2} a^G_{pq} g^{\lambda\mu} g^{\beta\nu} \mathcal{F}^p_{\lambda\beta} \mathcal{F}^q_{\mu\nu} \sqrt{\mid g \mid}\omega, \qquad g = \det(g_{\mu\nu}), \tag{3.13}$$

where $\mathcal{F}^r_{\lambda\mu}$ are components of the canonical splitting (3.9) and a^G is a G-invariant bilinear form on the Lie algebra $\mathfrak{g}_r$.

On-shell, the strong equality (3.11) becomes the weak Noether conservation law

$$0 \approx d_\lambda[(\partial_\mu \xi^r + c^r_{qp} a^q_\mu \xi^p)\partial^{\lambda\mu}_r \mathcal{L}] \tag{3.14}$$

of the Noether current

$$\mathfrak{T}^\lambda_\xi = -(\partial_\mu \xi^r + c^r_{qp} a^q_\mu \xi^p)\partial^{\lambda\mu}_r \mathcal{L}. \tag{3.15}$$

In accordance with the strong equalities (3.12b) and (3.12c), the Noether current (3.15) is brought into the superpotential form

$$\mathfrak{T}^\lambda_\xi = \xi^r \delta^\lambda_r \mathcal{L} + d_\mu U^{\mu\lambda}, \qquad U^{\mu\lambda} = \xi^p \partial^{\lambda\mu}_p \mathcal{L}.$$

The corresponding integral relation (2.13) reads

$$\int\limits_{N^{n-1}} s^* \mathfrak{T}^\lambda \omega_\lambda = \int\limits_{\partial N^{n-1}} s^* (\xi^p \partial^{\mu\lambda}_p) \omega_{\mu\lambda}, \tag{3.16}$$

where N^{n-1} is a compact oriented $(n-1)$-dimensional submanifold of X with the boundary ∂N^{n-1}. One can think of (3.16) as being the integral relation between the Noether current (3.15) and the gauge field, generated by this current. In electromagnetic theory seen as a $U(1)$ gauge theory, the similar relation between an electric current and the electromagnetic field generated by this current is well known. However, it is free from gauge parameters due to the peculiarity of Abelian gauge models.

4. Energy-momentum conservation laws in gauge theory

Let us turn now to energy-momentum conservation laws in gauge theory.

Let B be a principal connection on a principal bundle $P \to X$. Given a vector field τ on X, there exists its lift

$$\widetilde{\tau}_B = \tau^\lambda \partial_\lambda + [\partial_\mu(\tau^\lambda B^r_\lambda) + c^r_{qp} a^q_\mu (\tau^\lambda B^p_\lambda) - a^r_\lambda \partial_\mu \tau^\lambda] \partial^\mu_r. \tag{4.1}$$

onto the connection bundle $C \to X$ (1.13) [5, 7, 12, 13]. Comparing the expressions (3.8) and (4.1), one easily observes that the lift $\widetilde{\tau}_B$ is a generator of gauge transformations of C with gauge parameters $\xi^\lambda = \tau^\lambda$, $\xi^r = \tau^\lambda B^r_\lambda$.

Let us discover the energy-momentum current along the lift (4.1). We assume that a Lagrangian L of gauge theory also depends on a background metric on X. This metric is described by a section of the tensor bundle $\overset{2}{\vee} TX$ provided with the holonomic coordinates $(x^\lambda, \sigma^{\mu\nu})$. Following Remark 2.6, we define L on the total configuration space

$$J^1 Y = J^1(C \underset{X}{\times} \overset{2}{\vee} TX). \tag{4.2}$$

Given a vector field τ on X, there exists its canonical lift

$$\widetilde{\tau} = \tau^\lambda \partial_\lambda + (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha}) \partial_{\alpha\beta} \tag{4.3}$$

(1.11) onto the tensor bundle $\overset{2}{\vee} TX$. It is a generator of general covariant transformations of $\overset{2}{\vee} TX$. Combining (4.1) and (4.3), we obtain the lift

$$\begin{aligned}\widetilde{\tau}_Y = \widetilde{\tau}_1 + \widetilde{\tau}_2 = {} & [\tau^\lambda \partial_\lambda + (\partial_\nu \tau^\alpha \sigma^{\nu\beta} + \partial_\nu \tau^\beta \sigma^{\nu\alpha}) \partial_{\alpha\beta} - a^r_\lambda \partial_\mu \tau^\lambda \partial^\mu_r] + \\ & [\partial_\mu(\tau^\lambda B^r_\lambda) + c^r_{qp} a^q_\mu (\tau^\lambda B^p_\lambda)] \partial^\mu_r\end{aligned} \tag{4.4}$$

of a vector field τ on X onto the product Y. Note that the decomposition (4.4) of the lift $\widetilde{\tau}_Y$ is local. One can think of the summands $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$ as being local generators of general covariant transformations (cf. (1.11)) and vertical gauge transformations (cf. (3.10)) of the product Y, respectively.

