Construction of Special Solutions of Nonintegrable Systems with the Help of the Painleve Test

S.Yu. Vernov[1]
Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, Russia

The Hénon–Heiles system in the general form has been considered. In two nonintegrable cases with the help of the Painlevé test new special solutions have been found as Laurent series, depending on three parameters. The obtained series converge in some ring. One of parameters determines a location of the singularity point, other parameters determine coefficients of series. For some values of these parameters the obtained Laurent series coincide with the Laurent series of the known exact solutions. It is established, that in other nonintegrable cases the similar special solutions do not exist.

1. The Painlevé property and integrability

A Hamiltonian system in a $2s$–dimensional phase space is called *completely integrable* (Liouville integrable) if it possesses s independent integrals which commute with respect to the associated Poisson bracket. When this is the case, the equations of motion are (in principal, at least) separable and solutions can be obtained by the method of quadratures.

When we study some mechanical or field theory problem, we imply that time and space coordinates are real, whereas the integrability of motion equations is connected with the behavior of their solutions as functions of complex time and (in the case of the field theory) complex spatial coordinates.

S.V. Kovalevskaya was the first, who proposed [1] to consider time as a complex variable and to demand that solutions of the motion equations have to be single-valued, meromorphic functions on the whole complex (time) plane. This idea gave a remarkable result: S.V. Kovalevskaya discovered a new integrable case (nowadays known as the Kovalevskaya's case) for the motion of a heavy rigid body about a fixed point [1] (see also [2, 3]). The work of S.V. Kovalevskaya has shown the importance of application of the analytical theory of differential equations to physical problems. The essential stage of development of this theory was a classification of ordinary differential equations (ODE's) in order of types of singularities of their solutions. This classification has been made by P. Painlevé.

Let us formulate the Painlevé property for ODE's. Solutions of a system of ODE's are regarded as analytic functions, may be with isolated singular points [4, 5]. A singular point of a solution is said *critical* (as opposed to *noncritical*) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions[2]. The *general solution* of an ODE of order N is the set of all solutions mentioned in the existence theorem of Cauchy, i.e. determined by the initial values. It depends on N arbitrary independent constants. A *special (particular) solution* is any solution obtained from the general solution by giving values to the arbitrary constants. A

[1]E-mail address: svernov@theory.sinp.msu.ru
[2]Solutions of a system with a time-independent Hamiltonian can have only movable singularities.

singular solution is any solution which is not special, i.e. which does not belong to the general solution.

Definition. A system of ODE's has ***the Painlevé property*** *if its general solution has no movable critical singular point* [6, 7].

If a system has not the Painlevé property, but, after some change of variables, the obtained system possesses this property, then the initial system is said to have *the weak Painlevé property.*

There exist two differences between the structure of solutions of linear differential equations and nonlinear ones. Linear ODE's have no singular solution and their general solutions have no movable singularity.

Investigations of many dynamical systems, Hamiltonian [8–10] or dissipative (for example, the Lorenz systems [10–13]), show that a system is completely integrable only for such values of parameters, at which it has the Painlevé property (or the weak Painlevé property). Arguments, which clarify the connection between the Painlevé analysis and the existence of motion integrals, are presented in [14, 15]. If the system misses the Painlevé property (has complex or irrational "resonances"), then the system cannot be "algebraically integrable" [16, 17]. At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is not an algorithm for construction of the additional integral by the Painlevé analysis. It is easy to give an example of an integrable system without the Painlevé property [18]: $H = \frac{1}{2}p^2 + f(x)$, where $f(x)$ is a polynomial which power is not lower than five. The given system is trivially integrable, but its general solution is not a meromorphic function. The study of complex-time singularities is a useful tool for the analysis of not only integrable systems, but also chaotic dynamics [19].

The ***Painlevé test*** is any algorithm, which checks some necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODE's with Painlevé property [7], is known as the α-method. The method of S.V. Kovalevskaya is not as general as the α–method, but much more simple. The remarkable property of this test is that it can be checked in a finite number of steps. This test can only detect the occurrence of logarithmic and algebraic branch points. To date there is no general finite algorithmic method to detect the occurrence of essential singularities[3].

In 1980, motivated by the work of S.V. Kovalevskaya [1], M.J. Ablowitz, A. Ramani and H. Segur [21] developed a new algorithm of the Painlevé test for ODE's. They also were the first to point out the connection between the nonlinear partial differential equations (PDE's), which are soluble by the inverse scattering transform method, and ODE's with the Painlevé property. Subsequently the Painlevé property for PDE was defined and the corresponding Painlevé test (the WTC procedure) was constructed [22, 23] (see also [20, 24–32]). With the help of this test it has been found, that all PDE's, which are solvable by the inverse scattering transforms, have the Painlevé property, may be, after some change of variables. For many integrable PDE's, for example, the Korteweg–de-Vries [10] and the sine–Gordon [24] equations, the Bäcklund transformations and the Lax representations result from the WTC procedure [23, 25, 26, 32]. For certain nonintegrable PDE's special solutions were constructed using this algorithm [33, 34].

The algorithm for finding special solutions for ODEs in the form of a finite expansion in powers of unknown function $\varphi(t-t_0)$ has been constructed in [35, 36]. The function $\varphi(t-t_0)$ and coefficients have to satisfy some system of ODE, often more simple than an initial one. This method has been used to construct exact special solutions for some ODE's [37, 38]. With the help of the perturbative Painleve test [31] four-parameter generalization of an exact three-parameter solution of the Bianchi IX (Mixmaster) cosmological model has been constructed [39].

[3]Different variants of the Painlevé test are compared in [20, R. Conte paper].

