# ‘Alice’ String as Source of the Kerr Spinning Particle

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Kerr geometry has twofoldedness which can be cured by a truncation of the ‘negative’ sheet of metric. It leads to the models of disk-like sources of the Kerr solution and to a class of disk-like or bag-like models of the Kerr spinning particle. There is an alternative way: to retain the ‘negative’ sheet as the sheet of advanced fields. In this case the source of spinning particle is the Kerr singular ring which can be considered as a twofold ‘Alice’ string. This string can have electromagnetic excitations in the form of traveling waves generating spin and mass of the particle. Model of this sort was suggested in 1974 as a ‘microgeon with spin’. Recent progress in the obtaining of the nonstationary and radiating Kerr solutions enforces us to return to this model and to consider it as a model for the light spinning particles. We discuss here the real and complex Kerr geometry and some unusual properties of the oscillating solutions in the model of ‘Alice’ string source.

## 1. Introduction

It is known, that Kerr geometry displays some properties of spinning particle and diverse classical models of spinning particle were considered on the base of Kerr geometry [1, 2, 3, 4, 5]. Since the Kerr geometry has a topological twofoldedness of space-time, there appears an alternative: either to remove this twofoldedness or to give it a physical interpretation. The both approaches have paid attention and were considered in literature, it seems however that the final preference for the solution of this problem is not formed yet. It is also possible that the both versions of the Kerr source can be valid and applicable for different models. The approach which was the most popular last time is the truncation of the negative sheet of the Kerr geometry. It leads to the appearance of a (relativistically rotating) disk-like source of the Kerr solution [2] and to a class of the disk-like [4] or bag-like [7, 8] models of the Kerr spinning particle.

Alternative way is to retain the negative sheet treating it as the sheet of advanced fields. In this case the source of spinning particle turns out to be the Kerr singular ring and its electromagnetic excitations in the form of traveling waves generating spin and mass of the particle. Model of this sort can be considered as a twofold ‘Alice’ string and was suggested in 1974 as a model of ‘microgeon with spin’[3]. Recent progress in the obtaining of the nonstationary and radiating Kerr solutions [6] enforced us to return to this model and to consider it as a plausible classical model for the light spinning particles.

In this paper we discuss the real and complex Kerr structures of the Kerr geometry and a way for obtaining the exact solutions for microgeon. We display also the CPT-invariance of Kerr geometry and consider the physical interpretation of the negative sheet leading to some unusual properties of the oscillating solutions of this kind.

In this paper we use the Kerr-Schild approach to the Kerr geometry which is based on the Kerr-Schild form of metrics $g^{\mu\nu} = \eta^{\mu\nu} - 2hk^\mu k^\nu$, where $\eta^{\mu\nu}$ is the auxiliary Minkowski background. It allows to give an exact meaning to the complex representation of the Kerr geometry and simplifies the treatment of the complex null cones that is necessary to consider the complex retarded-time construction for oscillating solutions.

## 2. Main peculiarities of the real Kerr geometry

**The Kerr singular ring** is one of the most remarkable peculiarities of the Kerr solution. It is a branch line of space on two sheets: "negative" and "positive" where the fields change their signs and directions. **The Kerr twisting PNC** is the second remarkable structure of the Kerr geometry. It is described by a vector field $k^\mu$ which determines the Kerr-Schild ansatz for metric

$$g_{\mu\nu} = \eta_{\mu\nu} + 2hk_\mu k_\nu, \tag{1}$$

where $\eta_{\mu\nu}$ is metric of auxiliary Minkowski space-time and

$$h = \frac{mr - e^2/2}{r^2 + a^2\cos^2\theta}. \tag{2}$$

This is a remarkable simple form showing that all the complicatedness of the Kerr solution is included in the form of the field $k_\mu(x)$ which is tangent to the Kerr PNC. This form shows also that metric is singular at $r = \cos\theta = 0$, that are the focal points of the oblate spheroidal coordinate system.

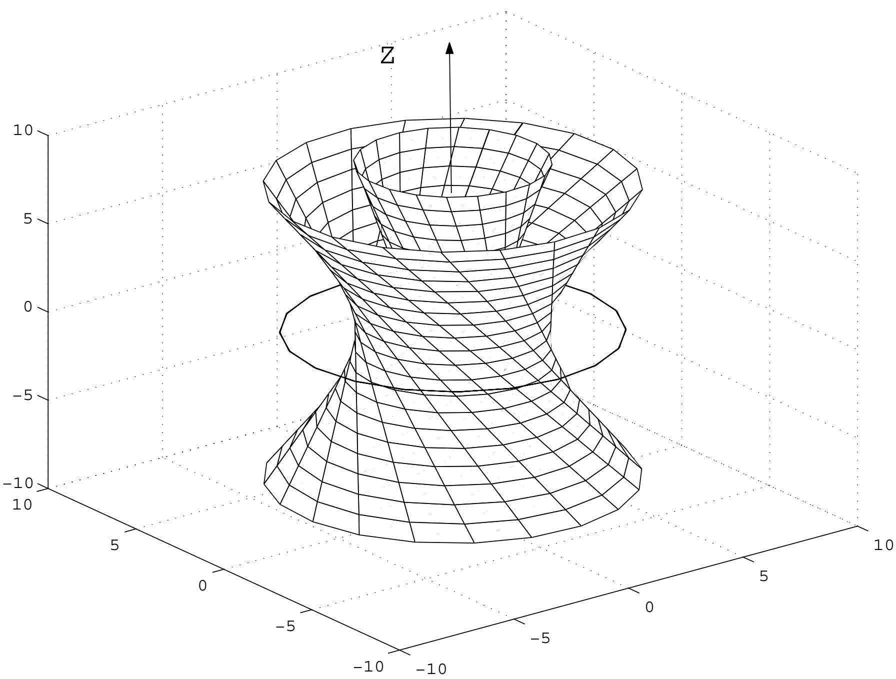


Figure 1: *The Kerr singular ring and 3-D section of the Kerr principal null congruence (PNC). Singular ring is a branch line of space, and PNC propagates from "negative" sheet of the Kerr space to "positive " one, covering the space-time twice.*

Field $k^\mu$ is null with respect to $\eta_{\mu\nu}$ as well as with respect to the metric $g_{\mu\nu}$. The Kerr singular ring and a part of the Kerr PNC are shown on the Fig. 1. The Kerr PNC consists of the linear generators of the surfaces $\theta = const.$ The shown on the fig.1 region $z < 0$ corresponds to a "negative" sheet of space ($r < 0$) where we set the null rays to be "in"-going.

