# MODEL OF THE UNIVERSE EVOLUTION FROM INFLATION STAGE UP TO POST-FRIEDMANN ERA

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It is proposed to consider dark matter as the perfect dilaton-spin fluid (the particles of which are endowed with intrinsic spin and dilaton charge) within the framework of the gravitational theory with the Weyl–Cartan geometrical structure. The modified Friedmann–Lemaître equation (with a cosmological term) is obtained for the homogeneous and isotropic Universe filled with the dilaton-spin dark matter. On basis of this equation we develop the nonsingular model of evolution of the Universe starting from an inflation-like stage (for superrigid equation of state), passing radiation dominated and matter dominated decelerating stages and turning into the post-Friedmann accelerating era.

## 1. Introduction

The basis concept of the modern fundamental physics consists in the preposition that the spacetime geometrical structure is compatible with the properties of matter filling the spacetime. It means that the matter dynamics determines the metric and the connection of the spacetime manifold and in turn is determined by the spacetime geometric properties. Therefore the possible deviation from the geometrical structure of the General Relativity spacetime should be stipulated by the existence of matter with unusual properties, which fills spacetime, generates its structure and interacts with it. As examples of such matter there were considered the perfect media with intrinsic degrees of freedom, such as the perfect fluid with spin and non-Abelian colour charge [1], the perfect hypermomentum fluid (see [2]–[4] and the references therein), the perfect dilaton-spin fluid [5]. All these fluids are the generalization of the perfect Weyssenhoff–Raabe spin fluid [6]. The modern observations [7, 8] lead to the conclusion about the existence of dark (nonluminous) matter with the density exceeding by one order of magnitude the density of baryonic matter, from which stars and luminous components of galaxies are formed. It is dark matter interacting with the equal by order of magnitude positive vacuum energy or with quintessence [9, 10] that realizes the dynamics of the Universe in modern era. Another important consequence of the modern observations is an understanding of the fact that the end of the Friedmann era occurs when the expansion with deceleration is succeeded by the expansion with acceleration, the transition to the unrestrained exponential expansion being possible.

The hypothesis about the existence of dark matter into galaxies was proposed by Zwicky in his pioneering work [11]. But the essence of dark matter is yet unknown. The hypothesis that dark matter was endowed with a new kind of gravitational charge, which generates a short-range gravitational interaction of Proca type, was proposed in [12]. This interaction is best appreciated in terms of the existence of Weyl–Cartan spacetime. Independently in [13] it was shown that Weyl–Cartan geometry was generated by a perfect dilaton-spin fluid and the corresponding non-singular cosmological model was constructed. Then in [14] the hypothesis on the perfect dilaton-spin fluid as the model of dark matter was proposed within the framework of the gravitational theory with Weyl–Cartan geometrical structure.

Within the framework of these ideas the modified Friedmann–Lemeître (FL) equation with a cosmological term for the homogeneous and isotropic Universe filled with the dilaton-spin dark matter was constructed for an arbitrary equation of state of dark matter [13]–[15]. In present paper the solutions of this equation for the various equations of state are received. The inflation-like solution is obtained in case of the superrigid equation of state of dark matter at the early stage of the Universe. The evolution of the Universe starts from very small, but non-zero size, then passes Friedmann decelerating stage and turns into the post-Friedmann accelerating era.

Throughout the paper the signature of the metric is assumed to be $(+,+,+,-)$ and the conventions $c=1$, $\hbar=1$ are used.

## 2. Weyl–Cartan space

Let us consider a connected 4-dimensional oriented differentiable manifold $\mathcal{M}$ equipped with a metric $\breve{g}$ of index 1, a linear connection $\Gamma$ and a volume 4-form $\eta$. Then a Weyl–Cartan space $CW_4$ is defined as the space equipped with a curvature 2-form $\mathcal{R}^{\alpha}{}_{\beta}$ and a torsion 2-form $\mathcal{T}^{\alpha}$ with the metric tensor and the connection 1-form obeying the condition [1]

$$-\mathrm{D}g_{\alpha\beta}=\mathcal{Q}_{\alpha\beta}=\frac{1}{4}g_{\alpha\beta}\mathcal{Q}\ ,\qquad \mathcal{Q}=g^{\alpha\beta}\mathcal{Q}_{\alpha\beta}=Q_{\alpha}\theta^{\alpha}\ , \tag{2.1}$$

where $\mathcal{Q}_{\alpha\beta}$ – a nonmetricity 1-form, $\mathcal{Q}$ – a Weyl 1-form and $\mathrm{D}=\mathrm{d}+\Gamma\wedge\ldots$ – the exterior covariant differential. Here $\theta^{\alpha}$ $(\alpha=1,2,3,4)$ – cobasis of 1-forms of the $CW_4$-space ($\wedge$ – the exterior product operator).

A curvature 2-form $\mathcal{R}^{\alpha}{}_{\beta}$ and a torsion 2-form $\mathcal{T}^{\alpha}$,

$$\mathcal{R}^{\alpha}{}_{\beta}=\frac{1}{2}R^{\alpha}{}_{\beta\gamma\lambda}\theta^{\gamma}\wedge\theta^{\lambda}\ ,\qquad \mathcal{T}^{\alpha}=\frac{1}{2}T^{\alpha}{}_{\beta\gamma}\theta^{\beta}\wedge\theta^{\gamma}\ ,\qquad T^{\alpha}{}_{\beta\gamma}=-2\Gamma^{\alpha}{}_{[\beta\gamma]}\ .$$

are defined by virtue of the Cartan's structure equations,

$$\mathcal{R}^{\alpha}{}_{\beta}=\mathrm{d}\Gamma^{\alpha}{}_{\beta}+\Gamma^{\alpha}{}_{\gamma}\wedge\Gamma^{\gamma}{}_{\beta}\ ,\qquad \mathcal{T}^{\alpha}=\mathrm{D}\theta^{\alpha}=\mathrm{d}\theta^{\alpha}+\Gamma^{\alpha}{}_{\beta}\wedge\theta^{\beta}\ . \tag{2.2}$$

The Bianchi identities for the curvature 2-form, the torsion 2-form and the Weyl 1-form are valid [16],

$$\mathrm{D}\mathcal{R}^{\alpha}{}_{\beta}=0\ ,\qquad \mathrm{D}\mathcal{T}^{\alpha}=\mathcal{R}^{\alpha}{}_{\beta}\wedge\theta^{\beta}\ ,\qquad \mathrm{d}\mathcal{Q}=2\mathcal{R}^{\gamma}{}_{\gamma}\ . \tag{2.3}$$

It is convenient to use the auxiliary fields of 3-forms $\eta_{\alpha}$, 2-forms $\eta_{\alpha\beta}$, 1-forms $\eta_{\alpha\beta\gamma}$ and 0-forms $\eta_{\alpha\beta\gamma\lambda}$ with the properties,

$$\eta_{\alpha}=\vec{e}_{\alpha}\rfloor\eta=*\theta_{\alpha}\ ,\qquad \eta_{\alpha\beta\gamma}=\vec{e}_{\gamma}\rfloor\eta_{\alpha\beta}=*(\theta_{\alpha}\wedge\theta_{\beta}\wedge\theta_{\gamma})\ , \tag{2.4}$$

$$\eta_{\alpha\beta}=\vec{e}_{\beta}\rfloor\eta_{\alpha}=*(\theta_{\alpha}\wedge\theta_{\beta})\ ,\qquad \eta_{\alpha\beta\gamma\lambda}=\vec{e}_{\lambda}\rfloor\eta_{\alpha\beta\gamma}=*(\theta_{\alpha}\wedge\theta_{\beta}\wedge\theta_{\gamma}\wedge\theta_{\lambda})\ . \tag{2.5}$$

Here $*$ is the Hodge operator and $\rfloor$ is the operation of contraction (interior product) which obeys to the Leibnitz antidifferentiation rule,

$$\vec{v}\rfloor(\Phi\wedge\Psi)=(\vec{v}\rfloor\Phi)\wedge\Psi+(-1)^{p}\Phi\wedge(\vec{v}\rfloor\Psi)\ , \tag{2.6}$$

where $\Phi$ is a $p$-form.

[1] Our notations differ by some details from the notations accepted in [16].