2. The Hénon–Heiles Hamiltonian

In the 1960s the models of the star motion in an axial-symmetric and time-independent potential have been developed [40, 41, 42] to show either existence or absence of the third integral for some polynomial potentials. The motion equations admit two well-known integrals (energy and angular momentum) and would be solved by the method of quadratures if the third integral of motion is known. However, for many polynomial potentials the obtained system has not the second integral as a polynomial function. Due to the symmetry of the potential the considered system is equivalent to two-dimensional one.

To clarify the question about the existence of the third integral Hénon and Heiles [42] considered the behavior of numerically integrated trajectories. Emphasizing that their choice of potential does not proceed from experimental data, they have proposed the Hamiltonian

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + x^2 + y^2\Big) + x^2 y - \frac{1}{3}y^3, \tag{1}$$

because: on the one hand, it is analytically simple; this makes the numerical computations of trajectories easy; on the other hand, it is sufficiently complicated to give trajectories which are far from trivial. Indeed, for low energies the Hénon–Heiles system appears to be integrable, in so much as trajectories (numerically integrated) always lay on well-defined two-dimensional surfaces. On the other hand, for high energies many of these integral surfaces are destroyed, it points on absence of the third integral. Subsequent investigations [43, 44] show, that in the complex t-plane singular points of solutions of the motion equations group in self-similar spirals. It turns out extremely complicated distributions of singularities, forming a boundary, across which the solutions can not be analytically continued.

The generalized Hénon–Heiles system is described by the Hamiltonian:

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + \lambda x^2 + y^2\Big) + x^2 y - \frac{C}{3}\,y^3 \tag{1'}$$

and the corresponding system of the motion equations:

$$\left\{\begin{array}{lcl} x_{tt} & = & -\lambda x - 2xy, \\ y_{tt} & = & -y - x^2 + Cy^2, \end{array}\right. \tag{2}$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, λ and C are numerical parameters.

Due to the Painlevé analysis the following integrable cases of (2) have been found:

(i) $C = -1,\qquad \lambda = 1,$
(ii) $C = -6,\qquad \lambda$ is an arbitrary number,
(iii) $C = -16,\qquad \lambda = \frac{1}{16}.$

In contradiction to the case (i) the cases (ii) and (iii) are nontrivial, so the integrability of these cases had to be proved additionally. In the 1980's the required second integrals were constructed [45–48]. For integrable cases of the Hénon–Heiles system the Bäcklund transformations [35, 36] and the Lax representations [27, 28, 50] have been found.

The three integrable cases of the Hénon–Heiles system correspond precisely to the stationary flows of the only three integrable cases of *fifth-order polynomial nonlinear evolution* equations of scale weight 7 (respectively the Sawada–Kotega, the fifth-order Korteweg–de Vries and the Kaup–Kupershmidt equations) [36, 49, 50].

The Hénon–Heiles system is a model, not only actively investigated by various mathematical methods, but also widely used in physics, in particular, in gravitation [52, 53] and plasma theory [54]. The models, described by the Hamiltonian (1′) with some additional nonpolynomial terms, are actively studied [50, 55–57] as well.

3. Nonintegrable cases

The general solutions of the Hénon–Heiles system are known only in integrable cases [57], in other cases not only four-, but even three-parameter exact solutions have yet to be found. The aim of this paper is to find new three-parameter special solutions as Laurent series.

The procedure for transformation the Hamiltonian to a normal form and for construction the second independent integral in the form of formal power series in the phase variables x, x_t, y and y_t (Gustavson integral) has been realized for the Hénon–Heiles system both in the original ($\lambda = 1$, $C = 1$) [41] (see also [58]) and in the general forms [59, 60]. Using the Bruno algorithm [61, 62] V.F. Edneral has constructed the Poincaré–Dulac normal form and found [63, 64] (provided that all phase variables are small) local families of periodic solutions. Recently it has been found that a local series around the singularities in the complex (time) plane can be transformed to some local series around the singularities at the fixed points in phase space and analyzed via normal forms theory [65, 66].

The Hénon–Heiles system as a system of two second order ODE's is equivalent to the fourth order equation[4]:

$$y_{tttt} = (2C-8)y_{tt}y - (4\lambda+1)y_{tt} + 2(C+1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda-6)y^2 - 4\lambda y - 4H, \qquad (3)$$

where H is the energy of the system.

To find a special solution of the given equation one can assume that y satisfies some more simple equation. For example, the well-known solutions in terms of the Weierstrass elliptic functions [67] satisfy the following first-order differential equation:

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}}, \qquad (4)$$

where $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{D}}$ are some constants. $\tilde{\mathcal{D}}$ is proportional to energy H (arbitrary parameter), therefore, the obtained solutions are two-parameter ones.

E.I. Timoshkova [68] generalized equation (4):

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}} + \tilde{\mathcal{G}}y^{5/2} + \tilde{\mathcal{E}}y^{3/2} \qquad (5)$$

and found new one-parameter solutions of the Hénon–Heiles system in nonintegrable cases ($C = -\frac{4}{3}$ or $C = -\frac{16}{5}$, λ is an arbitrary number). These solutions (i.e. solutions with $\tilde{\mathcal{G}} \neq 0$ or $\tilde{\mathcal{E}} \neq 0$) are derived only at $\tilde{\mathcal{D}} = 0$, therefore, substitution $y = \varrho^2$ gives:

$$\varrho_t^2 = \frac{1}{4}\Big(\tilde{\mathcal{A}}\varrho^4 + \tilde{\mathcal{G}}\varrho^3 + \tilde{\mathcal{B}}\varrho^2 + \tilde{\mathcal{E}}\varrho + \tilde{\mathcal{C}}\Big). \qquad (6)$$