Twisting vortex of the null rays propagates through the singular ring $r = \cos\theta = 0$ and get "out" on the "positive" sheet of space ($z > 0$). Indeed, the Kerr congruence covers the space-time twice, and this picture shows only the half of PNC corresponding to $0 > \theta > \pi/2$. It has to be completed by the part for $\pi/2 > \theta > \pi$ which is described by another system of the linear generators (having opposite twist). Two PNC directions for the each point $x^\mu \in M^4$ correspond to the known twofoldedness of the Kerr geometry and to the algebraically degenerated metrics of type D.

As it is explicitly seen from the expression for $h$, the Kerr gravitational field has twovaluedness, $h(r) \neq h(-r)$, and so do also the other fields on the Kerr background. The oblate coordinate system turns out to be very useful since it covers also the space twice, for $r > 0$ and $r < 0$ with the branch line on the Kerr singular ring.

## 3. Kerr singular ring as the 'Alice' string

In the case $e^2 + a^2 >> m^2$, corresponding to parameters of elementary particles, the horizons of the Kerr-Newman solution disappear and the Kerr singular ring turns out to be naked. The naked Kerr singular ring was considered in the model of spinning particle - microgeon [3] - as a waveguide providing a circular propagation of an electromagnetic or fermionic wave excitation. Twofoldedness of the Kerr geometry admits integer and half integer wave excitations with $n = 2\pi a/\lambda$ wave periods on the Kerr ring of radius $a$. It is consistent with the corresponding values of the angular momentum $J = n\hbar$ and mass $m$ in accordance with the main relation for the Kerr parameters $m = J/a$. Radius of the Kerr ring $a = n\hbar/m$ turns out to be of the order of the corresponding compton size. It was recognized soon [5] that this construction can be considered as a closed string with a spin excitations. This proposal was natural since singular lines were used in many physical models of the dual relativistic strings. The most well known example of this kind is the Nielsen-Olesen model representing a vortex line in superconductor, another example is the Witten superconducting cosmic string.

The Kerr ring displays a few stringy properties. First, the compton size of Kerr ring $a$ is a typical size of the relativistic string excitations. Because of that the radius of interaction of the Kerr spinning particle is not determined by mass parameter $m$, as it has the place for other gravitational solutions, but it is extended to the compton distances $\sim a$. In the same time the contact stringy character of interaction is provided with a very small effective cross-section. If we assume the existence of a stringy tension $T$, so that $E = m = 2\pi Ta$, then, in the combination with the Kerr relation $J = ma$, one obtains the Regge relation $J = (2\pi T)^{-1} m^2$. Finally, as it was shown in [9] by the analysis of the axidilatonic generalization of the Kerr solution [10], the field near the Kerr singular ring of this solution is similar to the field around a heterotic string. Note also, that the Kerr ring is a relativistic light-like object, that can be seen from the analysis of the Kerr null congruence near the ring.

The light-like rays of the Kerr congruence are tangent to the ring, as it is shown on the Fig. 2. In stringy terms the Kerr string contains only modes of one (say 'left') direction. However, the equations for the usual bosonic closed strings do not admit such solutions in four dimensions. There are a few ways to avoid this obstacle. One of them is to assume that the model contains the right modes too, but they are moving in the fifth compactified direction. Another way is to consider the traveling waves as the missing 'right' excitations. In this connection one should mention one more stringy structure of the Kerr geometry, complex euclidean string [12], which is related to its complex representation.

Complex representation of Kerr geometry, has been found useful in diverse problems [12, 13, 14]. In the Kerr-Schild approach it allows one to get a retarded-time description of the nonstationary Maxwell fields and twisting algebraically special solutions of the Einstein equations [6]. Twisting solutions are represented in this approach as the retarded-time fields which are similar to Lienard-Wiechard fields, however they are generated by a *complex* source moving along a *complex world line* $x_0(\tau)$ in complex Minkowski space-time $CM^4$.

The objects described by the complex world lines occupy an intermediate position between particle and string. Like the string they form the two-dimensional surfaces or the world-sheets in