Let a Lagrangian L on the total configuration space (4.2) be invariant under general covariant transformations and vertical gauge transformations. Hence, its Lie derivative along the vector field $\widetilde{\tau}_Y$ (4.4) equals zero. Then using the formula (2.21) on the shell

$$\sigma^{\mu\nu} = g^{\mu\nu}(x), \qquad \delta^\mu_r \mathcal{L} = 0,$$

we obtain the weak identity

$$0 \approx (\partial_\nu \tau^\alpha g^{\nu\beta} + \partial_\nu \tau^\beta g^{\nu\alpha} - \partial_\lambda g^{\alpha\beta}\tau^\lambda)\partial_{\alpha\beta}\mathcal{L} - d_\lambda \mathfrak{T}^\lambda_B, \tag{4.5}$$

where

$$\mathfrak{T}^\lambda_B = [\partial^{\lambda\nu}_r \mathcal{L}(\tau^\mu a^r_{\mu\nu} + \partial_\nu \tau^\mu a^r_\mu) - \tau^\lambda \mathcal{L}] + [-\partial^{\lambda\nu}_r \mathcal{L}(\partial_\nu(\tau^\mu B^r_\mu) + c^r_{qp} a^q_\nu (\tau^\mu B^p_\mu))] \tag{4.6}$$

is the energy-momentum current along the vector field (4.1). The weak identity (4.5) takes the form

$$0 \approx \partial_\lambda \tau^\mu t^\lambda_\mu \sqrt{\mid g \mid} - \tau^\mu \{_\mu{}^\beta{}_\lambda\} t^\lambda_\beta \sqrt{\mid g \mid} - d_\lambda \mathfrak{T}^\lambda_B, \tag{4.7}$$

where t^λ_μ is the metric energy-momentum tensor (1.15) and $\{_\mu{}^\beta{}_\lambda\}$ are the Christoffel symbols of g. Accordingly, the current $\mathfrak{T}^\lambda_B$ (4.6) is brought into the form

$$\mathfrak{T}^\lambda_B = \tau^\mu t^\lambda_\mu \sqrt{\mid g \mid} + \tau^\alpha (B^r_\alpha - a^r_\alpha)\delta^\lambda_r \mathcal{L} + d_\mu(\tau^\alpha (B^r_\alpha - a^r_\alpha)\partial^{\lambda\mu}_r \mathcal{L}). \tag{4.8}$$

Substituting $\mathfrak{T}^\lambda_B$ (4.8) into the weak identity (4.7), we obtain the covariant conservation law (1.14) independent of the choice of the connection B in the lift (4.1).

5. The case of broken gauge invariance

On a local coordinate chart, the conservation law (1.14) issues directly from the local decomposition (4.4). Namely, the current $\mathfrak{T}_B$ is decomposed locally into the sum $\mathfrak{T}_{\widetilde{\tau}_1} + \mathfrak{T}_{\widetilde{\tau}_2}$ of the energy-momentum current $\mathfrak{T}_{\widetilde{\tau}_1}$ along the the projectible vector field $\widetilde{\tau}_1$ and the Noether current $\mathfrak{T}_{\widetilde{\tau}_2}$ along the vertical vector field $\widetilde{\tau}_2$. Since the Noether current $\mathfrak{T}_{\widetilde{\tau}_2}$ is reduced to a superpotential, it does not contribute to the energy-momentum conservation law (1.14) if a Lagrangian L is invariant under vertical gauge transformations. However, if L is not gauge-invariant, the conservation law (1.14) takes the local form

$$\mathbf{L}_{J^1\widetilde{\tau}_2} L = \tau^\mu \nabla_\lambda (t^\lambda_\mu \sqrt{\mid g \mid}) \tag{5.1}$$

on each coordinate chart. Of course, one can choose $B = 0$ and restart the conservation law (1.14) on a given coordinate chart without fail. However, if P is a non-trivial principal bundle, no principal connection on P vanishes everywhere. In this case, no energy-momentum of gauge fields is conserved.

Turn now to the above mentioned example of the Chern–Simons Lagrangian [2, 7].