Let us consider in detail these two nonintegrable cases. Substituting $y = \varrho^2$ in (3) we obtain:

$$\begin{aligned} \varrho_{tttt}\varrho &= 4\varrho_{ttt}\varrho_t - 3\varrho_{tt}^2 + (2C-8)\varrho_{tt}\varrho^3 - (4\lambda+1)\varrho_{tt}\varrho - +(6C-4)\varrho_t^2\varrho^2 \\ &\quad -(4\lambda+1)\varrho_t^2 + \frac{10C}{3}\varrho^6 + (2C\lambda-3)\varrho^4 - 2\lambda\varrho^2 - 2H. \end{aligned} \qquad (7)$$

It is easy to verify that solutions of equation (6) with $\tilde{\mathcal{G}} \neq 0$ or $\tilde{\mathcal{E}} \neq 0$ satisfy equation (7) if

[4] For given $y(t)$ the function $x^2(t)$ is a solution of a linear equation. System (2) is invariant under exchange x to $-x$.

1. $C = -\dfrac{4}{3}$

$$\begin{aligned}
&\tilde{\mathcal{A}} = -\frac{4}{3}, \qquad \tilde{\mathcal{B}} = \frac{s(\lambda) - 8\lambda - 93}{66},\\
&\tilde{\mathcal{C}} = \frac{8984 s(\lambda)\lambda - 3669 s(\lambda) + 760256\lambda^2 - 990696\lambda + 273105}{81312},\\
&\tilde{\mathcal{G}} = \pm\frac{\sqrt{10}(-13 s(\lambda) + 952\lambda - 945)}{\sqrt{33}(498960\lambda^2 - 336105\lambda + 31185)}\sqrt{R_1(\lambda)}, \qquad \tilde{\mathcal{E}} = \mp\frac{\sqrt{10}}{231\sqrt{33}}\sqrt{R_1(\lambda)};\\
&\tilde{\mathcal{A}} = -\frac{4}{3}, \qquad \tilde{\mathcal{B}} = \frac{-s(\lambda) - 8\lambda - 93}{66},\\
&\tilde{\mathcal{C}} = \frac{-8984 s(\lambda)\lambda + 3669 s(\lambda) + 760256\lambda^2 - 990696\lambda + 273105}{81312},\\
&\tilde{\mathcal{G}} = \pm\frac{\sqrt{10}(13 s(\lambda) + 952\lambda - 945)}{\sqrt{33}(498960\lambda^2 - 336105\lambda + 31185)}\sqrt{R_2(\lambda)}, \qquad \tilde{\mathcal{E}} = \pm\frac{\sqrt{10}}{231\sqrt{33}}\sqrt{R_2(\lambda)},
\end{aligned} \tag{8.1}$$

where

$$s(\lambda) = \sqrt{7(1216\lambda^2 - 1824\lambda + 783)},$$

$$R_1(\lambda) \equiv (20008\lambda^2 - 37173\lambda + 13581)s(\lambda) + 2099776\lambda^3 - 4911144\lambda^2 + 3943233\lambda - 1006425,$$

$$R_2(\lambda) \equiv -(20008\lambda^2 - 37173\lambda + 13581)s(\lambda) + 2099776\lambda^3 - 4911144\lambda^2 + 3943233\lambda - 1006425.$$

2. $C = -\dfrac{16}{5}$

$$\begin{aligned}
&\tilde{\mathcal{A}} = -\frac{32}{15}, \qquad \tilde{\mathcal{B}} = \frac{4q(\lambda) - 1680\lambda - 784}{1309},\\
&\tilde{\mathcal{C}} = \frac{(14160\lambda - 685)q(\lambda) + 1413120\lambda^2 - 2454000\lambda + 168855}{3916528},\\
&\tilde{\mathcal{G}} = \pm\frac{13q(\lambda) + 224\lambda + 1239}{\sqrt{1122}(3769920\lambda^2 - 2539460\lambda + 235620)}\sqrt{P_1(\lambda)}, \qquad \tilde{\mathcal{E}} = \mp\frac{5}{5236\sqrt{1122}}\sqrt{P_1(\lambda)};\\
&\tilde{\mathcal{A}} = -\frac{32}{15}, \qquad \tilde{\mathcal{B}} = -\frac{4q(\lambda) + 1680\lambda + 784}{1309},\\
&\tilde{\mathcal{C}} = \frac{-(14160\lambda - 685)q(\lambda) + 1413120\lambda^2 - 2454000\lambda + 168855}{3916528},\\
&\tilde{\mathcal{G}} = \pm\frac{-13q(\lambda) - 224\lambda - 1239}{\sqrt{1122}(3769920\lambda^2 - 2539460\lambda + 235620)}\sqrt{P_2(\lambda)}, \qquad \tilde{\mathcal{E}} = \mp\frac{5}{5236\sqrt{1122}}\sqrt{P_2(\lambda)},
\end{aligned} \tag{8.2}$$

where

$$q(\lambda) \equiv \sqrt{35(2048\lambda^2 - 1280\lambda + 387)},$$

$$\begin{aligned}P_1(\lambda) \equiv &-(4174336\lambda^2 + 1642672\lambda - 115389)q(\lambda) + 1578967040\lambda^3 -\\ &- 712893440\lambda^2 + 332211600\lambda - 18740295,\end{aligned}$$

$$\begin{aligned}P_2(\lambda) \equiv &(4174336\lambda^2 + 1642672\lambda - 115389)q(\lambda) + 1578967040\lambda^3 -\\ &- 712893440\lambda^2 + 332211600\lambda - 18740295.\end{aligned}$$

The general solution of (6) has one arbitrary parameter and can be expressed in elliptic functions. The Timoshkova's substitution gives four one-parameter sets of solutions for each value of λ. The case of $C = -\frac{16}{5}$ and $\lambda = \frac{1}{9}$ has been considered in [69].

