Colour-Spin and Dilaton-Spin Perfect Fluids as the Sources of Post-Riemannian Spacetime

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Perfect fluids with intrinsic degrees of freedom, namely, the perfect colour-spin and dilaton-spin fluids are considered as the sources of the possible post-Riemannian spacetime structure. We develop the variational theories of the perfect fluids with intrinsic degrees of freedom. We derive the expressions for the corresponding energy-momentum tensors. The homogeneous and isotropic Universe filled with the dilaton matter as the dark matter is considered. The modified Friedmann–Lemaitre equation is obtained, from which the absence of the initial singularity in the cosmological solution follows. Also the existence of two points of inflection of the scale factor function is established, the first of which corresponds to the early stage of the Universe and the second one corresponds to the modern era when the expansion with deceleration is replaced by the expansion with acceleration. The possible equations of state for the self-interacting cold dark matter are found on the basis of the modern observational data. The inflation-like solution is obtained.

1. Introduction

As it is proposed in the modern fundamental physics, the spacetime geometrical structure is compatible with the properties of matter filling the spacetime. As a result of this fact the matter dynamics exhibits the constraints on a metric and a connection of the spacetime manifold. The possible post-Riemannian spacetime structure and the corresponding cosmology should appear as a result of existence of some unusual matter, filling spacetime, generating the spacetime geometrical structure and interacting with it.

As an example of such matter we consider the perfect fluid with intrinsic degrees of freedom, namely, the perfect colour-spin, dilaton-spin and hypermomentum fluids. All of these fluids generalize the Weyssenhoff–Raabe perfect spin fluid. Every particle of the first fluid is endowed with intrinsic spin and non-Abelian colour charge [1]–[7]. This sort of matter generates a Riemann–Cartan geometrical structure of spacetime. Every particle of the second fluid is endowed with spin and dilaton charge [8]. This fluid generates a Weyl–Cartan geometry of spacetime. Particles of the latter sort of fluid are endowed with intrinsic hypermomentum (see [9] and references therein). This sort of fluid generates a metric-affine geometrical structure.

We develop the variational theories of the perfect fluids with intrinsic degrees of freedom, using the exterior form language. As a result we derive the expressions for the energy-momentum tensors and the generalized Euler-type hydrodynamic equations of motion of the perfect fluids with intrinsic degrees of freedom on post-Riemannian geometrical background.

The modern observations [10], [11] lead to the conclusion about the existence of dark (non-luminous) matter with the density exceeding by one order of magnitude the density of baryonic matter, from which stars and luminous components of galaxies are formed. It is the dark matter interacting with the equal by order of magnitude positive vacuum energy or with quintessence that realizes the dynamics of the Universe in modern era. Another important consequence of the modern observations is the understanding of the fact that the end of the Friedmann era occurs when the expansion with deceleration is succeeded by the expansion with acceleration.

As dark matter we propose to consider the dilaton matter, the model of which is realized as the perfect dilaton-spin fluid. Then we develop the variational formalism of the gravitational field in a Weyl–Cartan spacetime in the exterior form language. The homogeneous and isotropic Universe filled with the dilaton matter as the dark matter is considered and one of the field equations is represented as the Einstein-like equation which leads to the modified Friedmann–Lemaitre equation. From this equation the absence of the initial singularity in the cosmological solution follows. Also the existence of two points of inflection of the scale factor function is established, the first of which corresponds to the early stage of the Universe and the second one corresponds to the modern era when the expansion with deceleration is replaced by the expansion with acceleration. The possible equations of state for the self-interacting cold dark matter are found on the basis of the modern observational data. The inflation-like solution is obtained.

Throughout the paper the conventions $c = 1$, $\hbar = 1$ are used.

2. Colour-spin perfect fluid in a Riemann–Cartan space

As the dynamic variables describing the fluid in a Riemann–Cartan space U_4, we take the 0-form fields ψ and their conjugates $\bar{\psi}$, which are transformed by the representations of the direct product of the Lorentz group and $SU(3)$ colour group. An element of the fluid possesses a 4-dimensional velocity vector $\vec{u} = u^a \vec{e}_a$, to which the 3- and 1-forms of velocity are corresponded, respectively, $u = \vec{u} \rfloor \eta = u^a \eta_a$ and $*u = u_a \theta^a = g(\vec{u}, \cdots)$, with $*u \wedge u = -\eta$, the latter being the conventional condition for the squared velocity, $g(\vec{u}, \vec{u}) = -1$. Here θ^a are the basis 1-forms, η is the volume 4-form, $\rfloor$ denotes the inner product, and $*$ denotes the Hodge dualization. We also introduce the 3- and 2-form fields $\eta_b = \vec{e}_b \rfloor \eta = *\theta_b$ and $\eta_{ab} = \vec{e}_b \rfloor \eta_a = *(\theta_a \wedge \theta_b)$, respectively. The basis $\vec{e}_a$ is supposed to be nonholonomic orthogonal with $g(\vec{e}_a, \vec{e}_b) = g_{ab} = \mathrm{diag}(1, 1, 1, -1)$.