Let $P \to X^3$ be a principal bundle over a 3-dimensional manifold X whose structure group G is a semisimple Lie group. The Chern–Simons Lagrangian on the configuration space J^1C of principal connections on P reads

$$L_{\rm CS} = \frac{1}{2k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} a^m_\alpha (\mathcal{F}^n_{\lambda\mu} - \frac{1}{3} c^n_{pq} a^p_\lambda a^q_\mu)\omega, \tag{5.2}$$

where $\varepsilon^{\alpha\lambda\mu}$ is the skew-symmetric Levi–Civita tensor and k is a coupling constant. In comparison with the Yang–Mills Lagrangian (3.13), the Chern–Simons Lagrangian (5.2) is independent of any metric on X and is not gauge-invariant. At the same time, the Euler–Lagrange operator

$$\mathcal{E}_{L_{\rm CS}} = \frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \mathcal{F}^n_{\lambda\mu} \theta^m_\alpha \wedge \omega. \tag{5.3}$$

is gauge-invariant. Therefore, let us follow Remark 2.5 in order to study Lagrangian conservation laws in the Chern–Simons model.

Given a generator ξ_C (3.10) of vertical gauge transformations, we obtain

$$\mathbf{L}_{J^1\xi_C} L_{\rm CS} = \frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \partial_\alpha \xi^m a^n_{\lambda\mu} \omega. \tag{5.4}$$

Since $C \to X$ is an affine bundle, the Lie derivative (5.4) is brought into the form

$$\mathbf{L}_{J^1\xi_C} L_{\mathrm{CS}} = d_H \phi,$$

where

$$\phi = \frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \partial_\alpha \xi^m a^n_\mu \omega_\lambda$$

is a horizontal 2-form on $C \to X$. Then we obtain the weak conservation law (2.16) where

$$\mathfrak{T}^\lambda = -\frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \xi_C{}^n_\mu a^m_\alpha$$

is the Nöther current. Moreover, this conservation law takes the superpotential form

$$0 \approx d_\lambda(\xi^\lambda \mathcal{L}_{\mathrm{CS}} + d_\mu U^{\mu\lambda}), \qquad U^{\mu\lambda} = \frac{2}{k} a^G_{mn} \varepsilon^{\alpha\mu\lambda} \xi^n a^m_\alpha.$$

Turn now to the energy-momentum conservation law in the Chern–Simons model. Let τ be a vector field on the base X and $\widetilde{\tau}_B$ (4.1) its lift onto the connection bundle C by means of a principal connection B. We obtain

$$\mathbf{L}_{J^1\widetilde{\tau}_B} L_{\mathrm{CS}} = \frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \partial_\alpha (\tau^\nu B^m_\nu) a^n_{\lambda\mu} \omega.$$

Then the corresponding conservation law (2.16) takes the form

$$0 \approx -d_\lambda[\mathfrak{T}^\lambda_B + \frac{1}{k} a^G_{mn} \varepsilon^{\alpha\lambda\mu} \partial_\alpha (\tau^\nu B^m_\nu) a^n_\mu],$$

where $\mathfrak{T}_B$ is the energy-momentum current (4.6) along the vector field $\widetilde{\tau}_B$. It follows that the energy-momentum current of the Chern–Simons model is not conserved because the Lagrangian (5.2) is not gauge-invariant, but there exists another conserved quantity.

Another example of non-invariant Lagrangians is a Lagrangian of gauge fields in the presence of a background field (see Remark 2.6). Let us focus on the physically relevant case of gauge theory with spontaneous symmetry breaking. It is gauge theory on a principal bundle $P \to X$ whose structure group G is reduced to its closed subgroup H, i.e., there exists a principal subbundle P^σ of P with the structure group H. Moreover, by the well-known theorem [7, 10], there is one-to-one correspondence between the H-principal subbundles P^σ of P and the global sections σ of the quotient bundle $P/H \to X$. These sections are called Higgs fields. The total Lagrangian L of gauge potentials and Higgs fields on the configuration space $J^1(C \times P/H)$ is gauge invariant. Therefore, we can appeal to Remark 2.6 in order to obtain the energy-momentum conservation law of gauge potentials in the presence of a background Higgs field σ. The key point is that, due to the equivariance condition (3.7), any principal connection on the reduced bundle $P^\sigma \to X$ gives rise to a principal connection A_σ on $P \to X$ whose integral section is the Higgs field σ. Let us consider the lift

$$\widetilde{\tau} = \tau + u_1 + u_2 \tag{5.5}$$

onto $C \times P/H$ of a vector field τ on X such that: $\tau + u_1$ is the lift $\widetilde{\tau}_{A_\sigma}$ (4.1) of τ onto C, and $\tau + u_2$ is the horizontal lift of τ onto P/H by means of the connection A_σ. Since the Lagrangian L is gauge-invariant, its Lie derivative along the vector field $\widetilde{\tau}$ (5.5) vanishes. Therefore, we come to the weak identity (2.21) whose left-hand side also vanishes, and we obtain again the energy-momentum conservation law (1.14).

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