4. Results of the Painlevé test

The Ablowitz–Ramani–Segur algorithm of the Painlevé test appears very useful to find asymptotic solutions as a formal Laurent series.

We assume that the behavior of solutions in a sufficiently small neighborhood of the singularity is algebraic, it means that x and y tend to infinity as some powers of $t - t_0$:

$$x = a_\alpha (t - t_0)^\alpha \qquad \text{and} \qquad y = b_\beta (t - t_0)^\beta, \tag{9}$$

where α, β, a_α and b_β are some constants. We assume that real parts of α or β are less then zero, and, of course, $a_\alpha \neq 0$ and $b_\beta \neq 0$.

If α and β are integer numbers, then substituting

$$x = a_\alpha (t - t_0)^\alpha + \sum_{k=1}^{N_{max}} a_{k+\alpha} (t - t_0)^{k+\alpha} \qquad \text{and} \qquad y = b_\beta (t - t_0)^\beta + \sum_{k=1}^{N_{max}} b_{k+\beta} (t - t_0)^{k+\beta} \tag{10}$$

one can transform the ODE system into a set of linear algebraic systems in coefficients a_k and b_k. With the help of some computer algebra system, for example, the system **REDUCE** [70, 71], these systems can be solved step by step and solutions can be automatically found with any accuracy. But previously one has to determine values of constants α, β, a_α and b_β and to analyze systems with zero determinants. Such systems correspond to new arbitrary constants or have no solutions. Powers at which new arbitrary constants enter are called *resonances*. The Painlevé test gives all information about possible dominant behaviors and resonances (see, for example, [10]). Moreover, the results of the Painlevé analysis point out cases, in which it is useful to include into expansion terms with fractional powers of $t - t_0$.

For the generalized Hénon-Heiles system there exist two possible dominant behaviors and resonance structures [10, 44, 72]:

Case 1:	*Case 2*: $(\beta < \Re e(\alpha) < 0)$
$\alpha = -2$, $\beta = -2$, $a_\alpha = \pm 3\sqrt{2 + C}$, $b_\beta = -3$, $r = -1,\ 6,\ \frac{5}{2} - \frac{\sqrt{1-24(1+C)}}{2},\ \frac{5}{2} + \frac{\sqrt{1-24(1+C)}}{2}$	$\alpha = \frac{1 \pm \sqrt{1-48/C}}{2}$, $\beta = -2$, $a_\alpha = c_1$ **(an arbitrary number)**, $b_\beta = \frac{6}{C}$, $r = -1,\ 0,\ 6,\ \mp\sqrt{1 - \frac{48}{C}}$

In the Table the values of r denote resonances: $r = -1$ corresponds to arbitrary parameter t_0; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter c_1. Other values of r determine powers of t, to be exact, $t^{\alpha+r}$ for x and $t^{\beta+r}$ for y, at which new arbitrary parameters enter (as solutions of systems with zero determinants).

For integrability of system (2) all values of α and r have to be integer (or rational) and all systems with zero determinants have to have solutions at any values of included in them free parameters. It is possible only in the integrable cases (i) — (iii).

Those values of C, at which α and r are integer (or rational) numbers either only in the *Case 1* or only in the *Case 2*, are of interest for search of special solutions.

Let's consider all cases, when there exist special (no singular) solutions, representable as a three-parameter Laurent series (may be, multiplied on $\sqrt{t - t_0}$). From the requirement that all values of r but one are integer and positive we obtain, that the general solution can be represented as a

Laurent series or at $C = -1$ and $C = -\frac{4}{3}$ (the *Case 1*), or at $C = -\frac{16}{5}$, $C = -6$ and $C = -16$ (the *Case 2*, $\alpha = \frac{1-\sqrt{1-48/C}}{2}$). And also at $C = -2$, when two types of singular behaviour coincide. Let's consider all these possibilities.

At $C = -2$ (in the *Case 1*) $a_\alpha = 0$. It is the consequence of the fact that, contrary to our assumption, the behaviour of the solution in the neighborhood of a singular point is not algebraic, because its dominant term includes logarithm [10]. At $C = -6$ and any value λ the exact four-parameter solutions are known. In cases $C = -1$ and $C = -16$ the substitution of unknown function as Laurent series gives the equations in λ: accordingly $\lambda = 1$ and $\lambda = \frac{1}{16}$, hence, in nonintegrable cases special three-parameter local solutions have to include logarithmic terms. Single-valued three-parameter solutions can exist only in two nonintegrable cases, at $C = -\frac{16}{5}$ and at $C = -\frac{4}{3}$. It is remarkable, that new solutions have been obtained [68] namely in these nonintegrable cases. In this paper we show how these one-parameter exact periodic solutions can be generalized to three-parameter Laurent series solutions.