The properties (2.4), (2.5) lead to the following useful relations,

$$\theta^\sigma \wedge \eta_\alpha = \delta^\sigma_\alpha \eta \ , \qquad \theta^\sigma \wedge \eta_{\alpha_1 \dots \alpha_p} = (-1)^{p-1} p \delta^\sigma_{[\alpha_1} \eta_{\alpha_2 \dots \alpha_p]} \ , \tag{2.7}$$

$$\theta^\sigma \wedge \theta^\rho \wedge \eta_{\alpha\beta} = 2\delta^\sigma_{[\alpha} \delta^\rho_{\beta]} \eta \ , \tag{2.8}$$

$$\theta^\sigma \wedge \theta^\rho \wedge \eta_{\alpha\beta\gamma} = 3\theta^\sigma \wedge \delta^\rho_{[\alpha} \eta_{\beta\gamma]} = 6\delta^\sigma_{[\alpha} \delta^\rho_\beta \eta_{\gamma]} \ , \tag{2.9}$$

$$\theta_{[\alpha} \wedge *(\theta_{\beta]} \wedge \theta_\gamma) = -\frac{1}{2} \eta_{\alpha\beta} \wedge \theta_\gamma \ , \tag{2.10}$$

$$\vec{e}_\lambda \rfloor (\theta^{\alpha_1} \wedge \dots \wedge \theta^{\alpha_p}) = p \delta^{[\alpha_1}_\lambda \theta^{\alpha_2} \dots \wedge \theta^{\alpha_p]} \ . \tag{2.11}$$

In space $CW_4$ the equality $\mathrm{D}\eta = 0$ is fulfilled and the following formulae are valid [16],

$$\mathrm{D}\eta_{\alpha\beta\gamma\lambda} = -\frac{1}{2} \mathcal{Q} \eta_{\alpha\beta\gamma\lambda} \ , \qquad \mathrm{D}\eta_{\alpha\beta\gamma} = -\tfrac{1}{2} \mathcal{Q} \wedge \eta_{\alpha\beta\gamma} + \mathcal{T}^\lambda \eta_{\alpha\beta\gamma\lambda} \ , \tag{2.12}$$

$$\mathrm{D}\eta_{\alpha\beta} = -\frac{1}{2} \mathcal{Q} \wedge \eta_{\alpha\beta} + \mathcal{T}^\gamma \wedge \eta_{\alpha\beta\gamma} \ , \qquad \mathrm{D}\eta_\alpha = -\tfrac{1}{2} \mathcal{Q} \wedge \eta_\alpha + \mathcal{T}^\beta \wedge \eta_{\alpha\beta} \ . \tag{2.13}$$

In a Weyl–Cartan space the following decomposition of the connection 1-form is valid,

$$\Gamma^\alpha{}_\beta = \overset{C}{\Gamma}{}^\alpha{}_\beta + \Delta^\alpha{}_\beta \ , \qquad \Delta_{\alpha\beta} = \frac{1}{8}(2\theta_{[\alpha} Q_{\beta]} + g_{\alpha\beta} \mathcal{Q}) \ , \tag{2.14}$$

where $\overset{C}{\Gamma}{}^\alpha{}_\beta$ denotes a connection 1-form of a Riemann–Cartan space $U_4$ with curvature, torsion and a metric compatible with a connection. This decomposition of the connection induces corresponding decomposition of the curvature 2-form [5],

$$\mathcal{R}^\alpha{}_\beta = \overset{C}{\mathcal{R}}{}^\alpha{}_\beta + \overset{C}{\mathrm{D}} \Delta^\alpha{}_\beta + \Delta^\alpha{}_\gamma \wedge \Delta^\gamma{}_\beta = \overset{C}{\mathcal{R}}{}^\alpha{}_\beta + \frac{1}{4} \delta^\alpha_\beta \mathcal{R}^\gamma{}_\gamma + \mathcal{P}^\alpha{}_\beta \ , \tag{2.15}$$

$$\mathcal{P}_{\alpha\beta} = \frac{1}{4} \left( \mathcal{T}_{[\alpha} Q_{\beta]} - \theta_{[\alpha} \wedge \overset{C}{\mathrm{D}} Q_{\beta]} + \frac{1}{8} \theta_{[\alpha} Q_{\beta]} \wedge \mathcal{Q} - \frac{1}{16} \theta_\alpha \wedge \theta_\beta Q_\gamma Q^\gamma \right) \ , \tag{2.16}$$

where $\overset{C}{\mathrm{D}}$ is the exterior covariant differential with respect to the Riemann–Cartan connection 1-form $\overset{C}{\Gamma}{}^\alpha{}_\beta$ and $\overset{C}{\mathcal{R}}{}^\alpha{}_\beta$ is the Riemann–Cartan curvature 2-form. The decomposition (2.15) contains the Weyl segmental curvature 2-form $\mathcal{R}^\gamma{}_\gamma$ (2.3).

The Riemann–Cartan connection 1-form can be decomposed as follows [16],

$$\overset{C}{\Gamma}{}^\alpha{}_\beta = \overset{R}{\Gamma}{}^\alpha{}_\beta + \mathcal{K}^\alpha{}_\beta \ , \qquad \mathcal{T}^\alpha = \mathcal{K}^\alpha{}_\beta \wedge \theta^\beta \ , \tag{2.17}$$

$$\mathcal{K}_{\alpha\beta} = 2\vec{e}_{[\alpha} \rfloor \mathcal{T}_{\beta]} - \frac{1}{2} \vec{e}_\alpha \rfloor \vec{e}_\beta \rfloor (\mathcal{T}_\gamma \wedge \theta^\gamma) \ , \tag{2.18}$$

where $\overset{R}{\Gamma}{}^\alpha{}_\beta$ is a Riemann (Levi–Civita) connection 1-form and $\mathcal{K}^\alpha{}_\beta$ is a contortion 1-form.

The decomposition (2.17) of the connection induces the decomposition of the curvature as follows,

$$\overset{C}{\mathcal{R}}{}^\alpha{}_\beta = \overset{R}{\mathcal{R}}{}^\alpha{}_\beta + \overset{R}{\mathrm{D}} \mathcal{K}^\alpha{}_\beta + \mathcal{K}^\alpha{}_\gamma \wedge \mathcal{K}^\gamma{}_\beta \ , \tag{2.19}$$

where $\overset{R}{\mathcal{R}}{}^\alpha{}_\beta$ is the Riemann curvature 2-form and $\overset{R}{\mathrm{D}}$ is the exterior covariant differential with respect to the Riemann connection 1-form $\overset{R}{\Gamma}{}^\alpha{}_\beta$.

## 3. The model of dilaton-spin fluid

The perfect dilaton-spin fluid was introduced in [5], where the variational theory of this fluid in a Weyl–Cartan space was developed. The additional degrees of freedom of a fluid element are described with the help of the material frame attached to every fluid element and consisting of four 1-forms $l^p$ $(p = 1, 2, 3, 4)$, which have dual 3-forms $l_q$, the constraint $l^p \wedge l_q = \delta^p_q \eta$ being fulfilled. Each fluid element possesses a 4-velocity vector $\vec{u} = u^\alpha \vec{e}_\alpha$ which corresponds to a flow 3-form $u := \vec{u}\rfloor\eta = u^\alpha \eta_\alpha$ and a velocity 1-form $*u = u_\alpha \theta^\alpha = \breve{g}(\vec{u}, \cdots)$ with $*u \wedge u = -\eta$ that means the usual condition $\breve{g}(\vec{u}, \vec{u}) = -1$.

In case of the dilaton-spin fluid the spin dynamical variable of the Weyssenhoff–Raabe fluid is generalized and becomes the new dynamical variable $J^p{}_q$ named the dilaton-spin tensor:

$$J^p{}_q = S^p{}_q + \frac{1}{4}\delta^p_q J \,, \qquad S_{pq} = J_{[pq]} \,, \qquad J = J^p{}_p \,. \tag{3.1}$$

Here $S^p{}_q$ is the specific (per particle) spin tensor. Spin of particles is spacelike in its nature that is the fact of fundamental physical meaning. This leads to the classical Frenkel condition [17], which can be expressed in the exterior form language in two equivalent forms, $S^p{}_q l_p \wedge *u = 0$, $S^p{}_q l^q \wedge u = 0$. It should be mentioned that the Frenkel condition appears to be a consequence of the generalized conformal invariance of the Weyssenhoff perfect spin fluid Lagrangian [18]. Quantity $J$ in (3.1) is the specific (per particle) dilaton charge of the fluid element. The existence of the dilaton charge is the consequence of the extension of the Poincaré symmetry (with the spin tensor as the dynamical invariant) to the Poincaré–Weyl symmetry with the dilaton-spin tensor as the dynamical invariant.