The internal-energy density ε of the fluid depends on the extensive (additive) thermodynamic parameters: the number of particles per unit volume (concentration) n and the entropy per particle s, and on the quantities describing the internal degrees of freedom of the fluid particle, namely, the spin tensor $S_{ab} = \bar{\psi} M_{ab} \psi$ and the colour charge $J_m = \bar{\psi} I_m \psi$, where M_{ab} and I_m are the generators of the corresponding representations of the Lorentz and $SU(3)$ colour groups, respectively.

For the fluid element the first law of thermodynamics has the form,

$$d\varepsilon(n, s, S_{ab}, J_m) = \frac{\varepsilon + p}{n} dn + nT ds + \frac{1}{2} n \omega^{ab} dS_{ab} + n \omega^m dJ_m \,, \tag{2..1}$$

where p is the hydrodynamic pressure of the fluid. In (2..1) ω^{ab} and ω^m describe, respectively, the possible spin and colour charge exchanges between fluid elements. We suppose that the conservation laws of the particles number and of entropy, which can be expressed as $d(nu) = 0$ and $d(nsu) = 0$, are valid.

It is well-known that the spin tensor is spacelike in its nature that is the fact of fundamental physical meaning. This leads to the classical Frenkel condition, $S_{ab} u^b = 0$. In terms of the exterior forms, this condition can be written as $(\vec{e}_a \rfloor \mathcal{S}) \wedge u = 0$, where the spin 2-form $\mathcal{S} = (1/2) S_{ab} \theta^a \wedge \theta^b$ has been introduced.

The 4-form of Lagrangian density reads,

$$\begin{aligned}\mathcal{L}_{fluid} &= -\varepsilon(n,s,\psi,\bar{\psi})\eta + n\bar{\psi}D\psi \wedge u - \chi n J_m \mathcal{F}^m \wedge *\mathcal{S} \\ &+ n\lambda_1(*u \wedge u + \eta) + nu \wedge d\lambda_2 + n\lambda_3 u \wedge ds \\ &+ n\zeta^a(\vec{e}_a \rfloor \mathcal{S}) \wedge u \ , \end{aligned} \tag{2..2}$$

where $\mathcal{F}^m$ is the strength 2-form of the non-Abelian gauge colour field. Here n, s, ψ, $\bar{\psi}$ and u are regarded as independent variables. The constraints imposed on the independent variables are taken into account by using the Lagrange multipliers λ_1, λ_2, λ_3 and ζ^a.

Using the variational equation of motion of the perfect fluid one can deduce the evolution equation of the spin tensor

$$u \wedge DS_{ab} + 2S_{[a}{}^{c}u_{b]}\dot{u}_c\eta = -2S_{[a}{}^{f}\Pi^c_{b]}\Pi^d_f(\chi F^m{}_{cd}J_m + \omega_{cd})\eta \ , \tag{2..3}$$

where the projection tensor is $\Pi^c_b = \delta^c_b + u^c u_b$, and "dot" notation is introduced, $\dot{\Phi}^a{}_b = *(u \wedge D\Phi^a{}_b)$.

The total Lagrangian density of the theory has the form,

$$\mathcal{L}_{matter} = \mathcal{L}_{fluid} + \mathcal{L}_{field} \ , \quad \mathcal{L}_{field} = -\frac{\alpha}{2}\mathcal{F}^m \wedge *\mathcal{F}_m \ , \tag{2..4}$$

where $\mathcal{L}_{field}$ is the 4-form of the colour field Lagrangian density. The equations for the colour field are obtained by variation of (2..4) over 1-form potential A^m,

$$D(\alpha *\mathcal{F}_m + \chi n J_m *\mathcal{S}) = nJ_m u \ . \tag{2..5}$$

With the help of (2..4) we obtain the expression of the energy-momentum 3-form,

$$\Sigma_a = -\frac{\delta \mathcal{L}_{matter}}{\delta \theta^a} = \Sigma_a^{fluid} + \Sigma_a^{field} \ , \tag{2..6}$$

$$\Sigma_a^{fluid} = p\eta_a + n\left(\pi_a + \frac{p}{n}u_a\right)u + \chi n(e_a \rfloor \mathcal{F}^m J_m) \wedge *\mathcal{S} \ , \tag{2..7}$$

where the dynamic momentum per particle is introduced,

$$\pi_a = \varepsilon^* u_a - S_a{}^c\left(\dot{u}_c + u^b(\chi J_m F^m{}_{bc} + \omega_{bc})\right) \ . \tag{2..8}$$

Here ε^* denotes the effective energy per fluid particle, $\varepsilon^*\eta = \varepsilon_o\eta + \chi J_m\mathcal{F}^m \wedge *\mathcal{S}$, where $\varepsilon_o = \varepsilon/n$.