5. New solutions

5.1. Finding of solutions in the form of formal Laurent series

Let us consider the Hénon–Heiles system with $C = -\frac{16}{5}$. In the *Case 1* some values of r are not rational, so it is a nonintegrable system. To find special asymptotic solutions let us consider the *Case 2*. In this case $\alpha = -\frac{3}{2}$ and $r = -1,\ 0,\ 4,\ 6$, hence, in the neighborhood of the singular point t_0 we have to seek x in such form that x^2 can be expand into Laurent series, beginning from $(t-t_0)^{-3}$. Let $t_0 = 0$, substituting

$$x = \sqrt{t}\left(c_1 t^{-2} + \sum_{j=-1}^{\infty} a_j t^j\right) \qquad \text{and} \qquad y = -\frac{15}{8}t^{-2} + \sum_{j=-1}^{\infty} b_j t^j$$

in (2), we obtain the following sequence of linear system in a_k and b_k:

$$\left\{\begin{array}{l} \Big(k^2-4\Big)a_k + 2c_1 b_k = -\lambda a_{k-2} - 2\displaystyle\sum_{j=-1}^{k-1} a_j b_{k-j-2}, \\ \Big((k-1)k-12\Big)b_k = -\,b_{k-2} - \displaystyle\sum_{j=-2}^{k-1} a_j a_{k-j-3} - \frac{16}{5}\sum_{j=-1}^{k-1} b_j b_{k-j-2}. \end{array}\right. \tag{11}$$

If $k = 2$ or $k = 4$, then the determinant of (11) is equal to zero. To determine a_2 and b_2 we have the following system:

$$\left\{\begin{array}{l} c_1\Big(557056c_1^8 + (15552000\lambda - 4860000)c_1^4 + 864000000b_2 + \\ + 108000000\lambda^2 - 67500000\lambda + 10546875\Big) = 0, \\ 818176c_1^8 + \Big(15660000\lambda - 4893750\Big)c_1^4 - \\ -\,810000000b_2 - 6328125 = 0. \end{array}\right. \tag{12}$$

As one can see this system does not include terms, which are proportional to a_2, hence, a_2 is an arbitrary parameter (a constant of integration).

We discard the solution with $c_1 = 0$ and obtain the system in $\tilde{c}_1 \equiv c_1^4$ and b_2. System (12) has solutions only if

$$\begin{aligned} \tilde{c}_1 &= \frac{1125(4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 1680\lambda + 525)}{167552}, \\ \tilde{c}_1 &= \frac{1125(-4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)} - 1680\lambda + 525)}{167552}. \end{aligned} \tag{13}$$

We obtain new constant of integration a_2, but we must fix c_1, so number of constants of integration is equal to 2. It is easy to verify that b_4 is an arbitrary parameter, because the corresponding system is equivalent to one linear equation. So, using Painlevé test, we obtain four solutions which depend on three parameters, namely t_0, a_2 and b_4. With the help of some computer algebra system these solutions can be obtained with arbitrary accuracy.

At $C = -\frac{4}{3}$ the situation is similar. In the *Case 1* we have $r = -1,\ 1,\ 4,\ 6$. Substituting

$$x = \sqrt{6}t^{-2} + \sum_{k=-1}^{\infty} d_k t^k \qquad \text{and} \qquad y = -3t^{-2} + \sum_{k=-1}^{\infty} f_k t^k$$

in system (2), we receive a sequence of linear systems in d_k and f_k:

$$\begin{cases} \Big((k-1)k - 6\Big)d_k + 2\sqrt{6}f_k = -\lambda d_{k-2} - 2\sum\limits_{j=-1}^{k-1} d_j f_{k-j-2}, \\ 2\sqrt{6}d_k + \Big((k-1)k - 8\Big)f_k = -f_{k-2} - \sum\limits_{j=-1}^{k-1} d_j d_{k-j-2} - \dfrac{4}{3}\sum\limits_{j=-1}^{k-1} f_j f_{k-j-2}. \end{cases} \tag{14}$$

The systems corresponding to $k = -1,\ 2,\ 4$ have a zero determinant. The first system ($k = -1$) always has infinite number of solutions and f_{-1} is a parameter. We have to fix this parameter to solve the system corresponding to $k = 2$. This system has solutions (f_2 is a new arbitrary parameter) only if

$$\begin{aligned} f_{-1} &= \pm\sqrt{\frac{\sqrt{7(1216\lambda^2 - 1824\lambda + 783)} - 140\lambda + 105}{385}}, \\ f_{-1} &= \pm\sqrt{\frac{-\sqrt{7(1216\lambda^2 - 1824\lambda + 783)} - 140\lambda + 105}{385}}, \\ f_{-1} &= 0. \end{aligned} \tag{15}$$

Similarly to case $C = -\frac{16}{5}$ at $k = 4$ the system is reduced to one equation. Thus, at $C = -\frac{4}{3}$ we have five three-parameter (t_0, f_2 and f_4) solutions. If we choose $f_{-1} = 0$, then we obtain three-parameter solution which generalizes known two-parameter solution in terms of Weierstrass elliptic functions. Other solutions generalize four one-parameter solutions, found in [68]. So, we can made a conclusion that all one-parameter solutions for both $C = -\frac{16}{5}$ and $C = -\frac{4}{3}$ can be generalized.

5.2. Convergence of the obtained series

When a formal series is obtained the question about its convergence arises. The convergence of psi-series solutions of the generalized Hénon–Heiles system on some real time interval has been proved in [72]. For Laurent series solutions it is easy to find conditions, at which the obtained series converge at $0 < |t| \leqslant 1 - \varepsilon$, where ε is any positive number. Our series converge in the above-mentioned ring, if $\exists N$ such that $\forall n > N \quad |a_n| \leqslant M$ and $|b_n| \leqslant M$. Let $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for all $-1 < n < k$, then from (8) we obtain:

$$|a_k| \leqslant \frac{2M(k+1) + |\lambda| + 2|c_1|}{|k^2 - 4|} M, \qquad |b_k| \leqslant \frac{21Mk + 26M + 5}{5|k^2 - k - 12|} M. \tag{16}$$

It is easy to see that there exists such N that if $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n \leqslant N$, then $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n < \infty$. So one can prove the convergence, analyzing values of a finite number of the first coefficients of series.