It is important that only the first term of $J^p{}_q$ (the spin tensor) obeys to the Frenkel condition [5]. As the consequence of this fact the variational theory of the perfect dilaton-spin fluid developed in [5] is the limiting case of the variational theory of the perfect hypermomentum fluid developed in [4], but does not appear to be the limiting case of the variational theory of the hyperfluid developed in [3], in which the Frenkel condition is imposed on the full intrinsic hypermomentum tensor.

The measure of intrinsic motion contained in a fluid element is the quantity $\Omega^q{}_p$ which generalizes the intrinsic 'angular velocity' of the Weyssenhoff spin fluid theory,

$$\Omega^q{}_p \eta = u \wedge l^q_\alpha \mathrm{D} l^\alpha_p \,, \qquad \mathrm{D} l^\alpha_p = \mathrm{d} l^\alpha_p + \Gamma^\alpha{}_\beta l^\beta_p \,.$$

An element of the perfect dilaton-spin fluid possesses the additional intrinsic 'kinetic' energy density 4-form,

$$E = \frac{1}{2} n J^p{}_q \Omega^q{}_p \eta = \frac{1}{2} n S^p{}_q u \wedge l^q_\alpha \mathrm{D} l^\alpha_p + \frac{1}{8} n J u \wedge l^p_\alpha \mathrm{D} l^\alpha_p \,,$$

where $n$ is the fluid particles concentration.

The Lagrangian density 4-form of the perfect dilaton-spin fluid can be constructed from the quantities $\varepsilon$ and $E$ with regard to the constraints (the fluid particles number conservation law, $\mathrm{d}(nu) = 0$, the entropy $s$ conservation law along streamline of the fluid, $\mathrm{d}(nus) = 0$, two forms of Frenkel conditions), which should be introduced into the Lagrangian density by means of the Lagrange multipliers $\varphi$, $\tau$, $\chi$, $\chi^q$ and $\zeta_p$ ,

$$\begin{aligned}\mathcal{L}_{\text{fluid}} = {} & -\varepsilon(n, s, S^p{}_q, J)\eta + \frac{1}{2} n S^p{}_q u \wedge l^q_\alpha \mathrm{D} l^\alpha_p + \frac{1}{8} n J u \wedge l^p_\alpha \mathrm{D} l^\alpha_p + nu \wedge \mathrm{d}\varphi \\ & + n\tau u \wedge \mathrm{d}s + n\chi(* \wedge u + \eta) + n\chi^q S^p{}_q l_p \wedge *u + n\zeta_p S^p{}_q l^q \wedge u \,. \end{aligned} \tag{3.2}$$

The fluid motion equations and the evolution equation of the dilaton-spin tensor are derived by the variation of (3.2) with respect to the independent variables $n$, $s$, $S^p{}_q$, $J$, $u$, $l^q$ and the Lagrange multipliers, the thermodynamic principle (see [5]) being taken into account and master-formula

(4.7) being used. We shall consider the 1-form $l^q$ as an independent variable and the 3-form $l_p$ as a function of $l^q$. One can verify that the Lagrangian density 4-form (3.2) is proportional to the hydrodynamic fluid pressure, $\mathcal{L}_{\rm fluid} = p\eta$.

The variation with respect to the 1-forms $l^q$ yields the motion equations of the material frame, which lead to the dilaton charge conservation law $\dot{J} = 0$ and to the evolution equation of the spin tensor, $\dot{S}^\alpha{}_\beta + \dot{S}^\alpha{}_\gamma u^\gamma u_\beta + \dot{S}^\gamma{}_\beta u_\gamma u^\alpha = 0$ , where the 'dot' notation for the tensor object $\Phi$ is introduced, $\dot{\Phi}^\alpha{}_\beta = *(u \wedge {\rm D}\Phi^\alpha{}_\beta)$.

By means of the variational derivatives of the matter Lagrangian density (3.2) one can derive the external matter currents which are the sources of the gravitational field [5]. The variational derivative of the Lagrangian density (3.2) with respect to $\theta^\sigma$ yields the canonical energy-momentum 3-form,

$$\Sigma_\sigma = \frac{\delta \mathcal{L}_{\rm fluid}}{\delta \theta^\sigma} = p\eta_\sigma + (\varepsilon + p) u_\sigma u + n\dot{S}_{\sigma\rho} u^\rho u \ . \tag{3.3}$$

Here the Frenkel condition, the dilaton charge conservation law $\dot{J} = 0$ and the evolution equation of the spin tensor have been used. In case of the dilaton-spin fluid the energy density $\varepsilon$ in (3.3) contains the energy density of the dilaton interaction of the fluid.

The metric stress-energy 4-form can be derived in the same way,

$$\sigma^{\alpha\beta} = 2\frac{\delta \mathcal{L}_{\rm fluid}}{\delta g_{\alpha\beta}} = \left( p g^{\alpha\beta} + (\varepsilon + p) u^\alpha u^\beta + n\dot{S}^{(\alpha}{}_\gamma u^{\beta)} u^\gamma \right) \eta \ .$$

The dilaton-spin momentum 3-form can be obtained in the following way,

$$\mathcal{J}^\alpha{}_\beta = -\frac{\delta \mathcal{L}_{\rm fluid}}{\delta \Gamma^\beta{}_\alpha} = \frac{1}{2} n \left( S^\alpha{}_\beta + \frac{1}{4} J \delta^\alpha_\beta \right) u = \mathcal{S}_{\alpha\beta} + \frac{1}{4} \mathcal{J} \delta^\alpha_\beta \ . \tag{3.4}$$

In a Weyl–Cartan space the matter Lagrangian obeys the diffeomorphism invariance, the local Lorentz invariance and the local scale invariance that lead to the corresponding Noether identities [5],

$$\begin{gathered}
{\rm D}\Sigma_\sigma = (\vec{e}_\sigma \rfloor \mathcal{T}^\alpha) \wedge \Sigma_\alpha - (\vec{e}_\sigma \rfloor \mathcal{R}^\alpha{}_\beta) \wedge \mathcal{J}^\beta{}_\alpha - \frac{1}{8} (\vec{e}_\sigma \rfloor \mathcal{Q}) \sigma^\alpha{}_\alpha \ , \\
\left( {\rm D} + \frac{1}{4}\mathcal{Q} \right) \wedge \mathcal{S}_{\alpha\beta} = \theta_{[\alpha} \wedge \Sigma_{\beta]} \ , \qquad {\rm D}\mathcal{J} = \theta^\alpha \wedge \Sigma_\alpha - \sigma^\alpha{}_\alpha \ .
\end{gathered} \tag{3.5}$$

The Noether identity (3.5) represents the quasiconservation law for the canonical matter energy-momentum 3-form. This law leads to the equation of motion of the perfect dilaton-spin fluid in the form of the generalized hydrodynamic Euler-type equation of the perfect fluid [5],

$$\begin{aligned}
u \wedge {\rm D} \left( \pi_\sigma + \frac{p}{n} u_\sigma \right) = \frac{1}{n} \eta \vec{e}_\sigma \rfloor {\rm D}p - \frac{1}{8n} \eta (\varepsilon + p) Q_\sigma - (\vec{e}_\sigma \rfloor \mathcal{T}^\alpha) \wedge \left( \pi_\alpha + \frac{p}{n} u_\alpha \right) u \\
- \frac{1}{2} (\vec{e}_\sigma \rfloor \mathcal{R}^{\alpha\beta}) \wedge S_{\alpha\beta} u + \frac{1}{8} (\vec{e}_\sigma \rfloor \mathcal{R}^\alpha{}_\alpha) \wedge J u \ .
\end{aligned} \tag{3.6}$$