With the help of the energy-momentum quasi-conservation law in U_4 one can derive the equation of motion of the perfect spin fluid with colour charge in the form generalizing the Euler hydrodynamic equation [6], [7]. In this equation by going over to the limit of zero pressure we find that in U_4 space the equation of motion of a particle with a spin and a colour charge in an external non-Abelian colour gauge field has the form

$$\begin{aligned} u \wedge \mathcal{D}\pi_a &= -(\vec{e}_a \rfloor \mathcal{F}^m) \wedge J_m u + \chi(\vec{e}_a \rfloor \nabla)\mathcal{F}^m \wedge J_m *\mathcal{S} - \\ &-\frac{1}{2}(\vec{e}_a \rfloor \mathcal{R}^{bc}) \wedge S_{bc}u - (\vec{e}_a \rfloor \mathcal{T}^b) \wedge \pi_b \ . \end{aligned} \tag{2..9}$$

The first term on the right-hand side of this equation is a generalization of the Lorentz force to the case of a non-Abelian gauge field. The second term is a chromomagnetic analog of the Stern–Gerlach force acting on a magnetic moment in an electromagnetic field (this force is generated by the additional potential energy of a magnetic momentum in a magnetic field). The third term on the right-hand side equation (2..9) represents the Mathisson force arising from the interaction of the particle spin with the curvature of space, while the fourth term is so-called translational force, which is due to the interaction of the particle dynamical momentum with the torsion of space. The emergence of this force is peculiar to U_4 space.

Equation (2..9), supplemented with the evolution equation of the spin tensor (2..3) and the colour charge evolution equation [7], describe the motion of a particle with a spin and a colour charge in the presence of spin-chromomagnetic interaction. They generalize the well-known Wong equations [13] to the case of the $SU(3)$ colour group and take into account the spin of particles, which may be responsible for the possible interaction between the spin and the chromomagnetic component of the colour field and for the additional effect of the gravitational field, associated with the geometry of the Riemann–Cartan space, on the motion of particles.

Then we derive the evolution equation of the particle spin in a colour field in a Riemann–Cartan space. The particle-spin vector (Tamm–Pauli–Lyubanski vector) is defined as

$$\sigma^a = \frac{1}{2}\eta^{abcd}S_{bc}u_d \ , \qquad \sigma = \sqrt{\sigma^a\sigma_a} \ , \tag{2..10}$$

where η^{abcd} are the components of the Levi-Civita antisymmetric tensor (of the volume 4-form η). In Minkowski spacetime, the evolution of this vector in a slowly varying electromagnetic field is governed by the Bargmann–Michel–Telegdi equation [14]. However, this equation take no account of the effect of the spin on the trajectory of the particle. A more precise equation that describes the motion of the particle spin in a nonuniform electromagnetic field and which takes into account the effect of the spin on the motion of the particle was derived by Good who extended the Tamm equation (see [14]–[16]). By using equation (2..9) we will now extend the Tamm–Good and Bargmann–Michel–Telegdi equations to the case of the motion of such a particle in an external colour (generally nonuniform) field in U_4 [2], [7]:

$$\begin{aligned} u\wedge\mathcal{D}\sigma^a \;=\; & -\sigma^b(\chi\eta^a{}_b\wedge\mathcal{F}^m J_m + \omega^a{}_b\eta) - \frac{u^a\sigma^b}{m^*}[(\vec{e}_b\rfloor\mathcal{F}^m)\wedge J_m u \\ & -\pi^c(\chi\eta_{bc}\wedge\mathcal{F}^m J_m + \omega_{bc}\eta) - \chi(\vec{e}_b\rfloor\nabla)\mathcal{F}^m\wedge J_m *\mathcal{S} \\ & +\frac{1}{2}(\vec{e}_b\rfloor\mathcal{R}^{cd})\wedge S_{cd}u + (\vec{e}_b\rfloor\mathcal{T}^c)\wedge\pi_c u] \ . \end{aligned} \tag{2..11}$$

Note that, apart from terms involving field gradients, the equation of motions for a fluid particle (2..9) coincides in form with the corresponding equation of motion that follows from the string action functional [17]. This suggests that this equation of motion may prove to be valid not only for a fluid particle but also for an extended object like cosmic string.

The theory developed in this section can be applied to the describing of quark-gluon plasma [7]. According to the modern observations in cosmology [10], [11] the energy density of baryonic matter due to its small magnitude (in comparison with dark matter) does not realize the dynamics of the Universe. In our opinion quark-gluon plasma can play the essential role in local phenomena, i.e. in solving the gravitational collapse problem.

3. Dilaton-spin perfect fluid in a Weyl–Cartan space

Now we shall consider the variational theory of the dilaton-spin perfect fluid [8]. A Weyl–Cartan space is a space with a curvature 2-form $\mathcal{R}^a{}_b$, a torsion 2-form $\mathcal{T}^a$ and with the metric g and the connection Γ which obey the constraint,

$$-\mathcal{D}g_{ab} = \mathcal{Q}_{ab} = \frac{1}{4}g_{ab}\mathcal{Q} \ , \qquad \mathcal{Q} = g^{ab}\mathcal{Q}_{ab} = Q_a\theta^a \ , \tag{3..1}$$

where $\mathcal{D} = d + \Gamma\wedge\ldots$ is the covariant exterior differential with respect to the connection 1-form $\Gamma^a{}_b$, $\mathcal{Q}_{ab}$ is a nonmetricity 1-form and $\mathcal{Q}$ is a Weyl 1-form, which represents the gauge field, called dilaton field.

The additional degrees of freedom, which a fluid element possesses, are described with the help of the material frame (called directors) attached to every fluid element. In the exterior form language the material frame is realized as the coframe of 1-forms l^p ($p = 1, 2, 3, 4$), which have dual 3-forms l_q, the constraint $l^p \wedge l_q = \delta^p_q \eta$ being fulfilled.