6. Conclusion

Using the Painlevé analysis one can not only find integrable cases of dynamical systems, but also construct special solutions in nonintegrable cases.

We have found the special solutions of the Hénon–Heiles system with $C = -\frac{16}{5}$ and $C = -\frac{4}{3}$ as Laurent series, depending on three parameters. For some values of two parameters the obtained solutions coincide with the known exact periodic solutions. Any obstacle to the existence of three-parameter single valued solutions is absent, so, the probability of finding of new exact two- or may be even three-parameter solutions, which generalize the solutions found in [68], is high.

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References

[1] *S. Kowalevski (S.V. Kovalevskaya)*, Sur le problème de la rotation d'un corps solide autour d'un point fixe, *Acta Mathematica*, 1889, vol. 12, pp. 177–232; Sur une properiété du sustème d'équations différentielles qui définit la rotation d'un corps solide autour d'un point fixe, *Acta Mathematica*, 1890, vol. 14, pp. 81–93 {in French}. Reprinted in: *S.V. Kovalevskaya*, ***Scientific Works***, AS USSR Publ. House, Moscow, 1948 {in Russian}.

[2] *V.V. Golubev*, ***Lectures on the Integration of the Equation of Motion of a Heavy Rigid Body about a Fixed Point***, Gostekhizdat (State Pub. House), Moscow, 1953, reprinted: RCD, Moscow–Izhevsk, 2002 {in Russian}. Israel program for scientific translations, 1960 {in English}.

[3] *A. Goriely*, A brief history of Kovalevskaya exponents and modern developments, *Regular and Chaotic Dynamics*, 2000, vol. 5, № 1, pp. 3–16.

[4] *V.V. Golubev*, ***Lectures on Analytical Theory of Differential Equations***, Gostekhizdat (State Pub. House), Moscow–Leningrad, 1950 {in Russian}.

[5] *E. Hille*, ***Ordinary Differential Equations in the Complex Domain***, New York, Wiley, 1976.

[6] *P. Painlevé*, ***Leçons sur la théorie analytique des eq́uations différentelles (Leçons de Stockholm, 1895)***, Paris, 1896. Reprinted in: ***Oeuvres de P. Painlevé***, vol. 1, ed. du CNRS, Paris, 1973.

[7] *P. Painlevé,* Mémoire sur les eq́uations différentelles dont l'intégrale générale est uniforme, *Bull Soc. Math. France*, 1900, vol. 28, pp. 201–261; Sur les eq́uations différentelles du second order d'ordre supérieur dont l'intégrale générale est uniforme, *Acta Mathematica*, 1902, vol. 25, pp. 1–85. Reprinted in: ***Oeuvres de P. Painlevé***, vol. 1, ed. du CNRS, Paris, 1973.

[8] *T. Bountis, H. Segur, F. Vivaldi,* Integrable Hamiltonian Systems and the Painlevé Property, *Physical Review A*, 1982, vol. 25, pp. 1257–1264.

[9] *A. Ramani, B. Grammaticos, T. Bountis,* The Painlevé Property and Singularity Analysis of Integrable and Nonintegrable Systems, *Physics Reports*, 1989, vol. 180, pp. 159–245.

[10] *M. Tabor,* ***Chaos and Integrability in Nonlinear Dynamics***, Wiles, New York, 1989.

[11] *M. Tabor, J. Weiss,* Analytic structure of the Lorenz system, *Phys. Rev. A* , 1981, vol. 24, pp. 2157–2167.

[12] *G. Levine, M. Tabor,* Integrating the nonintegrable : analytic structure of the Lorenz system revisited, *Physica D*, 1988, vol. 33, pp. 189–210.

[13] *T. Sen, M. Tabor,* Lie symmetries of the Lorenz model, *Physica D*, 1990, vol. 44, pp. 313–339.

[14] *N. Ercolani, E.D. Siggia,* Painlevé Property and Integrability, *Phys. Lett. A*, 1986, vol. 119, pp. 112–116.

[15] *N. Ercolani, E.D. Siggia,* Painlevé Property and Geometry, *Physica D*, 1989, vol. 34, pp. 303–346.

[16] *H. Yoshida,* Necessary Conditions for the Existence of Algebraic First Integrals. I & II, *Celest. Mech.*, 1983, vol. 31, pp. 363–379, 381–399; A new Necessary Condition for the Integrability of Hamiltonian Systems with a Two-dimensional Homogeneous Potential, *Physica D*, 1999, vol. 128, pp. 53–69.

[17] *K. Nakagawa, H. Yoshida*, A necessary condition for the integrability of homogeneous Hamiltonian systems with two degrees of freedom, *J. Phys. A: Math. Gen.*, vol. 34, pp. 2137-2148.

[18] *V.V. Kozlov,* ***Symmetry, topology and resonances in Hamiltonian mechanics***, Izhevsk, UGU Publ. House, 1995 {in Russian}.

[19] *A. Goriely, M. Tabor,* The Role of Complex-time Singularities in Chaotic Dynamics, *Regular Chaotic Dynamics*, 1998, vol. 3, № 3, pp. 32–44.

[20] *R. Conte* (ed.), ***The Painlevé property, one century later*** {Proceedings of the Cargèse school (3–22 June, 1996)}, CRM series in mathematical physics, 810 pages, Springer–Verlag, Berlin, 1998, Springer, New York, 1999.

{*M. Musette*, Painlevé analysis for nonlinear partial differential equations, *The Painlevé property, one century later*, pp. 517–572, **arXiv:solv-int/9804003**;

R. Conte, The Painlevé approach to nonlinear ordinary differential equations, *The Painlevé property, one century later*, pp. 77–180, **arXiv:solv-int/9710020** }.