If one evaluates the component of the equation (3.6) along the 4-velocity by contracting with $u^\sigma$, then one gets the energy conservation law along a streamline of the fluid [5],

$${\rm d}\varepsilon = \frac{\varepsilon + p}{n} {\rm d}n \ . \tag{3.7}$$

## 4. Variational formalism in a Weyl–Cartan space

Let us consider a spacetime with a Weyl–Cartan geometrical structure and develop the variational formalism of the gravitational field in this spacetime. We represent the total Lagrangian density 4-form of the theory as follows

$$\mathcal{L} = \mathcal{L}_{\rm grav} + \mathcal{L}_{\rm fluid} \ , \tag{4.1}$$

where the gravitational field Lagrangian density 4-form reads,

$$\begin{aligned} \mathcal{L}_{\rm grav} \quad = \quad & 2f_0\Big(\frac{1}{2}\mathcal{R}^{\alpha}{}_{\beta} \wedge \eta_{\alpha}{}^{\beta} - \Lambda\eta + \frac{1}{4}\lambda\,\mathcal{R}^{\alpha}{}_{\alpha} \wedge *\mathcal{R}^{\beta}{}_{\beta} + \varrho_1\,\mathcal{T}^{\alpha} \wedge *\mathcal{T}_{\alpha} \\ & +\varrho_2\,(\mathcal{T}^{\alpha} \wedge \theta_{\beta}) \wedge *(\mathcal{T}^{\beta} \wedge \theta_{\alpha}) + \varrho_3\,(\mathcal{T}^{\alpha} \wedge \theta_{\alpha}) \wedge *(\mathcal{T}^{\beta} \wedge \theta_{\beta}) \\ & +\xi\,\mathcal{Q} \wedge *\mathcal{Q} + \zeta\,\mathcal{Q} \wedge \theta^{\alpha} \wedge *\mathcal{T}_{\alpha}\Big) + \Lambda^{\alpha\beta} \wedge \left(\mathcal{Q}_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{Q}\right) \ . \end{aligned} \tag{4.2}$$

Here $f_0 = 1/(2æ)$ ($æ = 8\pi G$), $\Lambda$ is the cosmological constant, $\lambda$, $\varrho_1$, $\varrho_2$, $\varrho_3$, $\xi$, $\zeta$ are the coupling constants, $\mathcal{L}_{\rm fluid}$ is the Lagrangian density 4-form of the perfect dilaton-spin fluid (3.2), and $\Lambda^{\alpha\beta}$ is the Lagrange multiplier 3-form with the evident properties,

$$\Lambda^{\alpha\beta} = \Lambda^{\beta\alpha} \ , \qquad \Lambda^{\gamma}{}_{\gamma} = 0 \ , \tag{4.3}$$

which are the consequences of the Weyl's condition (2.1).

In (4.2) the first term is the linear Hilbert–Einstein Lagrangian generalized to a Weyl–Cartan space. The second term is the Weyl quadratic Lagrangian, which is the square of the Weyl segmental curvature 2-form (2.3). Here the Weyl 1-form $\mathcal{Q}$, in contrast to the classical Weyl theory, represents the gauge field, which does not relate to an electromagnetic field, that was first pointed out by Utiyama [19]. We shall call the field of the Weyl 1-form $\mathcal{Q}$ a dilatation field (Weyl field). The term with the coupling constant $\zeta$ represents the contact interaction of the dilatation field with the torsion that can occur in a Weyl–Cartan space.

The gravitational field equations in a Weyl–Cartan spacetime can be obtained by a variational procedure of the first order. Let us vary the Lagrangian (4.1) with respect to the connection 1-form $\Gamma^{\alpha}{}_{\beta}$ ($\Gamma$-equation) and to the basis 1-form $\theta^{\alpha}$ ($\theta$-equation) independently, the constraints on the connection 1-form in a Weyl–Cartan space being satisfied by means of the Lagrange multiplier 3-form $\Lambda^{\alpha\beta}$.

The including into the Lagrangian density 4-form the term with the Lagrange multiplier $\Lambda^{\alpha\beta}$ means that the theory is considered in a Weyl–Cartan spacetime from the very beginning [13]–[15]. Another variational approach has been developed in [20] where the field equations in a Weyl–Cartan spacetime have been obtained as a limiting case of the field equations of the metric-affine gauge theory of gravity. These two approaches are not identical in general and coincide only in case when $\Lambda^{\alpha\beta}$ is equal to zero as a consequence of the field equations.

For the variational procedure it is efficiently to use the following general relations which can be obtained for the arbitrary 2-forms $\Phi^{\beta}{}_{\alpha}$, $\Phi_{\alpha}$ and the arbitrary 3-form $\Psi^{\alpha\beta}$ with the help of the Cartan structure equations (2.2) and the structure equation for the nonmetricity 1-form $\mathcal{Q}_{\alpha\beta}$ (2.1),

$$\delta\mathcal{R}^{\alpha}{}_{\beta} \wedge \Phi^{\beta}{}_{\alpha} = {\rm d}(\delta\Gamma^{\alpha}{}_{\beta} \wedge \Phi^{\beta}{}_{\alpha}) + \delta\Gamma^{\alpha}{}_{\beta} \wedge {\rm D}\Phi^{\beta}{}_{\alpha} \ , \tag{4.4}$$

$$\delta\mathcal{T}^{\alpha} \wedge \Phi_{\alpha} = {\rm d}(\delta\theta^{\alpha} \wedge \Phi_{\alpha}) + \delta\theta^{\alpha} \wedge {\rm D}\Phi_{\alpha} + \delta\Gamma^{\alpha}{}_{\beta} \wedge \theta^{\beta} \wedge \Phi_{\alpha} \ , \tag{4.5}$$

$$\delta\mathcal{Q}_{\alpha\beta} \wedge \Psi^{\alpha\beta} = {\rm d}(-\delta g_{\alpha\beta}\Psi^{\alpha\beta}) + \delta\Gamma^{\alpha}{}_{\beta} \wedge 2\Psi_{(\alpha}{}^{\beta)} + \delta g_{\alpha\beta}{\rm D}\Psi^{\alpha\beta} \ . \tag{4.6}$$

The subsequent derivation of the variations of the Lagrangian density 4-form (4.2) is based on the master formula derived the following Lemma, proved in [21].

*Lemma.* Let $\Phi$ and $\Psi$ be arbitrary $p$-forms defined on $n$ -dimensional manifold. Then the variational identity for the commutator of the variation operator $\delta$ and the Hodge star operator $*$ is valid,

$$\begin{aligned}\Phi\wedge\delta*\Psi &= \delta\Psi\wedge*\Phi\\ &+\delta g_{\sigma\rho}\left(\frac{1}{2}g^{\sigma\rho}\Phi\wedge*\Psi+(-1)^{p(n-1)+s+1}\theta^{\sigma}\wedge*(*\Psi\wedge\theta^{\rho})\wedge*\Phi\right)\\ &+\delta\theta^{\alpha}\wedge\Big((-1)^{p}\Phi\wedge*(\Psi\wedge\theta_{\alpha})+(-1)^{p(n-1)+s+1}*(*\Psi\wedge\theta_{\alpha})\wedge*\Phi\Big)\ . \qquad (4.7)\end{aligned}$$

This master formula gives the rule how to compute the commutator of the variation operator $\delta$ and the Hodge star operator $*$.