The perfect dilaton-spin fluid obeys the Frenkel condition for the spin tensor $S^p{}_q$, that in the exterior form language can be written as follows, $S^p{}_q l_p \wedge {*u} = 0$, $S^p{}_q l^q \wedge u = 0$.

In case of the dilaton-spin fluid the spin dynamical variable of the Weyssenhoff fluid $S^p{}_q$ is generalized and becomes the new dynamical variable named the dilaton-spin tensor $J^p{}_q$:

$$J^p{}_q = S^p{}_q + \frac{1}{4}\delta^p_q J\ , \qquad S^p{}_q = J^{[p}{}_{q]}\ , \qquad J = J^p{}_p\ , \tag{3..2}$$

where J is the specific (per particle) dilaton charge of a fluid element. The measure of intrinsic motion contained in a fluid element is the quantity $\Omega^q{}_p$ which generalizes the intrinsic "angular velocity" of the Weyssenhoff spin fluid, $\Omega^q{}_p\eta = u \wedge l^q_a \mathcal{D} l^a_p$. An element of the perfect dilaton-spin fluid possesses the additional intrinsic "kinetic" energy density 4-form $E = \frac{1}{2} n J^p{}_q \Omega^q{}_p \eta$. The internal energy density of the fluid ε obeys to the first thermodynamic principle,

$$d\varepsilon(n, s, S^p{}_q, J) = \frac{\varepsilon + p}{n} dn + nT ds + \frac{\partial \varepsilon}{\partial S^p{}_q} dS^p{}_q + \frac{\partial \varepsilon}{\partial J} dJ\ . \tag{3..3}$$

The perfect fluid Lagrangian density 4-form of the perfect dilaton-spin fluid has the form,

$$\begin{aligned}\mathcal{L}_{fluid} \;\; = \;\; & -\varepsilon(n, s, S^p{}_q, J)\eta + \frac{1}{2} n J^p{}_q \Omega^q{}_p \eta + nu \wedge d\varphi + n\tau u \wedge ds \\ & + n\lambda({*u} \wedge u + \eta) + n\chi^q S^p{}_q l_p \wedge {*u} + n\zeta_p S^p{}_q l^q \wedge u\ .\end{aligned} \tag{3..4}$$

The fluid motion equations and the evolution equation of the dilaton-spin tensor are derived by the variation of (3..4) with respect to the independent variables n, s, $S^p{}_q$, J, u, l^q and the Lagrange multipliers, the thermodynamic principle (3..3) being taken into account. We shall consider the 1-form l^q as an independent variable and the 3-form l_p as a function of l^q. As a consequence one obtains the evolution equation of the spin tensor: $\dot{S}^a{}_b + \dot{S}^a{}_c u^c u_b + \dot{S}^c{}_b u_c u^a = 0$. This equation generalizes to a Weyl–Cartan space the evolution equation of the spin tensor in the Weyssenhoff fluid theory.

By means of the variational derivatives of the matter Lagrangian density (3..4) one can derive the external matter currents which are the sources of a gravitational field. The variational derivative of the Lagrangian density (3..4) with respect to θ^a yields the canonical energy-momentum 3-form [8],

$$\Sigma_a = \frac{\delta \mathcal{L}_{fluid}}{\delta \theta^a} = p\eta_a + (\varepsilon + p) u_a u + n \dot{S}_{ab} u^b u\ . \tag{3..5}$$

It should be mentioned that in case of the dilaton-spin fluid the specific energy density ε in (3..5) contains the energy density of the dilaton interaction of the fluid.

In a Weyl–Cartan space the matter Lagrangian (3..4) obeys the diffeomorphism invariance that leads to the Noether identity, which represents the quasiconservation law for the canonical matter energy-momentum 3-form. This identity is fulfilled, if the equations of matter motion are valid, and therefore it represents in essence another form of the matter motion equations. With the help of this Noether identity one can obtain the equation of motion of the perfect dilaton-spin fluid in the form of the generalized hydrodynamic Euler-type equation of the perfect fluid [8],

$$\begin{aligned}u \wedge \mathrm{D}\left(\pi_a + \frac{p}{n} u_a\right) = & \frac{1}{n}\eta \vec{e}_a \rfloor \mathrm{D}p - \frac{1}{8n}\eta(\varepsilon + p) Q_a - (\vec{e}_a \rfloor \mathcal{T}^b) \wedge \left(\pi_b + \frac{p}{n} u_b\right) u \\ & - \frac{1}{2}(\vec{e}_a \rfloor \mathcal{R}^{bc}) \wedge S_{bc} u + \frac{1}{8}(\vec{e}_a \rfloor \mathcal{R}^b{}_b) \wedge J u\ .\end{aligned} \tag{3..6}$$

If we evaluate the component of the equation (3..6) along the 4-velocity by contracting one with u^a, then after some algebra we get the energy conservation law along a streamline of the fluid [8],

$$\mathrm{d}\varepsilon = \frac{\varepsilon + p}{n}\mathrm{d}n \; . \qquad (3..7)$$

On the basis of (3..6) one can prove the following Theorem [8], [9]:

> The motion of a test particle with spin and without dilaton charge in a Weyl–Cartan space coincides with the motion of this particle in the Riemann–Cartan space, the metric and torsion tensors of which coincide with the metric and torsion tensors of the Weyl–Cartan space.