[21] *M.J. Ablowitz, A. Ramani, H. Segur,* Nonlinear evolution equations and ordinary differential equations of Painlevé type, *Lett. Nuovo Cimento*, 1978, v. 23, pp. 333–338; A Connection between Nonlinear Evolution Equations and Ordinary Differential Equations of P-type. I & II, *J. Math. Phys.*, 1980, vol. 21, pp. 715–721, 1006–1015.

[22] *J. Weiss, M. Tabor, G. Carnevale,* The Painlevé Property for Partial Differential Equations, *J. Math. Phys.*, 1983, vol. 24, pp. 522–526.

[23] *J. Weiss,* The Painlevé Property for Partial Differential Equations. II: Bäcklund Transformation, Lax Pairs and the Schwarzian derivative, *J. Math. Phys.*, 1983, vol. 24, pp. 1405–1413.

[24] *J. Weiss,* The Sine-Gordon equations: Complete and Partial Integrability, *J. Math. Phys.*, 1984, vol. 25, pp. 2226–2235.

[25] *J. Weiss,* Bäcklund Transformation and the Painlevé Property, *J. Math. Phys.*, 1986, vol. 27, pp. 1293–1305.

[26] *R. Conte*, Universal invariance properties of Painlevé analysis and Bäcklund transformation in nonlinear partial differential equations, *Phys. Lett. A*, 1988, vol. 134, pp. 100–104.

[27] *A.C. Newell, M. Tabor, Y.B. Zeng,* A Unified Approach to Painlevé Expansions, *Physica D*, 1987, vol. 29, pp. 1–68.

[28] *H. Flaschka, A.C. Newell, M. Tabor,* Integrability, in *V.E. Zakharov* (ed.), ***What is Integrability?***, series in nonlinear dynamics, Springer–Verlag, Berlin, Springer, New York, 1991, pp. 73–114.

[29] *J.D. Gibbon, P. Radmore, M. Tabor, D. Wood,* The Painlevé property and Hirota's method, *Stud. Appl. Math.*, 1985, vol. 72, pp. 39–63.

[30] *R. Conte,* The Painlevé Analysis of nonlinear PDE and related topics: a computer algebra program, preprint, 1988; Invariant Painlevé Analysis of Partial Differential Equations, *Phys. Lett. A*, 1989, vol. 140, pp. 383–390.

[31] *R. Conte, A.P. Fordy, A. Pickering,* A Perturbative Painlevé Approach to Nonlinear Differential Equations, *Physica D*, 1993, vol. 69, pp. 33–58.

[32] *R. Conte, M. Musette,* Riccati Pseudopotential of AKNS two-family NLPDES by Painlevé Analysis, *TMF (Russ. J. Theor. Math. Phys.)*, 1994, vol. 99, pp. 478–486 {in English}.

[33] *F. Cariello, M. Tabor,* Painlevé Expansions for Nonintegrable Evolution Equations, *Physica D*, 1989, vol. 39, pp. 77–94.

[34] *R. Conte,* Exact solutions of nonlinear partial differential equations by singularity analysis, **arXiv:nlin.SI**/0009024, 2000.

[35] *J. Weiss,* Bäcklund Transformation and Linearizations of the Hénon–Heiles System, *Phys. Lett. A*, 1984, vol. 102, pp. 329–331.

[36] *J. Weiss,* Bäcklund Transformation and the Hénon–Heiles System, *Phys. Lett. A*, 1984, vol. 105, pp. 387–389.

[37] *J.D. Gibbon, A.C. Newell, M. Tabor, Y.B. Zeng*, Lax Pairs, Bäcklund Transformation and Special Solutions for Ordinary Differential Equations, *Nonlinearity* , 1988, vol. 1, pp. 481–490.

[38] *R. Sahadevan,* Painlevé Expansion and Exact Solution for Nonintegrable Evolution Equations, *TMF (Russ. J. Theor. Math. Phys.)* 1994, vol. 99, pp. 528–536 {in English}.

[39] *J. Springael, R. Conte, M. Musette,* On the exact solutions of the Bianchi IX cosmological model in the proper time, *Regular Chaotic Dynamics* {http://old.rcd.ru/rcd/}, 1998, vol. 3, № 1, pp. 3–8, **arXiv:solv-int/9804008,** 1998.

[40] *G. Contopoulos,* A Third Integral of Motion in a Galaxy, *Zeitschrift für Asrtophysik*, 1960, vol. 49, pp. 273–291; On the Existence of a Third Integral of Motion, *Astron. J.*, 1963, vol. 68, pp. 1–14; Resonance Cases and a Small Divisors in a Third Integral of Motion, *Astron. J.*, 1963, vol. 68, pp. 763–779.

[41] *A.G. Gustavson,* On Constructing Formal Integrals of a Hamiltonian System Near an Equilibrium Point, *Astronomical J.*, 1966, vol. 71, pp. 670–686.

[42] *M. Hénon, C. Heiles,* The Applicability of the Third Integral of Motion: Some Numerical Experiments, *Astronomical J.*, 1964, vol. 69, pp. 73–79.

[43] *Y.F. Chang, M. Tabor, J. Weiss, G. Corliss,* On the Analytic Structure of the Hénon–Heiles System, *Phys. Lett. A*, 1981, vol. 85, pp. 211–213.

[44] *Y.F. Chang, M. Tabor, J. Weiss,* Analytic Structure of the Hénon–Heiles Hamiltonian in integrable and nonintegrable regimes, *J. Math. Phys.*, 1982, vol. 23, pp. 531–538.