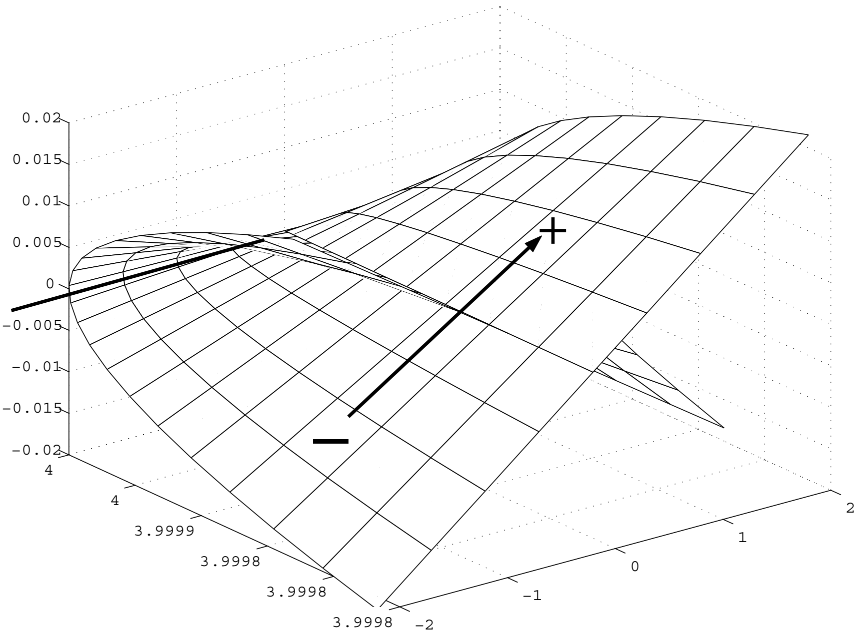


Figure 2: *The surface $\phi$ = const. formed by the light-like generators of the Kerr principal null congruence (PNC). The Kerr string (fat line) is tangent to this surface.*

the space-time [12, 26]. In many respects this source is similar to the "mysterious" $N = 2$ complex string of superstring theory [26].

It was shown [12], that analytical complex world lines are the solutions of the corresponding string equations. Below we shall show that a given complex world line can control excitations of the Kerr singular ring. Based on the recent progress in the obtaining nonstationary Kerr solutions [6], we shall try to get selfconsistent solutions for traveling waives accompanied by oscillating singular ring of the Kerr geometry.

## 4. Complex structure of Kerr geometry and related retarded-time construction

The light cones emanating from the word line of a source play usually a central role in the retarded-time constructions where the fields are defined by the values of a retarded time. In the case of complex world line, the corresponding light cone has to be complex that complicates the retarded-time scheme.

### 4.1. Appel complex solution

There exist the Newton and Coulomb analogues of the Kerr solution possessing the Kerr singular ring. It allows one understand the origin of this ring as well as the complex origin of the Kerr source. The corresponding Coulomb solution was obtained by Appel still in 1887 (!) by a method of complex shift [25].

A point-like charge $e$, placed on the complex z-axis $(x_0, y_0, z_0) = (0, 0, ia)$ gives a real Appel potential

$$\phi_a = Re \ e/\tilde{r}. \tag{3}$$

Here $\tilde{r}$ is in fact the Kerr complex radial coordinate $\tilde{r} = PZ^{-1} = r + ia\cos\theta$, where $r$ and $\theta$ are the oblate spheroidal coordinates. It may be expressed in the usual rectangular Cartesian coordinates $x, y, z, t$ as

$$\tilde{r} = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{1/2} = [x^2+y^2+(z-ia)^2]^{1/2}. \qquad (4)$$

Singular line of the solution corresponds to $r = \cos\theta = 0$, and it is seen that the Appel potential $\phi_a$ is singular at the ring $z = 0, \quad x^2+y^2 = a^2$. It was shown, that this ring is a branch line of space-time for two sheets similarly to the properties of the Kerr singular ring. Appel potential describes **exactly** the e.m. field of the Kerr-Newman solution [3].

As far the Appel source is shifted to a complex point of space $(x_o, y_o, z_o) \to (0, 0, ia)$, it can be considered as a mysterious "particle" propagating along a *complex world-line* $x_0^\mu(\tau)$ in $CM^4$ and parametrized by a complex time $\tau$. Complex source of the Kerr-Newman solution has just the same origin [11, 12] and leads to a complex retarded-time construction for Kerr geometry.

The appearance of the twisting Kerr congruence may be understood as a track of the null planes of the family of complex light cones emanated from the points of the complex world line $x_0^\mu(\tau)$ [20, 12] in the retarded-time construction. It is very instructive to consider the following splitting of the complex light cones.

### 4.2. Splitting of the complex light cone

The complex light cone $\mathcal{K}$ with the vertex at some point $x_0$, written in spinor form

$$\mathcal{K} = \{x : x = x_o^\mu(\tau) + \psi_L^A \sigma^\mu_{A\dot{A}} \tilde{\psi}_R^{\dot{A}}\} \qquad (5)$$

may be split into two families of null planes: "left" ($\psi_L$ =const; $\tilde{\psi}_R$ -var.) and "right"($\tilde{\psi}_R$ =const; $\psi_L$ -var.). These are the only two-dimensional planes which are wholly contained in the complex null cone. The rays of the principal null congruence of the Kerr geometry are the tracks of these complex null planes (right or left) on the real slice of Minkowski space.

The light cone equation in the Kerr-Schild metric coincides with the corresponding equation in Minkowski space because the null directions $k^\mu$ are null in both metrics, $g_{\mu\nu}$ and $\eta_{\mu\nu}$.

In the null Cartesian coordinates

$$\begin{aligned} 2^{\frac{1}{2}}\zeta &= x+iy, \qquad 2^{\frac{1}{2}}\bar{\zeta} = x-iy, \\ 2^{\frac{1}{2}}u &= z+t, \qquad 2^{\frac{1}{2}}v = z-t. \end{aligned} \qquad (6)$$

the light cone equation has the form $\zeta\bar{\zeta} + uv = 0$. As usually, in a complex extension to $CM^4$ the coordinates $u, v$ have to be considered as complex and coordinates $\zeta$ and $\bar{\zeta}$ as independent. On the real section, in $M^4$, coordinates $u$ and $v$ take the real values and $\zeta$ and $\bar{\zeta}$ are complex conjugate.