The important consequence of this statement means that fluids and particles without dilaton charge are not subjected to the influence of the possible Weyl structure of spacetime.

4. Dilaton-spin perfect fluid as a source of post-Riemannian spacetime

We shall consider the dilaton-spin fluid energy-momentum tensor (3..5) as a source of gravitational field [18], [19]. In this case the spacetime becomes a Weyl–Cartan space. The total Lagrangian density of the theory reads [20],

$$\begin{aligned}
&\mathcal{L} = \mathcal{L}_{grav} + \mathcal{L}_{fluid} \; , \\
&\mathcal{L}_{grav} = 2f_0\biggl(\frac{1}{2}\mathcal{R}^a{}_b \wedge \eta_a{}^b - \Lambda\eta + \frac{1}{4}\lambda\,\mathcal{R}^a{}_a \wedge *\mathcal{R}^b{}_b + \varrho_1\,\mathcal{T}^a \wedge *\mathcal{T}_a \\
&+\varrho_2\,(\mathcal{T}^a \wedge \theta_b) \wedge *(\mathcal{T}^b \wedge \theta_a) + \varrho_3\,(\mathcal{T}^a \wedge \theta_a) \wedge *(\mathcal{T}^b \wedge \theta_b) \\
&+\xi\,\mathcal{Q} \wedge *\mathcal{Q} + \zeta\,\mathcal{Q} \wedge \theta^a \wedge *\mathcal{T}_a\biggr) + \Lambda^{ab} \wedge \left(\mathcal{Q}_{ab} - \frac{1}{4}g_{ab}\mathcal{Q}\right) \; . \qquad (4..1)
\end{aligned}$$

Here $f_0 = 1/(2æ)$ ($æ = 8\pi G$), Λ is the cosmological constant, λ, ϱ_1, ϱ_2, ϱ_3, ξ, ζ are the coupling constants, and Λ^{ab} is the Lagrange multiplier 3-form with the evident properties: $\Lambda^{ab} = \Lambda^{ba}$, $\Lambda^c{}_c = 0$, which are the consequences of the Weyl's condition (3..1).

The field equations in Weyl–Cartan spacetime can be obtained by the variational procedure of the first order. Let us vary the Lagrangian (4..1) with respect to the basis 1-form θ^a and to the connection 1-form $\Gamma^a{}_b$ independently, the constraint imposed on the connection 1-form being taken into account by means of the Lagrange multiplier Λ^{ab}. It is useful to use the master formula (4..2) derived in the following Lemma, proved in [21].

> *Lemma.* Let Φ and Ψ be arbitrary p-forms defined on n-dimensional manifold. Then the variational identity for the commutator of the variation operator δ and the Hodge star operator $*$ is valid,

$$\begin{aligned}
&\Phi \wedge \delta * \Psi = \delta\Psi \wedge *\Phi \\
&+\delta g_{ab}\left(\frac{1}{2}g^{ab}\Phi \wedge *\Psi + (-1)^{p(n-1)+s+1}\theta^a \wedge *\left(*\Psi \wedge \theta^b\right) \wedge *\Phi\right) \\
&+\delta\theta^a \wedge \left((-1)^p\Phi \wedge *\left(\Psi \wedge \theta_a\right) + (-1)^{p(n-1)+s+1} * \left(*\Psi \wedge \theta_a\right) \wedge *\Phi\right) \; . \qquad (4..2)
\end{aligned}$$

The variation with respect to the connection 1-form $\Gamma^a{}_b$ yields the field equation (Γ-equation) [20], after antisymmetrization of which one obtains the expression for the torsion 2-form,

$$\begin{aligned}
&-\frac{1}{2}\mathcal{T}^c \wedge \eta_{abc} + \frac{1}{8}\mathcal{Q} \wedge \eta_{ab} + 2\varrho_1\theta_{[a} \wedge *\mathcal{T}_{b]} \\
&+2\varrho_2\theta_{[a} \wedge \theta_{|c|} \wedge *(\mathcal{T}^{|c|} \wedge \theta_{b]}) + 2\varrho_3\theta_a \wedge \theta_b \wedge *(\mathcal{T}^c \wedge \theta_c) \\
&+\zeta\theta_{[a} \wedge *(\mathcal{Q} \wedge \theta_{b]}) = \frac{1}{2}æ n S_{ab} u \; , \qquad æ = \frac{1}{2f_0} \; . \qquad (4..3)
\end{aligned}$$