[45] *J. Greene,* Preprint La Jolla Institute LJI–TN–81–122, 1981.

[46] *B. Grammaticos, B. Dorizzi, R. Padjen,* Painlevé Property and Integrals of Motion for the Hénon–Heiles System, *Phys. Lett. A*, 1982, vol. 89, pp. 111–113.

[47] *L.S. Hall,* A Theory of Exact and Approximate Configurational Invariants, *Physica D*, 1983, vol. 8, pp. 90–106.

[48] *A.P. Fordy,* Hamiltonian Symmetries of the Hénon–Heiles System, *Phys. Lett. A*, 1983, vol. 97, pp. 21–23.

[49] *A.P. Fordy,* The Hénon–Heiles System Revisited, *Physica D*, 1991, vol. 52, pp. 204–210.

[50] *M. Antonowicz, S. Rauch-Wojciechowski,* Bi-Hamiltonian formulation of the Hénon–Heiles system and its multidimensional extensions, *Phys. Lett. A*, 1992, vol. 163, pp. 167–172.

[51] *D.L. Rod,* J. of Differential Equations, 1973, vol. 14, P. 129.

[52] *Ji. Podolský, K. Veselý,* Chaos in $pp-$wave spacetime, *Phys. Rev. D*, 1998, vol. 58, 081501.

[53] *F. Kokubun,* Gravitational waves from the Hénon–Heiles system, *Phys. Rev. D,* 1998, vol. 57, pp. 2610–2612.

[54] *Y. Guo, C. Grotta Ragazza,* Pure Appl. Math, 1996, V. XLIX, P. 1145.

[55] *G. Tondo,* A connection between the Hénon–Heiles system and the Garnier system, *TMF (Russ. J. Theor. Math. Phys.)*, 1994, vol. 99, pp. 552–559.

[56] *F. Kokubun,* Gravitational waves from the Newtonian plus Hénon–Heiles system, *Phys. Lett. A*, 1998, vol. 245, pp. 358–362.

[57] *R. Conte, M. Musette, C. Verhoeven,* Integration of a generalized Hénon–Heiles Hamiltonian, *J. Math. Phys.*, 2002, vol. 43, pp. 1906–1915, **arXiv:nlin.SI/0112030,** 2001; General solution for Hamiltonians with extended cubic and quartic potensials, **arXiv:nlin.SI/0301011,** 2003.

[58] *J.K. Moser,* ***Lecture on Hamiltonian Systems***, *Memoirs of the AMS*, 1968, vol. 81, pp. 1–60 {in English}, 1973, "Mir", Moscow {in Russian}.

[59] *M. Braun,* On the Applicability of the Third Integral of Motion, *J. of Differential Equations*, 1973, vol. 13, pp. 300–318.

[60] *S. Kasperczuk,* Normal Forms, Lie-Poisson Structure and Reduction for the Hénon–Heiles System, *Celect. Mech. Dynamics Astr.*, 1995, vol. 63, pp. 245–253.

[61] *A.D. Bruno,* ***Local method in Nonlinear Differential Equations***, Springer Series in Soviet Mathematics, ISBN-3-540-18926-2, 1989.

[62] *A.D. Bruno,* ***Power Geometry in Algebraic and Differential Equations***, Moscow, Nayka, 1998 {in Russian}.

[63] *V.F. Edneral,* A Symbolic Approximation of Periodic Solutions of the Hénon–Heiles System by the Normal Form Method, *Mathematics and Computers in Simulations*, 1998, vol. 45, pp. 445–463.

[64] *V.F. Edneral,* Bifurcation Analysis of Low Resonant Case of the Generalized Hénon–Heiles System, *Proc. of CASC 2001 (Konstanz, Germany, 2001)*, ed. by Ganzha et al., Springer, 2001, pp. 167–176.

[65] *L. Brenig, A. Goriely,* Painlevé Analysis and Normal Forms, in E. Turnier (ed.), ***Computer Algebra and Differential Equations***, Cambridge University Press, Cambridge, 1994, pp. 211–238.

[66] *A. Goriely,* Painlevé Analysis and Normal Forms Theory, *Physica D*, 2001, vol. 152–153, pp. 124–144.

[67] *A. Erdelyi et al.* eds. ***Higher Transcendental Functions (based, in part, on notes left by H. Bateman)***. Vol. 3, MC Graw-Hill Book Company, New York, Toronto, London, 1955 {in English}, "Nauka", Moscow, 1967 {in Russian}, 1990 {in Russian}.

[68] *E.I. Timoshkova,* A New Class of Trajectories of Motion in the Hénon–Heiles Potential Field, *Rus. Astron. J.*, 1999, vol. 76, pp. 470–475 {in Russian}, *Astron. Rep.*, 1999, vol. 43, P. 406 {in English}.

[69] *S.Yu. Vernov,* The Painlevé Analysis and Special Solutions for Nonintegrable Systems, **arXiv:math-ph/0203003;** Construction of solutions for the generalized Henon-Heiles system with the help of the Painleve test, Preprint SINP MSU 2002-21/705, 21pp., **arXiv:math-ph/0209063,** 2002.

[70] *A.C. Hearn,* ***REDUCE. User's Manual. Vers. 3.6***, RAND Publ. CP78, 1995.

[71] *V.F. Edneral, A.P. Kryukov, A.Ya. Rodionov,* ***The Language for Algebraic Computation REDUCE***, Moscow, MSU Publ. House, 1989 {in Russian}.

[72] *S. Melkonian,* Psi-series Solutions of the Cubic Hénon–Heiles System and Their Convergence, *J. of Nonlin. Math. Phys.*, 1999, vol. 6, pp. 139–160, **arXiv:math.DS/9904186**.