The known splitting of the light cone on the complex null planes has a close connection to spinors and twistors. By introducing the projective spinor parameter $Y = \psi^1/\psi^0$ the equation of complex light cone with the vertex at point $x_0$,

$$(\zeta-\zeta_0)(\bar{\zeta}-\bar{\zeta}_0) = -(u-u_0)(v-v_0), \qquad (7)$$

splits into two linear equations [1]

$$\zeta - \zeta_0 = Y(v-v_0), \qquad (8)$$
$$-Y(\bar{\zeta}-\bar{\zeta}_0) = (u-u_0)\ , \qquad (9)$$

describing the "left" complex null planes (the null rays in the real space). Another splitting

$$-\tilde{Y}(\zeta-\zeta_0) = (u-u_0), \qquad (10)$$
$$(\bar{\zeta}-\bar{\zeta}_0) = \tilde{Y}(v-v_0)\ , \qquad (11)$$

gives the "right" complex null planes.

[1] It is a generalization of the Veblen and Ruse construction [27, 28] which has been used for the geometrical representation of spinors.

Thus, the equations of the "left" null planes (9) can be written in terms of the three parameters

$$Y, \qquad \lambda_1 = \zeta - Yv, \qquad \lambda_2 = u + Y\bar{\zeta}, \tag{12}$$

as follows

$$\lambda_1 = \lambda_1^0, \quad \lambda_2 = \lambda_2^0 \ , \tag{13}$$

where

$$\lambda_1^0 = \zeta_0 - Yv_0, \qquad \lambda_2^0 = u_0 + Y\bar{\zeta}_0 \tag{14}$$

note the values of these parameters at the point $x_0$. These three parameters are the projective twistor variables and very important for further consideration since the Kerr theorem is formulated in terms of these parameters. The above splitting of the complex light cone equation shows explicitly their origin. Note also that in the terms of the Kerr-Schild null tetrad

$$\begin{aligned} e^1 &= d\zeta - Y dv, \qquad e^2 = d\bar{\zeta} - \bar{Y} dv, \\ e^3 &= du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y\bar{Y} dv, \\ e^4 &= dv + he^3, \end{aligned} \tag{15}$$

the projective twistor parameters take the form

$$\begin{aligned} \lambda_1 &= x^\mu e^1_\mu, \\ \lambda_2 &= x^\mu (e^3_\mu - \bar{Y} e^1_\mu), \end{aligned} \tag{16}$$

and correspondingly

$$\begin{aligned} \lambda_1^0 &= x_0^\mu e^1_\mu, \\ \lambda_2^0 &= x_0^\mu (e^3_\mu - \bar{Y} e^1_\mu). \end{aligned} \tag{17}$$

The "left" complex null planes of the complex light cone at some point $x_0$ can be expressed in terms of the tetrad as follows

$$x_L = x_0(\tau) + \alpha e^1 + \beta e^3 \ , \tag{18}$$

and the null plane equations (13) follow then from (18) and the tetrad scalar products $e^{1\mu} e^1_\mu = e^{1\mu} e^3_\mu = e^{3\mu} e^3_\mu = 0$. Similar relations valid also for the "right" null planes with the replacement $e^1 \to e^2$.

The "left" null planes of the complex light cones form a complex Kerr congruence which generates all the rays of the principal null congruence on the real space. The ray with polar direction $\theta, \phi$ is the real track of the "left" plane corresponding to $Y = \exp i\phi \tan(\theta/2)$ and belonging to the cone which is placed at the point $x_0$ corresponding to $\sigma = a\cos(\theta)$. The parameter $\sigma = Im\tau$ has a meaning only in the range $-a \le \sigma \le a$ where the cones have real slices. Thus, the complex world line $x_0(t, \sigma)$ represents a restricted two-dimensional surface or strip, in complex Minkowski space, and is really a world-sheet [2].

The Kerr congruence arises as the real slice of the family of the "left" null planes ($Y = const.$) of the complex light cones which vertices lie on the complex world line $x_0(\tau)$.

The Kerr theorem can be linked to this retarded-time construction.

[2] It may be considered as a complex open string with a Euclidean parametrization $\tau = t + i\sigma, \bar{\tau} = t - i\sigma$, and with end points $x_0(t, \pm a)$ [12, 26].

## 5. Kerr theorem and the retarded-time construction

### 5.1. The Kerr theorem

Traditional formulation of the Kerr theorem is following.

Any geodesic and shear-free null congruence in Minkowski space is defined by a function $Y(x)$ which is a solution of the equation

$$F = 0, \tag{19}$$

where $F(\lambda_1, \lambda_2, Y)$ is an arbitrary analytic function of the projective twistor coordinates

$$Y, \qquad \lambda_1 = \zeta - Yv, \qquad \lambda_2 = u + Y\bar{\zeta}. \tag{20}$$

The vector field

$$e^3 = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv = Pk_\mu dx^\mu \tag{21}$$

determines the congruence then in the null cartesian coordinates $u,\ v,\ \bar{\zeta},\ \zeta$ [3].

In the Kerr-Schild backgrounds the Kerr theorem acquires a more wide contents [17, 18, 20]. It allows one to obtain the position of singular lines, caustics of the PNC, as a solution of the system of equations

$$F = 0; \quad \partial F/\partial_Y = 0\ , \tag{22}$$

and to determine the important parameters of the corresponding solutions:

$$\tilde{r} = - \quad dF/dY, \tag{23}$$

and

$$P = \partial_{\lambda_1} F - \bar{Y}\partial_{\lambda_2} F. \tag{24}$$

Parameter $\tilde{r}$ characterizes a complex radial distance, and for the stationary Kerr solution it is a typical complex combination $\tilde{r} = r + ia\cos\theta$. Parameter $P$ is connected with the boost of source. For details we refer reader to [6].