The torsion 2-form can be decomposed into the irreducible pieces (the traceless 2-form $\overset{(1)}{\mathcal{T}}{}^{a}$, the trace 2-form $\overset{(2)}{\mathcal{T}}{}^{a}$ and the pseudotrace 2-form $\overset{(3)}{\mathcal{T}}{}^{a}$) [22], [12]: $\mathcal{T}^a = \overset{(1)}{\mathcal{T}}{}^{a} + \overset{(2)}{\mathcal{T}}{}^{a} + \overset{(3)}{\mathcal{T}}{}^{a}$, where the torsion trace 2-form and the torsion pseudotrace 2-form of the pseudo-Riemannian 4-manifold are determined by the expressions, respectively,

$$\overset{(2)}{\mathcal{T}}{}^{a} = \tfrac{1}{3}\mathcal{T} \wedge \theta^a \ , \qquad \mathcal{T} = *(\theta_a \wedge *\mathcal{T}^a) = -(\vec{e}_a \rfloor \mathcal{T}^a) \ , \tag{4..4}$$

$$\overset{(3)}{\mathcal{T}}{}^{a} = \tfrac{1}{3} * (\mathcal{P} \wedge \theta^a) \ , \qquad \mathcal{P} = *(\theta_a \wedge \mathcal{T}^a) = \vec{e}_a \rfloor * \mathcal{T}^a \ , \tag{4..5}$$

where the torsion trace 1-form $\mathcal{T}$ and the torsion pseudotrace 1-form $\mathcal{P}$ are introduced.

The field equation (4..3) yields the consequences,

$$\mathcal{T} = \frac{3(\frac{1}{4} + \zeta)}{2(1 - \varrho_1 + 2\varrho_2)} \mathcal{Q} \ , \tag{4..6}$$

$$(1 - 4\varrho_1 - 4\varrho_2 - 12\varrho_3)\mathcal{P} = æ n\sigma \ , \tag{4..7}$$

the first of which gives the relation between the torsion trace 1-form $\mathcal{T}$ and the Weyl 1-form $\mathcal{Q}$, and the second one represents the torsion pseudotrace 1-form $\mathcal{P}$ via the Pauli–Lyubanski spin 1-form σ (2..10) of a fluid particle.

As a consequence of (4..7) the field equation (4..3) yields the equation for the traceless piece of the torsion 2-form,

$$(1 + 2\varrho_1 + 2\varrho_2) \overset{(1)}{\mathcal{T}}{}_{a} = æ n \left(S_{ab} u_c \theta^b \wedge \theta^c + \frac{2}{3}\sigma^b \eta_{ba} \right) = -\frac{2}{3} æ n S_{b(a} u_{c)} \theta^b \wedge \theta^c \ . \tag{4..8}$$

The symmetric part of the Γ-equation determines the Lagrange multiplier 3-form Λ_{ab}. It is very important that Λ_{ab} is in general not equal to zero.

By contracting the Γ-equation and after substituting (4..6) in the result, one finds the equation of the Proca type for the Weyl 1-form [20],

$$* \, \mathrm{d} * \mathrm{d}\mathcal{Q} + m^2 \mathcal{Q} = \frac{æ}{2\lambda} nJ * u \ , \qquad m^2 = 16\frac{\xi}{\lambda} + \frac{3(\varrho_1 - 2\varrho_2 + 8\zeta(1 + 2\zeta))}{4\lambda(1 - \varrho_1 + 2\varrho_2)} \ . \tag{4..9}$$

The equation (4..9) shows that the dilaton field $\mathcal{Q}$, in contrast to Maxwell field, possesses the non-zero rest mass and demonstrates a short-range nature, as it was pointed out by Utiyama [23], [24] (see also [25]). In the component form the Proca type equation for Weyl vector was used in [24], [26] and in the exterior form language in [12].

The equations (4..6), (4..7) and (4..8) solve the problem of the evaluation the torsion 2-form. With the help of the algebraic field equations (4..7) and (4..8) the traceless and pseudotrace pieces of the torsion 2-form are determined via the spin tensor and the flow 3-form u of the perfect dilaton-spin fluid in general case. With the help of the equation (4..6) one can determine the torsion trace 2-form via the dilaton field $\mathcal{Q}$, for which the differential field equation (4..9) is valid. Therefore the torsion trace 2-form can propagate in the theory under consideration.

5. Homogeneous and isotropic cosmology with dilaton matter

Now we consider the homogeneous and isotropic Universe filled with the perfect dilaton-spin fluid [18]–[20], which realizes the model of the dark matter with $J \neq 0$ in contrast to the baryonic and quark matter with $J = 0$. The metric of this cosmological model is the Robertson–Walker (RW) metric with scale factor $a(t)$,

$$\mathrm{d}s^2 = \frac{a^2(t)}{1 - kr^2}\mathrm{d}r^2 + a^2(t)r^2\Big(\mathrm{d}\theta^2 + (\sin\theta)^2\mathrm{d}\phi^2\Big) - \mathrm{d}t^2 \ . \tag{5..1}$$

As it was shown in [27]–[29], in the spacetime with the RW metric (5..1) the only nonvanishing components of the torsion are $T^1{}_{41} = T^2{}_{42} = T^3{}_{43}$ and T_{ijk} for $i = 1,\ 2,\ 3$. In this case from (4..4) we get that the only nonvanishing component of the trace 1-form is $T_4 = T_4(t)$ ($T_i = 0$ for $i = 1,\ 2,\ 3$). From (4..5) we also find, $\mathcal{P} = 3T_{[123]}\eta^{1234}\theta_4$. But the field equation (4..7) yields, $\mathcal{P}^4 \sim \sigma^4 = 0$, as a consequence of (2..10). Therefore the pseudotrace piece of the torsion 2-form vanishes. It is easy to calculate with the help of (4..4) that the traceless piece also vanishes. Therefore for the RW metric (5..1) the torsion 2-form consists only from the trace piece. We also put $S_{ab} = 0$. In this case dilaton-spin fluid becomes dilaton fluid.