The variational procedure is realized with the help of the computation rules,

$$**\Psi=(-1)^{p(n-p)+s}\Psi\ ,\qquad \Phi\wedge*\Psi=\Psi\wedge*\Phi\ , \qquad (4.8)$$

$$\vec{e}_{\alpha}\rfloor*\Psi=*(\Psi\wedge\theta_{\alpha})\ ,\qquad \theta^{\alpha}\wedge(\vec{e}_{\alpha}\rfloor\Psi)=p\Psi\ , \qquad (4.9)$$

where $\Psi$ and $\Phi$ are $p$-forms and $s=\mathrm{Ind}(\breve{g})$ is the index of the metric $\breve{g}$, which is equal to the number of negative eigenvalues of the diagonalized metric. The relations (4.8) and (4.9) lead to the consequences,

$$\vec{e}_{\alpha}\rfloor\Psi=(-1)^{n(p-1)+s}*(\theta_{\alpha}\wedge*\Psi)\ , \qquad (4.10)$$

$$*(\vec{e}_{\alpha}\rfloor\Psi)=(-1)^{p-1}\theta_{\alpha}\wedge*\Psi\ , \qquad (4.11)$$

$$*(\vec{e}_{\alpha}\rfloor*\Psi)=(-1)^{(n-1)(p+1)+s}\Psi\wedge\theta_{\alpha}\ . \qquad (4.12)$$

Let apply the master formula (4.7) to the variation of the Lagrangian density 4-form (4.2), the general relations (4.4)–(4.6) and the computation rules (2.7)–(2.13), (4.8)–(4.12) being used. The results of the variational procedure for every term of (4.2) have the following form, the exact forms being omitted,

$$\begin{aligned}2f_0\ :\ \ &\delta\Gamma^{\alpha}{}_{\beta}\wedge\left(-\frac{1}{4}\mathcal{Q}\wedge\eta_{\alpha}{}^{\beta}+\frac{1}{2}\mathcal{T}_{\lambda}\wedge\eta_{\alpha}{}^{\beta\lambda}+\frac{1}{2}\eta_{\alpha\gamma}\wedge\mathcal{Q}^{\beta\gamma}\right)\\ +\ &\delta g_{\sigma\rho}\left(\frac{1}{2}g^{\sigma\rho}\mathcal{R}^{\alpha}{}_{\beta}\wedge\eta_{\alpha}{}^{\beta}+\frac{1}{2}\theta^{\sigma}\wedge\theta_{\beta}\wedge*\mathcal{R}^{\beta\rho}\right)\\ +\ &\delta\theta^{\sigma}\wedge\left(\frac{1}{2}\mathcal{R}^{\alpha}{}_{\beta}\wedge\eta_{\alpha}{}^{\beta}{}_{\sigma}\right)\ , \qquad (4.13)\\ 2f_0\ \varrho_1\ :\ \ &\delta\Gamma^{\alpha}{}_{\beta}\wedge2\theta^{\beta}\wedge*\mathcal{T}_{\alpha}\\ +\ &\delta g_{\sigma\rho}\left(\mathcal{T}^{\sigma}\wedge*\mathcal{T}^{\rho}+\frac{1}{2}g^{\sigma\rho}\mathcal{T}_{\alpha}\wedge*\mathcal{T}^{\alpha}+\theta^{\sigma}\wedge*(*\mathcal{T}^{\alpha}\wedge\theta^{\rho})\wedge*\mathcal{T}_{\alpha}\right)\\ +\ &\delta\theta^{\sigma}\wedge(2\mathrm{D}*\mathcal{T}_{\sigma}+\mathcal{T}^{\alpha}\wedge*(\mathcal{T}_{\alpha}\wedge\theta_{\sigma})+*(*\mathcal{T}_{\alpha}\wedge\theta_{\sigma})\wedge*\mathcal{T}^{\alpha})\ , \qquad (4.14)\\ 2f_0\ \varrho_2\ :\ \ &\delta\Gamma^{\alpha}{}_{\beta}\wedge2\theta^{\beta}\wedge\theta_{\gamma}\wedge*(\mathcal{T}^{\gamma}\wedge\theta_{\alpha})+\delta g_{\sigma\rho}\Bigg(\frac{1}{2}g^{\sigma\rho}(\mathcal{T}^{\alpha}\wedge\theta_{\beta})\\ +\ &2\delta^{\sigma}_{\beta}\mathcal{T}^{\alpha}\wedge\theta^{\rho}-\theta^{\sigma}\wedge*(*(\mathcal{T}^{\alpha}\wedge\theta_{\beta})\wedge\theta^{\rho})\Bigg)\wedge*(\mathcal{T}^{\beta}\wedge\theta_{\alpha})\\ +\ &\delta\theta^{\sigma}\wedge\Bigg(2\mathrm{D}\Big(\theta_{\alpha}\wedge*(\mathcal{T}^{\alpha}\wedge\theta_{\sigma})\Big)-*(\mathcal{T}^{\beta}\wedge\theta_{\alpha}\wedge\theta_{\sigma})(\mathcal{T}^{\alpha}\wedge\theta_{\beta})\\ +\ &2\mathcal{T}^{\alpha}\wedge*(\mathcal{T}_{\sigma}\wedge\theta_{\alpha})-*\Big(*(\mathcal{T}^{\beta}\wedge\theta_{\alpha})\wedge\theta_{\sigma}\Big)\wedge*(\mathcal{T}^{\alpha}\wedge\theta_{\beta})\Bigg)\ , \qquad (4.15)\\ 2f_0\ \varrho_3\ :\ \ &\delta\Gamma^{\alpha}{}_{\beta}\wedge2\theta^{\beta}\wedge\theta_{\alpha}\wedge*(\mathcal{T}^{\gamma}\wedge\theta_{\gamma})+\delta g_{\sigma\rho}\Bigg(\frac{1}{2}g^{\sigma\rho}(\mathcal{T}^{\beta}\wedge\theta_{\beta})\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\mathcal{T}^\sigma\wedge\theta^\rho - \theta^\sigma\wedge *\Big(*(\mathcal{T}^\beta\wedge\theta_\beta)\wedge\theta^\rho\Big)\Big)\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha)\\
&+ \quad \delta\theta^\sigma\wedge\left(2\mathrm{D}\Big(\theta_\sigma\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha)\Big) - *(\mathcal{T}^\beta\wedge\theta_\beta\wedge\theta_\sigma)\mathcal{T}^\alpha\wedge\theta_\alpha\right.\\
&+ \quad \left. 2\mathcal{T}_\sigma\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha) - *\Big(*(\mathcal{T}^\beta\wedge\theta_\beta)\wedge\theta^\sigma\Big)\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha)\right)\,, \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
2f_0\,\lambda \quad &: \quad \delta\Gamma^\alpha{}_\beta\wedge\left(\frac{1}{2}\delta^\beta_\alpha \mathrm{D} * \mathcal{R}^\gamma{}_\gamma\right)\\
&+ \quad \delta g_{\sigma\rho}\left(\frac{1}{8}g^{\sigma\rho}\mathcal{R}^\alpha{}_\alpha\wedge *\mathcal{R}^\beta{}_\beta + \frac{1}{4}\theta^\sigma\wedge *(*\mathcal{R}^\alpha{}_\alpha\wedge\theta^\rho)\wedge *\mathcal{R}^\beta{}_\beta\right)\\
&+ \quad \delta\theta^\sigma\wedge\left(\frac{1}{4}\mathcal{R}^\beta{}_\beta\wedge *(\mathcal{R}^\alpha{}_\alpha\wedge\theta_\sigma) + \frac{1}{4} * (*\mathcal{R}^\alpha{}_\alpha\wedge\theta_\sigma)\wedge *\mathcal{R}^\beta{}_\beta\right)\,, \qquad (4.17)\\
2f_0\,\xi \quad &: \quad \delta\Gamma^\alpha{}_\beta\wedge(4\delta^\beta_\alpha * \mathcal{Q})\\
&+ \quad \delta g_{\sigma\rho}\left(2g^{\sigma\rho}\mathrm{D} * \mathcal{Q} + \frac{1}{2}g^{\sigma\rho}\mathcal{Q}\wedge *\mathcal{Q} - *(*\mathcal{Q}\wedge\theta^\rho)\theta^\sigma\wedge *\mathcal{Q}\right)\\
&+ \quad \delta\theta^\sigma\wedge(-\mathcal{Q}\wedge *(\mathcal{Q}\wedge\theta_\sigma) - *(*\mathcal{Q}\wedge\theta_\sigma)\wedge *\mathcal{Q})\,, \qquad (4.18)\\
2f_0\,\zeta \quad &: \quad \delta\Gamma^\alpha{}_\beta\wedge\Big(2\delta^\beta_\alpha\theta^\gamma\wedge *\mathcal{T}_\gamma + \theta^\beta\wedge *(\mathcal{Q}\wedge\theta_\alpha)\Big)\\
&+ \quad \delta g_{\sigma\rho}\Big(g^{\sigma\rho}\mathcal{T}^\alpha\wedge *\mathcal{T}_\alpha - g^{\sigma\rho}\theta^\alpha\wedge D * \mathcal{T}_\alpha + \frac{1}{2}g^{\sigma\rho}\mathcal{Q}\wedge\theta^\alpha\wedge *\mathcal{T}_\alpha\\
&+ \quad \theta^\sigma\wedge *(*\mathcal{T}_\alpha\wedge\theta^\rho)\wedge *(\mathcal{Q}\wedge\theta^\alpha) + \mathcal{T}^\sigma\wedge *(\mathcal{Q}\wedge\theta^\rho)\Big)\\
&+ \quad \delta\theta^\sigma\wedge\Big(\mathrm{D} * (\mathcal{Q}\wedge\theta_\sigma) + \mathcal{Q}\wedge\theta^\alpha\wedge *(\mathcal{T}_\alpha\wedge\theta_\sigma)\\
&+ \quad *(*\mathcal{T}_\alpha\wedge\theta_\sigma)\wedge *(\mathcal{Q}\wedge\theta^\alpha) - \mathcal{Q}\wedge *\mathcal{T}_\sigma\Big)\,, \qquad (4.19)\\
2f_0\,\Lambda \quad &: \quad \delta g_{\sigma\rho}\left(\frac{1}{2}g^{\sigma\rho}\eta\right) + \delta\theta^\sigma\wedge\eta_\sigma\,. \qquad (4.20)
\end{aligned}
$$