Working in $CM^4$ one has to consider $Y$ and $\bar{Y}$ functionally independent, as well as the null coordinates $\zeta$ and $\bar{\zeta}$. Coordinates $u, v$ and congruence turn out to be complex. The corresponding complex null tetrad (82) may be considered as a basis of $CM^4$. The Kerr theorem determines in this case only the "left" complex structure - function $F(Y)$. The real congruence appears as an intersection with a complex conjugate "right" structure.

### 5.2. Quadratic function F(Y) and interpretation of parameters

It is instructive to consider first stationary case. Stationary congruences having Kerr-like singularities contained in a bounded region have been considered in papers [5, 19, 24]. It was shown that in this case function $F$ must be at most quadratic in $Y$,

$$F \equiv a_0 + a_1 Y + a_2 Y^2 + (qY + c)\lambda_1 - (pY + \bar{q})\lambda_2, \tag{25}$$

where coefficients $c$ and $p$ are real constants and $a_0, a_1, a_2, q, \bar{q}$, are complex constants. Killing vector of the solution is determined as

$$\hat{K} = c\partial_u + \bar{q}\partial_\zeta + q\partial_{\bar{\zeta}} - p\partial_v. \tag{26}$$

Writing the function F in the form

$$F = AY^2 + BY + C, \tag{27}$$

[3] The field $k_\mu$ is a normalized form of $e^3_\mu$ with $k_\mu \Re e \dot{x}^\mu_0 = 1$.

one can find two solutions of the equation $F=0$ for the function $Y(x)$

$$Y_{1,2} = (-B \pm \Delta)/2A, \tag{28}$$

where $\Delta = (B^2 - 4AC)^{1/2}$.

On the other hand from (23)

$$\tilde{r} = -\partial F/\partial Y = -2AY - B, \tag{29}$$

and consequently

$$\tilde{r} = PZ^{-1} = \mp\Delta. \tag{30}$$

Two roots reflect the known twofoldedness of the Kerr geometry. They correspond to two different directions of congruences on positive and negative sheets of the Kerr space-time. The expression (24) yields

$$P = pY\bar{Y} + \bar{q}\bar{Y} + qY + c\ . \tag{31}$$

### 5.3. Link to the complex world line of source

The stationary and boosted Kerr geometries are described by a straight complex world line with a real 3-velocity $\vec{v}$ in $CM^4$:

$$x_0^\mu(\tau) = x_0^\mu(0) + \xi^\mu \tau; \qquad \xi^\mu = (1, \vec{v})\ . \tag{32}$$

The gauge of the complex parameter $\tau$ is chosen in such a way that $Re\ \tau$ corresponds to the real time $t$.

The quadratic in $Y$ function $F$ can be expressed in this case in the form [5, 19, 20, 14]

$$F \equiv (\lambda_1 - \lambda_1^0)\hat{K}\lambda_2 - (\lambda_2 - \lambda_2^0)\hat{K}\lambda_1\ , \tag{33}$$

where the twistor components $\lambda_1,\ \lambda_2$ with zero indices denote their values on the points of the complex world-line $x_0(\tau)$, (14), and $\hat{K}$ is a Killing vector of the solution

$$\hat{K} = \partial_\tau x_0^\mu(\tau)\partial_\mu = \xi^\mu \partial_\mu\ . \tag{34}$$

Application $\hat{K}$ to $\lambda_1$ and $\lambda_2$ yields the expressions

$$\begin{aligned} \hat{K}\lambda_1 &= \partial_\tau x_0^\mu(\tau) e_\mu^1, \\ \hat{K}\lambda_2 &= \partial_\tau x_0^\mu (e_\mu^3 - \bar{Y} e_\mu^1). \end{aligned} \tag{35}$$

From (24) one obtains in this case

$$P = \hat{K}\rho = \partial_\tau x_0^\mu(\tau) e_\mu^3\ , \tag{36}$$

where

$$\rho = \lambda_2 + \bar{Y}\lambda_1 = x^\mu e_\mu^3, \tag{37}$$

Comparing (36) and (31) one obtains the correspondence in terms of $p,\ c,\ q,\ \bar{q}$,

$$\hat{K}\lambda_1 = pY + \bar{q}, \qquad \hat{K}\lambda_2 = qY + c, \tag{38}$$

that allows one to set the relation between parameters $p, c, q, \bar{q}$ and $\xi^\mu$ showing that these parameters are connected with the boost of the source.

The complex initial position of complex world line $x_0^\mu(0)$ in (32) gives six more parameters to solution, which are connected with coefficients $a_0,\ a_1\ a_2$ . It can be decomposed as $\vec{x}_0(0) = \vec{c} + i\vec{d}$, where $\vec{c}$ and $\vec{d}$ are real 3-vectors with respect to the space O(3)-rotation. The real part, $\vec{c}$, defines the initial position of source, and the imaginary part, $\vec{d}$, defines the value and direction of angular momentum (or the size and orientation of singular ring).

It can be easily shown that in the rest frame, when $\vec{v}=0$, $\quad \vec{d} = \vec{d_0}$, the singular ring lies in the plane orthogonal to $\vec{d}$ and has a radius $a = |\vec{d_0}|$. The corresponding angular momentum is $\vec{J} = m\vec{d_0}$.

### 5.4. L-projection and complex retarded-time parameter

In the form (25) all the coefficients are constant while the form (33) has an extra explicit linear dependence on $\tau$ via terms $\lambda_1^0(x_0(\tau))$ and $\lambda_2^0(x_0(0))$. However, this dependence is really absent. As consequence of the relations $\lambda_1^0(x_0(\tau)) = \lambda_1^0(x_0(0)) + \tau\hat{K}\lambda_1, \quad \lambda_2^0(x_0(\tau)) = \lambda_2^0(x_0(0)) + \tau\hat{K}\lambda_2$ the terms proportional to $\tau$ cancels and these forms are equivalent.