The variation of (4..1) with respect to the basis 1-form θ^a gives the second field equation (θ-equation). The source of this equation is the fluid canonical energy-momentum 3-form (3..5). Now we decompose the field θ-equation into Riemannian and non-Riemannian parts and then transform the result to the component form. The terms with the derivatives of the dilaton field, like $\overset{R}{\nabla}_a Q^a$ and $\overset{R}{\nabla}_b Q^a$, and the same derivatives of the torsion trace 1-form in a remarkable manner mutually compensate each other and vanish. The terms with $\mathrm{d}\mathcal{Q}$ also vanish, as the equality $\mathrm{d}\mathcal{Q} = 0$ is valid identically for the RW metric (5..1) that can be easy verified in the component basis. As a consequence of this fact we can derive the Weyl 1-form $\mathcal{Q}$ algebraically from the equation (4..9).

Then we can represent the field θ-equation as an Einstein-like equation (the condition $S_{ab} = 0$ being used),

$$\overset{R}{R}_{ab} - \frac{1}{2} g_{ab} \overset{R}{R} = æ\Big((\varepsilon_{\mathrm{e}} + p_{\mathrm{e}})u_a u_b + p_{\mathrm{e}} g_{ab}\Big) , \tag{5..2}$$

where $\overset{R}{R}_{ab}$, $\overset{R}{R}$ are a Ricci tensor and a curvature scalar of a Riemann space, respectively, ε_{e} and p_{e} are an energy density and a pressure of an effective perfect fluid:

$$\varepsilon_{\mathrm{e}} = \varepsilon + \varepsilon_{\mathrm{v}} - \alpha æ \left(\frac{nJ}{2\lambda m^2}\right)^2 , \qquad p_{\mathrm{e}} = p + p_{\mathrm{v}} - \alpha æ \left(\frac{nJ}{2\lambda m^2}\right)^2 , \tag{5..3}$$

$$\alpha = \frac{3\left(\frac{1}{4} + \zeta\right)^2}{4(1 - \varrho_1 + 2\varrho_2)} + \xi - \frac{3}{64} . \tag{5..4}$$

and $\varepsilon_{\mathrm{v}} = \Lambda/æ$ and $p_{\mathrm{v}} = -\Lambda/æ$ are an energy density and a pressure of a vacuum with the equation of state, $\varepsilon_{\mathrm{v}} = -p_{\mathrm{v}} > 0$.

The field equation (5..2) yields the modified Friedmann–Lemaitre (FL) equation,

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{æ}{3}\left(\varepsilon + \varepsilon_{\mathrm{v}} - \alpha æ \left(\frac{Jn}{2\lambda m^2}\right)^2\right) . \tag{5..5}$$

The integration of the continuity equation $\mathrm{d}(nu) = 0$ (d – the operator of exterior differentiation) for RW metric (5..1) yields the matter conservation law $na^3 = N = \mathrm{const}$. As an equation of state of the dilaton fluid we choose the equation of state $p = \gamma\varepsilon$, $0 \le \gamma < 1$. Then the integration of the energy conservation law (3..7) for RW metric (5..1) yields the condition,

$$\varepsilon\, a^{3(1+\gamma)} = \mathcal{E}_\gamma = \mathrm{const} , \qquad \mathcal{E}_\gamma > 0 . \tag{5..6}$$

We put $k = 0$ in (5..5) in accordance with the modern observational evidence [10], [11], [30], [31], from which one should conclude that the Universe is spatially flat in cosmological scale. In this case we can conclude from the equation (5..5) that extremum points of the scale factor ($\dot a = 0$) correspond to zero points of the equation,

$$a^6 + \frac{\mathcal{E}_\gamma}{\varepsilon_{\mathrm{v}}}\left(a^{3(1-\gamma)} - \frac{\mathcal{E}}{\mathcal{E}_\gamma}\right) = 0 , \qquad \mathcal{E} = \alpha æ \left(\frac{JN}{2\lambda m^2}\right)^2 . \tag{5..7}$$

If the condition, $0 < \mathcal{E} << 1$ is valid, then in case $a << 1$ the equation (5..7) has one zero point,

$$a_0 \approx \left(\frac{\mathcal{E}}{\mathcal{E}_\gamma}\right)^{\frac{1}{3(1-\gamma)}} = \left(\frac{\alpha æ}{\mathcal{E}_\gamma}\left(\frac{JN}{2\lambda m^2}\right)^2\right)^{\frac{1}{3(1-\gamma)}} , \tag{5..8}$$

and none zero points in case $a >> 1$ (as $\mathcal{E}_\gamma > 0$).

In order to clarify, whether there is a minimum or a maximum in the extremum point, one can take the other component of the equation (5..2):

$$\frac{\ddot{a}}{a} = -\frac{æ}{6}(\varepsilon_{\rm e} + 3p_{\rm e}) = \frac{æ}{3a^6}\left(\varepsilon_{\rm v} a^6 - \frac{1}{2}(1+3\gamma)\mathcal{E}_\gamma a^{3(1-\gamma)} + 2\mathcal{E}\right) . \tag{5..9}$$

At the extremum point (5..8) (in case $a << 1$) one gets $\ddot{a} > 0$. Therefore the value a_0 corresponds to the minimum point of the scale factor $a(t)$.