The variation of the term with the Lagrange multiplier in (4.2) has the form,

$$
\delta\Lambda^{\alpha\beta}\wedge\left(\mathcal{Q}_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}\mathcal{Q}\right) + \delta\Gamma^\alpha{}_\beta\wedge(-2\Lambda_\alpha{}^\beta) + \delta g_{\sigma\rho}\left(-\mathrm{D}\Lambda^{\sigma\rho} - \frac{1}{4}\Lambda^{\sigma\rho}\wedge\mathcal{Q}\right)\,. \qquad (4.21)
$$

The variation of the total Lagrangian density 4-form (4.1) with respect to the Lagrange multiplier 3-form $\Lambda^{\alpha\beta}$ yields according to (4.21) the Weyl's condition (2.1) for the nonmetricity 1-form $Q_{\alpha\beta}$.

The variation of (4.1) with respect to $\Gamma^\alpha{}_\beta$ can be obtained by combining all terms in (4.13)–(4.21) proportional to the variation of $\Gamma^\alpha{}_\beta$ and taking into account the same variation of the fluid Lagrangian density 4-form (3.2), which is the dilaton-spin momentum 3-form (3.4). This variation yields the field $\Gamma$-equation,

$$
\begin{aligned}
&\frac{1}{4}\lambda\delta^\beta_\alpha \mathrm{d} * \mathrm{d}\mathcal{Q} - \frac{1}{8}\mathcal{Q}\wedge\eta_\alpha{}^\beta + \frac{1}{2}\mathcal{T}_\gamma\wedge\eta_\alpha{}^{\beta\gamma} + 2\varrho_1\theta^\beta\wedge *\mathcal{T}_\alpha\\
&+2\varrho_2\theta^\beta\wedge\theta_\gamma\wedge *(\mathcal{T}^\gamma\wedge\theta_\alpha) + 2\varrho_3\theta^\beta\wedge\theta_\alpha\wedge *(\mathcal{T}^\gamma\wedge\theta_\gamma)\\
&+4\xi\delta^\beta_\alpha * \mathcal{Q} + \zeta\Big(\theta^\beta\wedge *(\mathcal{Q}\wedge\theta_\alpha) + 2\delta^\beta_\alpha\theta^\gamma\wedge *\mathcal{T}_\gamma\Big) - \frac{1}{f_0}\Lambda_\alpha{}^\beta\\
&\qquad = \frac{1}{4f_0}n\left(S^\beta{}_\alpha + \frac{1}{4}J\delta^\beta_\alpha\right)u\,, \qquad (4.22)
\end{aligned}
$$

the condition (2.1) being taken into account after the variational procedure has been performed. The variation of (4.1) with respect to the basis 1-form $\theta^\sigma$ can be obtained in the similar way that gives the second field equation ($\theta$-equation),

$$\begin{aligned}
\frac{1}{2}\mathcal{R}^{\alpha}{}_{\beta}\wedge\eta_{\alpha}{}^{\beta}{}_{\sigma} - \Lambda\eta_\sigma + \varrho_1\Big(2\mathrm{D} *\mathcal{T}_\sigma + \mathcal{T}^\alpha\wedge *(\mathcal{T}_\alpha\wedge\theta_\sigma) + *(*\mathcal{T}_\alpha\wedge\theta_\sigma)\wedge *\mathcal{T}^\alpha\Big)\\
+\ \varrho_2\Big(2\mathrm{D}\Big(\theta_\alpha\wedge *(\mathcal{T}^\alpha\wedge\theta_\sigma)\Big) - *(\mathcal{T}^\beta\wedge\theta_\alpha\wedge\theta_\sigma)(\mathcal{T}^\alpha\wedge\theta_\beta)\\
+\ 2\mathcal{T}^\alpha\wedge *(\theta_\alpha\wedge\mathcal{T}_\sigma) - *\Big(*(\mathcal{T}^\beta\wedge\theta_\alpha)\wedge\theta_\sigma\Big)\wedge *(\mathcal{T}^\alpha\wedge\theta_\beta)\Big)\\
+\ \varrho_3\Big(2\mathrm{D}\Big(\theta_\sigma\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha)\Big) + 2\mathcal{T}_\sigma\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha)\\
-\ *(\mathcal{T}^\beta\wedge\theta_\beta\wedge\theta_\sigma)\mathcal{T}^\alpha\wedge\theta_\alpha - *\Big(*(\mathcal{T}^\beta\wedge\theta_\beta)\wedge\theta_\sigma\Big)\wedge *(\mathcal{T}^\alpha\wedge\theta_\alpha)\Big)\\
+\ \lambda\left(\frac{1}{4}\mathcal{R}^{\beta}{}_{\beta}\wedge *(\mathcal{R}^{\alpha}{}_{\alpha}\wedge\theta_\sigma) + \frac{1}{4} * (*\mathcal{R}^{\alpha}{}_{\alpha}\wedge\theta_\sigma)\wedge *\mathcal{R}^{\beta}{}_{\beta}\right)\\
+\ \zeta\Big(\mathrm{D} *(\mathcal{Q}\wedge\theta_\sigma) + \mathcal{Q}\wedge\theta^\alpha\wedge *(\mathcal{T}_\alpha\wedge\theta_\sigma)\\
-\ \mathcal{Q}\wedge *\mathcal{T}_\sigma + *(*\mathcal{T}_\alpha\wedge\theta_\sigma)\wedge *(\mathcal{Q}\wedge\theta^\alpha)\Big)\\
+\ \xi\Big(-\mathcal{Q}\wedge *(\mathcal{Q}\wedge\theta_\sigma) - *(*\mathcal{Q}\wedge\theta_\sigma) *\mathcal{Q}\Big) = -\frac{1}{2f_0}\Sigma_\sigma\ . \qquad (4.23)
\end{aligned}$$

Here $\Sigma_\sigma$ is the fluid canonical energy-momentum 3-form (3.3). In (4.23) the condition (2.1) is used after the variational procedure has been performed.

The result of the variation of the total Lagrangian density 4-form (4.1) with respect to the metric components $g_{\alpha\beta}$ ($g$-equation) is not independent and is a consequence of the field $\Gamma$- and $\theta$-equations. For the metric-affine theory of gravitation it was pointed out in [16]. In the Weyl–Cartan theory of gravitation it can be justified as follows. In this theory, as the consequence of the scale invariance, the metric of the tangent space can be chosen in the form [19], $g_{ab} = \sigma(x)g^M_{ab}$, where $g^M_{ab}$ is the metric tensor of the Minkowski space and $\sigma(x)$ is an arbitrary function to be varied when the $g$-equation is derived. Therefore the $g$-equation appears only in the trace form. But the total Lagrangian density 4-form (4.1) also obeys the diffeomorphism invariance and therefore the Noether identity analogous to the identity (3.5) is valid, from which the trace of the $g$-equation can be derived via the $\Gamma$- and $\theta$-equations. The quantity $\vec{e}_\sigma\rfloor\mathcal{Q}$ in (3.5) does not vanish identically in general, otherwise we should have a Riemann–Cartan spacetime, in which case we could choose $\sigma(x) = \mathrm{const} = 1$ and the g-equation would not appear.