Parameter $\tau$ may be defined for each point $x$ of the Kerr space-time and plays the role of a complex retarded time parameter. Its value for a given point $x$ may be defined by L-projection, using the solution $Y(x)$ and forming the twistor parameters $\lambda_1, \quad \lambda_2$ which fix a left null plane.

$L$-projection of the point $x$ on the complex world line $x_0(\tau)$ is determined by the condition

$$(\lambda_1 - \lambda_1^0)|_L = 0, \qquad (\lambda_2 - \lambda_2^0)|_L = 0 \ , \tag{39}$$

where the sign $|_L$ means that the points $x$ and $x_0(\tau)$ are synchronized by the left null plane (18),

$$x - x_0(\tau_L) = \alpha e^1 + \beta e^3. \tag{40}$$

The condition (39) in representation (17) has the form

$$(x^\mu - x_0^\mu)e_\mu^1|_L = 0, \qquad (x^\mu - x_0^\mu)(e_\mu^3 - \bar{Y}e_\mu^1)|_L = 0, \tag{41}$$

which shows that the points $x^\mu$ and $x_0^\mu$ are connected by the left null plane spanned by null vectors $e^1$ and $e^3$.

This left null plane belongs simultaneously to the "in" fold of the light cone connected to the point $x$ and to the "out" fold of the light cone emanating from point of complex world line $x_0$. The point of intersection of this plane with the complex world-line $x_0(\tau)$ gives a value of the "left" retarded time $\tau_L$, which is in fact a complex scalar function on the (complex) space-time $\tau_L(x)$.

By using the null plane equation (39) one can express $\Delta$ of (30) in the form

$$\Delta|_L = (u-u_0)\dot{v}_0 + (\zeta-\zeta_0)\dot{\bar{\zeta}}_0 + (\bar{\zeta}-\bar{\zeta}_0)\dot{\zeta}_0 + (v-v_0)\dot{u}_0 = \frac{1}{2}\partial_\tau(x-x_0)^2 = \tau_L - t + \vec{v}\vec{R}, \tag{42}$$

where

$$\vec{v} = \dot{\vec{x}}_0, \qquad \vec{R} = \vec{x} - \vec{x}_0. \tag{43}$$

It gives a retarded-advanced time equation

$$\tau = t \mp \tilde{r} + \vec{v}\vec{R}, \tag{44}$$

and a simple expression for the solutions $Y(x)$

$$Y_1 = [(u-u_0)\dot{v}_0 + (\zeta-\zeta_0)\dot{\bar{\zeta}}_0]/[(v-v_0)\dot{\bar{\zeta}}_0 - (\bar{\zeta}-\bar{\zeta}_0)\dot{v}_0], \tag{45}$$

and

$$Y_2 = [(u-u_0)\dot{\zeta}_0 - (\zeta-\zeta_0)\dot{u}_0]/[(u-u_0)\dot{v}_0 + (\zeta-\zeta_0)\dot{\bar{\zeta}}_0]. \tag{46}$$

For the stationary Kerr solution $\tilde{r} = r + ia\cos\theta$, and one sees that the second root $Y_2(x)$ corresponds to a transfer to negative sheet of metric: $r \to -r; \quad \vec{R} \to -\vec{R}$ with a simultaneous complex conjugation $ia \to -ia$.

Introducing the corresponding operations:

$$P: r \to -r, \quad \vec{R} \to -\vec{R} \tag{47}$$

$$C: x_0 \to \bar{x}_0, \tag{48}$$

and also the transfer "*out*" $\to$ "*in*"

$$T: t - \tau \to \tau - t. \tag{49}$$

One can see that the roots and corresponding Kerr congruences are CPT-invariant.

## 5.5. Nonstationary case. Real slice

In nonstationary case this construction acquires new peculiarities:

i/ coefficients of function $F$ turn out to be variable and dependent on a retarded-time parameter,

ii/ $\partial_\tau x_0^\mu = \xi^\mu$ can take complex values, that implies complex values for function $P$ and was an obstacle for obtaining the real solutions [20].

iii/ $K$ is not Killing vector more.

To form the real slice of space-time, we have to consider, along with the "left" complex structure, generated by a "left" complex world line $x_0$, parameter $Y$ and by the left null planes, an independent "right" structure with the "right" complex world line $\bar{x}_0$, parameter $\bar{Y}$ and the right null planes, spanned by $e^2$ and $e^3$. These structures can be considered as functionally independent in $CM^4$, but they have to be complex conjugate on the real slice of space-time.

First note, that for a real point of space-time $x$ and for the corresponding real null direction $e^3$, the values of function

$$\rho(x) = x^\mu e^3_\mu(x) \tag{50}$$

are real. Next, one can determine the values of $\rho$ at the points of the left and right complex world lines $x_0^\mu$ and $\bar{x}_0^\mu$ by L- and R-projections

$$\rho_L(x_0) = x_0^\mu e^3_\mu(x)|_L, \tag{51}$$

and

$$\rho_R(\bar{x}_0) = \bar{x}_0^\mu e^3_\mu(x)|_R. \tag{52}$$

For the "right" complex structure, points $x$ and $\bar{x}_0(\bar{\tau})$ are to be synchronized by the right null plane $x - \bar{x}_0(\bar{\tau}_R) = \alpha e^2 + \beta e^3$. As a consequence of the conditions $e^{1\mu} e^3_\mu = e^{3\mu} e^3_\mu = 0$, we obtain

$$\rho_L(x_0) = x_0^\mu e^3_\mu(x)|_L = \rho(x). \tag{53}$$

So far as the parameter $\rho(x)$ is real, parameter $\rho_L(x_0)$ will be real, too. Similarly,

$$\rho_R(\bar{x}_0) = \bar{x}_0^\mu e^3_\mu(x)|_R = \rho(x), \tag{54}$$

and consequently,

$$\rho_L(x_0) = \rho(x) = \rho_R(\bar{x}_0). \tag{55}$$

By using (17) and (50) one obtains

$$\rho = \lambda_2 + \bar{Y}\lambda_1. \tag{56}$$

Since L-projection (39) determines the values of the left retarded-time parameter $\tau_L = (t_0 + i\sigma)|_L$, the real function $\rho$ acquires a dependence on the retarded-time parameter $\tau_L$. It should be noted that the real and imaginary parts of $\tau|_L$ are not independent because of the constraint caused by L-projection.