In case $a >> 1$ we can neglect the last term in (5..9) and get the equation,

$$\frac{\ddot{a}}{a} = \frac{æ}{3}\left(\varepsilon_{\rm v} - \frac{1}{2}(1+3\gamma)\varepsilon(t)\right) , \tag{5..10}$$

where $\varepsilon(t)$ is the current value of dilaton fluid energy density. This equation is valid for the most part of the history of the Universe.

Consider now the conditions under which the points of inflection of the function $a(t)$ can exist. To this end let us equate the right side of (5..9) to zero ($\ddot{a} = 0$) and find zero points of the equation,

$$a^6 - \frac{(1+3\gamma)\mathcal{E}_\gamma}{2\varepsilon_{\rm v}}\left(a^{3(1-\gamma)} - \frac{4}{1+3\gamma}a_0^{3(1-\gamma)}\right) = 0 . \tag{5..11}$$

As the last term in (5..11) is very small, then this equation has two types of zero points: very small by magnitude with the value a_1 and large by magnitude with the value a_2:

$$a_1 \approx a_0\left(\frac{4}{1+3\gamma}\right)^{\frac{1}{3(1-\gamma)}} , \qquad a_2 \approx \left(\frac{(1+3\gamma)\mathcal{E}_\gamma}{2\varepsilon_{\rm v}}\right)^{\frac{1}{3(1+\gamma)}} . \tag{5..12}$$

From a_1 up to the a_2 one has $\ddot{a} < 0$ and the expansion with deceleration occurs up to the end of the Friedmann era. The point of inflection a_2 corresponds to the modern era when the Friedmann expansion with deceleration is replaced by the expansion with acceleration that means the beginning of the "second inflation" era.

By equating to zero the equation (5..10) we get the correlation between the vacuum energy density $\varepsilon_{\rm v}$ and the dilaton matter energy density ε at the point of inflection a_2,

$$\varepsilon = \frac{2\varepsilon_{\rm v}}{1+3\gamma} . \tag{5..13}$$

For the nonrelativistic cold matter ($\gamma = \frac{2}{3}$) the formula (5..13) yields,

$$\Omega_{\rm cdm} = \frac{2}{3}\Omega_\Lambda , \qquad \Omega_{\rm cdm} = \frac{\varepsilon}{\varepsilon_{\rm tot}} , \qquad \Omega_\Lambda = \frac{\varepsilon_{\rm v}}{\varepsilon_{\rm tot}} , \qquad \varepsilon_{\rm tot} = \frac{3H^2}{8\pi G} .$$

that fits to the boundary of the modern observational data [30],

$$\Omega_\Lambda = 0.66 \pm 0.06 , \qquad \Omega_{\rm cdm} h_0^2 = 0.17 \pm 0.02 ,$$

with $H_0 = 100\, h_0 = 65\ km\ s^{-1}\, Mpc^{-1}$ [10].

Curiously, that if one takes the generally accepted data of [10], $\Omega_{\rm cdm} = \frac{1}{3}$, $\Omega_{\Lambda} = \frac{2}{3}$, and substitute to (5..13), then one gets the value $\gamma = 1$, that corresponds to the equation of state of the superrigid matter. Therefore we can conclude that the theory together with the observational data gives the approximate range for the viable values of the factor γ in the equation of state of the dilaton matter, $\frac{2}{3} \leq \gamma \leq 1$.

It is interesting to investigate the limiting case $\gamma = 1$. For this case the equations (5..6)–(5..7) are valid, but the zero point of the equation (5..7) in case $\gamma = 1$ is

$$a_{\rm e} = \left(\frac{\mathcal{E} - \mathcal{E}_1}{\varepsilon_{\rm v}} \right)^{\frac{1}{6}} , \qquad (5..14)$$

where $\mathcal{E}_1 = \mathcal{E}_{\gamma=1}$ is the integration constant of the equation (5..6). The value $a_{\rm e}$ corresponds to the minimum point of the scale factor, $a_{\rm e} = a_{\rm min}$.

In case $\gamma = 1$, $k = 0$ the equation (5..5) reads,

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{æ}{3a^6} \left(\varepsilon_{\rm v} a^6 - \mathcal{E}_1 + \mathcal{E} \right) = \frac{\Lambda}{3a^6} \left(a^6 - a^6_{\rm min} \right) ,$$

and can be exactly integrated. The solution corresponding to the initial data $t = 0$, $a = a_{\rm min}$ reads,

$$a = a_{\rm min} (\cosh \sqrt{3\Lambda}\, t)^{\frac{1}{3}} , \qquad a_{\rm min} = \left(\frac{\alpha æ^2}{\Lambda} \left(\frac{JN}{2\lambda m^2} \right)^2 - \frac{æ\mathcal{E}_1}{\Lambda} \right)^{\frac{1}{6}} .$$

This solution describes the inflation-like stage of the evolution of the Universe, which continues until the equation of state of the dilaton matter will change and will become differ from the equation of state of the superrigid matter.

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