## 5. The analysis of the field $\Gamma$-equation

Let us give the detailed analysis of the $\Gamma$-equation (4.22). The antisymmetric part of this equation determines the torsion 2-form $\mathcal{T}^\alpha$. The symmetric part determines the Lagrange multiplier 3-form $\Lambda_\alpha{}^\beta$ and the Weyl 1-form $\mathcal{Q}$.

After antisymmetrization the equation (4.22) gives the following equation for the torsion 2-form,

$$\begin{aligned}
-\frac{1}{2}\mathcal{T}^\gamma\wedge\eta_{\alpha\beta\gamma} + \frac{1}{8}\mathcal{Q}\wedge\eta_{\alpha\beta} + 2\varrho_1\theta_{[\alpha}\wedge *\mathcal{T}_{\beta]} + 2\varrho_2\theta_{[\alpha}\wedge\theta_{|\gamma|}\wedge *(\mathcal{T}^{|\gamma|}\wedge\theta_{\beta]})\\
+2\varrho_3\theta_\alpha\wedge\theta_\beta\wedge *(\mathcal{T}^\gamma\wedge\theta_\gamma) + \zeta\theta_{[\alpha}\wedge *(\mathcal{Q}\wedge\theta_{\beta]}) = \frac{1}{2}æ n S_{\alpha\beta}u\ , \qquad æ = \frac{1}{2f_0}\ . \qquad (5.1)
\end{aligned}$$

The torsion 2-form can be decomposed into the irreducible pieces (the traceless 2-form $\overset{(1)}{\mathcal{T}}{}^{\alpha}$, the trace 2-form $\overset{(2)}{\mathcal{T}}{}^{\alpha}$ and the pseudotrace 2-form $\overset{(3)}{\mathcal{T}}{}^{\alpha}$) [16, 20],

$$\mathcal{T}^{\alpha} = \overset{(1)}{\mathcal{T}}{}^{\alpha} + \overset{(2)}{\mathcal{T}}{}^{\alpha} + \overset{(3)}{\mathcal{T}}{}^{a} \ . \tag{5.2}$$

Here the torsion trace 2-form and the torsion pseudotrace 2-form of the pseudo-Riemannian 4-manifold are determined by the expressions, respectively,

$$\overset{(2)}{\mathcal{T}}{}^{\alpha} = \frac{1}{3}\mathcal{T} \wedge \theta^{\alpha} \ , \qquad \mathcal{T} = *(\theta_{\alpha} \wedge *\mathcal{T}^{\alpha}) = -(\vec{e}_{\alpha} \rfloor \mathcal{T}^{\alpha}) \ , \tag{5.3}$$

$$\overset{(3)}{\mathcal{T}}{}^{\alpha} = \frac{1}{3} * (\mathcal{P} \wedge \theta^{\alpha}) \ , \qquad \mathcal{P} = *(\theta_{\alpha} \wedge \mathcal{T}^{\alpha}) = \vec{e}_{\alpha} \rfloor * \mathcal{T}^{\alpha} \ , \tag{5.4}$$

where the torsion trace 1-form $\mathcal{T}$ and the torsion pseudotrace 1-form $\mathcal{P}$ are introduced.

The irreducible pieces of torsion satisfy to the conditions [16],

$$\overset{(1)}{\mathcal{T}}{}^{\alpha} \wedge \theta_{\alpha} = 0 \ , \qquad \overset{(2)}{\mathcal{T}}{}^{\alpha} \wedge \theta_{\alpha} = 0 \ , \tag{5.5}$$

$$\vec{e}_{\alpha} \rfloor \overset{(1)}{\mathcal{T}}{}^{\alpha} = 0 \ , \qquad \vec{e}_{\alpha} \rfloor \overset{(3)}{\mathcal{T}}{}^{\alpha} = 0 \ . \tag{5.6}$$

Using the computational rules (2.4)–(2.11) and (4.11) let us derive two efficient identities,

$$\begin{aligned}
\mathcal{T}^{\gamma} \wedge \eta_{\alpha\beta\gamma} \ &= \ \mathcal{T}^{\gamma} \wedge (\vec{e}_{\gamma} \rfloor \eta_{\alpha\beta}) = \vec{e}_{\gamma} \rfloor (\mathcal{T}^{\gamma} \wedge \eta_{\alpha\beta}) - (\vec{e}_{\gamma} \rfloor \mathcal{T}^{\gamma}) \wedge \eta_{\alpha\beta} \\
&= \vec{e}_{\gamma} \rfloor \Big(\mathcal{T}^{\gamma} \wedge *(\theta_{\alpha} \wedge \theta_{\beta})\Big) + \mathcal{T} \wedge \eta_{\alpha\beta} \\
&= \Big(\vec{e}_{\gamma} \rfloor (\theta_{\alpha} \wedge \theta_{\beta})\Big) \wedge *\mathcal{T}^{\gamma} + \theta_{\alpha} \wedge \theta_{\beta} \wedge (\vec{e}_{\gamma} \rfloor * \mathcal{T}^{\gamma}) + \mathcal{T} \wedge \eta_{\alpha\beta} \\
&= -\theta_{[\alpha} \wedge *\mathcal{T}_{\beta]} + \theta^{\alpha} \wedge \theta_{\beta} \wedge \mathcal{P} + \mathcal{T} \wedge \eta_{\alpha\beta} \ ,
\end{aligned} \tag{5.7}$$

$$\theta_{\gamma} \wedge *(\mathcal{T}^{\gamma} \wedge \theta_{\alpha}) \ = \ *\Big(\vec{e}_{\gamma} \rfloor (\mathcal{T}^{\gamma} \theta_{\alpha})\Big) = *(-\mathcal{T} \wedge \theta_{\alpha} + \mathcal{T}_{\alpha}) = *\mathcal{T}_{\alpha} - 3 * \overset{(2)}{\mathcal{T}}{}_{\alpha} \ . \tag{5.8}$$

Using the identities (5.7), (5.8), one can represent the field equation (5.1) as follows,

$$\begin{aligned}
&(1 + 2\varrho_1 + 2\rho_2)\theta_{[\alpha} \wedge *\mathcal{T}_{\beta]} + \left(-\frac{1}{2} + 2\varrho_3\right) \theta_{\alpha} \wedge \theta_{\beta} \wedge \mathcal{P} - 6\varrho_2 \theta_{[\alpha} \wedge * \overset{(2)}{\mathcal{T}}{}_{\beta]} \\
&-\frac{1}{2}\mathcal{T} \wedge \eta_{\alpha\beta} + \frac{1}{8}\mathcal{Q} \wedge \eta_{\alpha\beta} + \zeta \theta_{[\alpha} \wedge *(\mathcal{Q} \wedge \theta_{\beta]}) = \frac{1}{2} æ n S_{\alpha\beta} u \ .
\end{aligned} \tag{5.9}$$

Multiplying the equation (5.9) by $\theta^{\beta}$ from the right externally, using the computation rules (4.8)–(4.12) and then the Hodge star operation, one gets in consequence of the Frenkel condition (see Section 3) the relation between the torsion trace 1-form $\mathcal{T}$ (5.3) and the Weyl 1-form $\mathcal{Q}$,

$$\mathcal{T} = \frac{3(\frac{1}{4} + \zeta)}{2(1 - \varrho_1 + 2\varrho_2)} \mathcal{Q} \ . \tag{5.10}$$

As a consequence of (5.10) and the relation (2.10) it can be proved the equality for the trace 2-form,

$$(1 + 2\varrho_1 - 4\varrho_2)\theta_{[\alpha} \wedge * \overset{(2)}{\mathcal{T}}{}_{\beta]} - \frac{1}{2}\mathcal{T} \wedge \eta_{\alpha\beta} + \frac{1}{8}\mathcal{Q} \wedge \eta_{\alpha\beta} + \zeta \theta_{[\alpha} \wedge *(\mathcal{Q} \wedge \theta_{\beta]}) = 0 \ . \tag{5.11}$$

Then as a consequence of (5.2), (5.4), (5.11) and the conditions (5.5) the field equation (5.9) is transformed as follows,

$$(1+2\varrho_1+2\varrho_2)\,\theta_{[\alpha}\wedge*\overset{(1)}{\mathcal{T}}{}_{\beta]}-\frac{1}{6}(1-4\varrho_1-4\varrho_2-12\varrho_3)\theta_\alpha\wedge\theta_\beta\wedge\mathcal{P}=\frac{1}{2}æ n S_{\alpha\beta}u\;.$$