It means that the real functions $\rho$ and $\rho_0$ turns out to be functions of real retarded-time parameter $t_0 = \Re e\ \tau_L$, while $\lambda_1^0$ and $\lambda_2^0$ can also depend on $\sigma$.

These parameters are constant on the left null planes that yields the relations

$$(\sigma|_L),_2 = (\sigma|_L),_4 = 0, \quad (t_0|_L),_2 = (t_0|_L),_4 = 0\ . \tag{57}$$

In analogue with the above considered stationary case, one can restrict function $F$ by the quadratic in $Y$ expression

$$F \equiv (\lambda_1 - \lambda_1^0)K_2 - (\lambda_2 - \lambda_2^0)K_1, \tag{58}$$

It can be shown [6] that the functions $K_1$ and $K_2$ are linear in $Y$ and depend on the retarded-time $t_0$. It leads to the form (25) which coefficients shall depend on the retarded-time

$$K_1(t_0) = \partial_{t_0}\lambda_1^0, \qquad K_2(t_0) = \partial_{t_0}\lambda_2^0 \; . \tag{59}$$

In tetrad representation (17) it takes the form

$$K_1 = \partial_{t_0} x_0^\mu e^1_\mu, \qquad K_2 = \partial_{t_0} x_0^\mu (e^3_\mu - \bar{Y} e^1_\mu), \tag{60}$$

and

$$P = \bar{Y} K_1 + K_2, \tag{61}$$

that yields for function $P$ the real expression

$$P = \partial_{t_0}(x_0^\mu e^3_\mu)|_L = \partial_{t_0}\rho_L \; . \tag{62}$$

It is seen that $\rho(t_0) = \rho_L(t_0)$ plays the role of a potential for $P$, similarly to some nonstationary solutions presented in [16].

It seems that the extra dependence of function $F$ from the non-analytic retarded-time parameters $t_0$ contradicts to the Kerr theorem, however the non-analytic part disappears by L-projection and analytic dependence on $Y, \quad l_1, \quad l_2$ is reconstructed. Note, that all the *real* retarded-time derivatives on the real space-time are non-analytic and have to involve the conjugate right complex structure. In particular, the expressions (60) acquire the form

$$K_1 = e^1_\mu \Re e \dot{x}_0^\mu, \qquad K_2 = (e^3_\mu - \bar{Y} e^1_\mu) \Re e \dot{x}_0^\mu, \tag{63}$$

where $\dot{x}_0^\mu = \partial_{t_0} x_0^\mu$ .

## 6. Field equations

Field equations for Einstein-Maxwell system in the Kerr-Schild class were obtained in [17]. Electromagnetic field is given by tetrad components of selfdual tensor

$$\mathcal{F}_{12} = AZ^2, \tag{64}$$

$$\mathcal{F}_{31} = \gamma Z - (AZ),_1 \; . \tag{65}$$

The equations for electromagnetic field are

$$A,_2 - 2Z^{-1}\bar{Z}Y,_3 A = 0, \tag{66}$$

$$\mathcal{D}A + \bar{Z}^{-1}\gamma,_2 - Z^{-1}Y,_3 \gamma = 0. \tag{67}$$

Gravitational field equations are

$$M,_2 - 3Z^{-1}\bar{Z}Y,_3 M = A\bar{\gamma}\bar{Z}, \tag{68}$$

$$\mathcal{D}M = \frac{1}{2}\gamma\bar{\gamma}, \tag{69}$$

where

$$\mathcal{D} = \partial_3 - Z^{-1}Y,_3 \partial_1 - \bar{Z}^{-1}\bar{Y},_3 \partial_2 \; . \tag{70}$$

Solutions of this system were given in [17] only for stationary case for $\gamma = 0$. In this paper we give a preliminary analysis of the nonstationary solutions for $\gamma \neq 0$. The principal new point is the existence of retarded-time parameter $t_0$, which satisfies

$$(t_0),_2 = (t_0),_4 = 0 \; . \tag{71}$$

The equation (66) takes the form

$$(AP^2)_{,2} = 0 \ , \tag{72}$$

and has the general solution

$$A = \psi(Y, t_0)/P^2. \tag{73}$$

Action of operator $\mathcal{D}$ on the variables $Y, \bar{Y}$ and $\rho$ is following

$$\mathcal{D}Y = \mathcal{D}\bar{Y} = 0, \qquad \mathcal{D}\rho = 1 \ . \tag{74}$$

From these relations and (62) we have $\mathcal{D}\rho = \partial\rho/\partial t_0 \mathcal{D}t_0 = P\mathcal{D}t_0 = 1$, that yields

$$\mathcal{D}t_0 = P^{-1}. \tag{75}$$

As a result the equation (67) takes the form

$$\dot{A} = -(\gamma P)_{,\bar{Y}} \ , \tag{76}$$

where $\dot{(\ )} \equiv \partial_{t_0}$. Note, that the derivatives $\partial_Y$ and $\mathcal{D}$ commute

$$\mathcal{D}\partial_Y - \partial_Y \mathcal{D} = 0, \tag{77}$$

that allows to obtain by integration

$$\gamma P = -\partial_{t_0}(\psi \ln P + \phi(Y))/P_{\bar{Y}}. \tag{78}$$

The equations (68) and (69) take the simple form

$$m_{,\bar{Y}} = P^3 A \bar{\gamma}, \tag{79}$$

and

$$\frac{1}{P}\dot{M} = \frac{1}{2}\gamma\bar{\gamma}. \tag{80}$$

We consider now the simplest solution corresponding to the case $\dot{A} = \dot{P} = 0$ without oscillations of the ring. In this case $\gamma = \chi(Y, t_0)/P$ that describes a null electromagnetic radiation $\mathcal{F}_{31} = \gamma Z - (AZ)_{,1}$ propagating along the Kerr PNC direction $e^3$. In accordance with (80) it has to lead to a loss of mass by radiation with the stress-energy tensor $\kappa T_{\mu\nu} = \frac{1}{2}\gamma\bar{\gamma}e^3_\mu e^3_\nu$, similarly to the Vaidia "shining star" solution [16, 29]. However, the Kerr twofoldedness shows us that the loss of mass on the positive sheet of metric is really compensated by an opposite process on the negative sheet with an in-flow of radiation. The structure of the Kerr PNC shows that there are no discharges or sources of this flow. One can to assume that the $\gamma$ is really a vacuum zero-point field, or the field of the null vacuum fluctuations, which has a resonance on the Kerr singular ring. The zero-point field has a semi classical nature since it has a quantum origin, but a classical exhibition as the well known Casimir effect. Therefore, this radiation can be interpreted as a specific exhibition of the Casimir effect. To improve the situation with the loss of mass by this field one can use the known recipe for Casimir effect [31]. The classical energy-momentum tensor has to be regularized

$$T^{\mu\nu}_{reg} = :T^{\mu\nu}: \equiv T^{\mu\nu} - <0|T^{\mu\nu}|0> \tag{81}$$

under the condition $\nabla_\mu T^{\mu\nu}_{reg} = 0$. On the classical level of the Einstein-Maxwell equations, this procedure corresponds exactly to the subtraction of the term $\gamma\bar{\gamma}$ from (80).

Since this radiative term $\gamma\bar{\gamma}$ will also appear in all other oscillating solutions, it is tempting to conjecture that on the quantum level this procedure is equivalent to the postulate on the absence of radiation by oscillations.

Preliminary treatment of the other oscillating solutions shows one more peculiarity of this system: the appearance of an imaginary contribution to the mass parameter. Such contributions are not admissible in the standard Kerr-Schild formalism, but the imaginary mass term appears in some other formulations, in particular, in the form of the NUT-parameter of the Kerr-NUT solution.

By a nonzero value of this parameter there appears one more 'axial' singular filament (Dirac monopole string) which threads the Kerr singular ring and is extended to infinity. Note, that such an 'axial' string was also observed in the supersymmetric extension of the Kerr-Newman solution by the treating the fermionic traveling waves [13]. As it was mentioned in [13], this second singular filament, being topologically coupled to the Kerr singular ring, acquires an interesting interpretation as a carrier of the de Broglie wave. In the space-time with topologically nontrivial boundaries this filament can play the role of guide in the wave-pilot construction.

More detailed analysis of the system of these equations will be given elsewhere.


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## Appendix A. Basic relations of the Kerr-Schild formalism

Following the notations of the work [17], the Kerr-Schild null tetrad $e^a = e^a_\mu dx^\mu$ is determined by relations:

$$\begin{aligned} e^1 &= d\zeta - Y dv, \qquad e^2 = d\bar\zeta - \bar Y dv, \\ e^3 &= du + \bar Y d\zeta + Y d\bar\zeta - Y\bar Y dv, \\ e^4 &= dv + h e^3, \end{aligned} \tag{82}$$

and

$$g_{ab} = e^\mu_a e_{b\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{83}$$

Vectors $e^3, e^4$ are real, and $e^1, e^2$ are complex conjugate.

The Ricci rotation coefficients are given by

$$\Gamma^a_{\ bc} = - \quad e^a_{\mu;\nu} e^\mu_b e^\nu_c. \tag{84}$$

The PNC have the $e^3$ direction as tangent. It will be geodesic if and only if $\Gamma_{424} = 0$ and shear free if and only if $\Gamma_{422} = 0$. The corresponding complex conjugate terms are $\Gamma_{414} = 0$ and $\Gamma_{411} = 0$.

The inverse (dual) tetrad has the form

$$\begin{aligned} \partial_1 &= \partial_\zeta - \bar Y \partial_u; \\ \partial_2 &= \partial_{\bar\zeta} - Y \partial_u; \\ \partial_3 &= \partial_u - h\partial_4; \\ \partial_4 &= \partial_v + Y\partial_\zeta + \bar Y \partial_{\bar\zeta} - Y\bar Y \partial_u, \end{aligned} \tag{85}$$

where $\partial_a \equiv ,_a \equiv e^\mu_a \partial,_\mu$.

Parameter $Z = Y,_1 = \rho + i\omega$ is a complex expansion of congruence $\rho =$ *expansion* and $\omega =$ *rotation*. $Z$ is connected with a complex radial distance $\tilde r$ by relation

$$\tilde r = P Z^{-1}. \tag{86}$$

It was obtained in [17] that connection forms in Kerr-Schild metrics are

$$\Gamma_{42} = \Gamma_{42a} e^a = -dY - hY_{,4}\, e^4. \tag{87}$$

The congruence $e^3$ is geodesic if $\Gamma_{424} = -Y_{,4}\,(1-h) = 0$, and is shear free if $\Gamma_{422} = -Y_{,2} = 0$. Thus, the function $Y(x)$ with conditions

$$Y_{,2} = Y_{,4} = 0, \tag{88}$$

defines a shear-free and geodesic congruence.

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