Contracting this equation with $g^{\beta\gamma}\vec{e}_\gamma$, we get with the help of the Leibnitz rule (2.6) the equation,

$$(1+2\varrho_1+2\varrho_2)*\overset{(1)}{\mathcal{T}}{}_\alpha-\frac{2}{3}(1-4\varrho_1-4\varrho_2-12\varrho_3)\theta_\alpha\wedge\mathcal{P}=æ n S_{\alpha\beta}u_\gamma\eta^{\beta\gamma}\;. \tag{5.12}$$

By contracting this equation with $g^{\alpha\beta}\vec{e}_\beta$ we get the equation

$$(1-4\varrho_1-4\varrho_2-12\varrho_3)\mathcal{P}=æ n\sigma\;, \tag{5.13}$$

which represents the torsion pseudotrace 1-form $\mathcal{P}$ via the Pauli–Lyubanski spin 1-form $\sigma$ of a fluid particle,

$$\sigma=-\frac{1}{2}S^{\alpha\beta}u^\gamma\eta_{\alpha\beta\gamma}=\frac{1}{2}S^{\alpha\beta}u^\gamma\eta_{\lambda\alpha\beta\gamma}\theta^\lambda\;. \tag{5.14}$$

As a consequence of (5.13) the field equation (5.12) yields the equation for the traceless piece of the torsion 2-form,

$$(1+2\varrho_1+2\varrho_2)\;\overset{(1)}{\mathcal{T}}{}_\alpha=æ n\left(S_{\alpha\beta}u_\gamma\theta^\beta\wedge\theta^\gamma+\frac{2}{3}\sigma^\beta\eta_{\beta\alpha}\right)=-\frac{2}{3}æ nS_{\beta(\alpha}u_{\gamma)}\theta^\beta\wedge\theta^\gamma\;. \tag{5.15}$$

Now let us calculate the symmetric part of the $\Gamma$-equation (4.22). Because of (4.3) the result can be represented as follows,

$$\begin{aligned}æ\Lambda_{\alpha\beta}=\varrho_1\theta_{(\alpha}\wedge*\mathcal{T}_{\beta)}+\varrho_2\theta_{(\alpha}\wedge\theta_{|\gamma|}\wedge*(\mathcal{T}^{|\gamma|}\wedge\theta_{\beta)})+\frac{1}{8}\lambda g_{\alpha\beta}\mathrm{d}*\mathrm{d}\mathcal{Q}+2\xi g_{\alpha\beta}*\mathcal{Q}\\+\zeta\left(\frac{1}{2}\theta_{(\alpha}\wedge*(\mathcal{Q}\wedge\theta_{\beta)})+g_{\alpha\beta}\theta_\gamma\wedge*\mathcal{T}_\gamma\right)-\frac{1}{32}æ ng_{\alpha\beta}Ju\;.\end{aligned} \tag{5.16}$$

By contracting the equation (5.16) on the indices $\alpha$ and $\beta$ and after substituting (5.10) in the result, one finds the equation of the Proca type for the Weyl 1-form,

$$*\mathrm{d}*\mathrm{d}\mathcal{Q}+m^2\mathcal{Q}=\frac{æ}{2\lambda}nJ*u\;,\qquad m^2=16\frac{\xi}{\lambda}+\frac{3(\varrho_1-2\varrho_2+8\zeta(1+2\zeta))}{4\lambda(1-\varrho_1+2\varrho_2)}\;. \tag{5.17}$$

The equation (5.17) shows that the dilatation field $\mathcal{Q}$, in contrast to Maxwell field, possesses the non-zero rest mass and demonstrates a short-range nature [19, 2, 12, 22].

By virtue of $\mathrm{d}(nu)=0$ and $\dot{J}=0$ (see Section 3) the equation (5.17) has the Lorentz condition as a consequence,

$$\mathrm{d}*\mathcal{Q}=0\;,\qquad \overset{R}{\nabla}_\alpha Q^\alpha=0\;,$$

where $\overset{R}{\nabla}_\alpha$ is the covariant derivative with respect to the Riemann connection. Here the latter relation is the component representation of the former one.

If we use (5.10) and (5.17), then the equation (5.16) takes the form,

$$\begin{aligned}æ\Lambda_{\alpha\beta}=\varrho_1\theta_{(\alpha}\wedge*\mathcal{T}_{\beta)}+\varrho_2\theta_{(\alpha}\wedge\theta_{|\gamma|}\wedge*(\mathcal{T}^{|\gamma|}\wedge\theta_{\beta)})\\+\frac{4\zeta(1-\varrho_1+2\varrho_2)}{3(1+4\zeta)}\theta_{(\alpha}\wedge*(\mathcal{T}\wedge\theta_{\beta)})-\frac{4\zeta+\varrho_1-2\varrho_2}{4(1+4\zeta)}g_{\alpha\beta}*\mathcal{T}\;.\end{aligned} \tag{5.18}$$

This equation determines the Lagrange multiplier 3-form $\Lambda_{\alpha\beta}$. It is very important that $\Lambda_{\alpha\beta}$ is in general not equal to zero.

The equations (5.10), (5.13), (5.15) and (5.18) solve the problem of the evaluation the torsion 2-form and the Lagrange multiplier 3-form. With the help of the algebraic field equations (5.13) and (5.15) the traceless and pseudotrace pieces of the torsion 2-form are determined via the spin tensor and the flow 3-form $u$ of the perfect dilaton-spin fluid in general case, when the conditions $1 + 2\varrho_1 + 2\varrho_2 \neq 0$ and $1 - 4\varrho_1 - 4\varrho_2 - 12\varrho_3 \neq 0$ are valid. With the help of the equation (5.10) one can determine the torsion trace 2-form via the Weyl field $\mathcal{Q}$, for which the differential field equation (5.17) is valid. Therefore the torsion trace 2-form can propagate in the theory under consideration.

## 6. Modified Friedmann–Lemaître equation for dilaton-spin dark matter

Let us consider the cosmological model with the Friedmann–Robertson–Walker (FRW) metric with scale factor $a(t)$,

$$\mathrm{d}s^2 = \frac{a^2(t)}{1 - kr^2}\mathrm{d}r^2 + a^2(t)r^2\Big(\mathrm{d}\theta^2 + (\sin\theta)^2\mathrm{d}\phi^2\Big) - \mathrm{d}t^2 \ , \tag{6.1}$$

the comoving frame of reference being chosen,

$$u^1 = u^2 = u^3 = 0 \ , \qquad u^4 = 1 \ . \tag{6.2}$$

In this model the homogeneous and isotropic Universe is filled with the perfect dilaton-spin fluid [14], which realizes the model of dark matter with $J \neq 0$ in contrast to the baryonic and quark matter with $J = 0$. The cosmological constant $\Lambda$ realizes the dark energy (cosmic vacuum).

As it was shown in [23, 24], in the spacetime with the FRW metric (6.1) the only nonvanishing components of the torsion are $T^1{}_{41} = T^2{}_{42} = T^3{}_{43}$ and $T_{ijk}$ for $i = 1,\ 2,\ 3$. In this case from (5.3) we get that the only nonvanishing component of the trace 1-form is $T_4 = T_4(t)$ ($T_i = 0$ for $i = 1,\ 2,\ 3$). From (5.4) we also find, $\mathcal{P} = 3T_{[123]}\eta^{1234}\theta_4$. But the field equation (5.13) yields, $\mathcal{P}^4 \sim \sigma^4 = 0$, as a consequence of (5.14) and (6.2). Therefore the pseudotrace piece of the torsion 2-form vanishes. It is easy to calculate with the help of (5.2), (5.3) that the traceless piece also vanishes. Therefore for the FRW metric (6.1) we have,

$$\overset{(1)}{\mathcal{T}}{}_\alpha = 0 \ , \qquad \overset{(3)}{\mathcal{T}}{}_\alpha = 0 \ , \tag{6.3}$$

and the torsion 2-form consists only from the trace piece that in the component representation reads,

$$T_{\lambda\alpha\beta} = -\frac{2}{3}g_{\lambda[\alpha}T_{\beta]} \ . \tag{6.4}$$

As a consequence of (5.13)–(5.15) and the identity,

$$u_\lambda S_{\alpha\beta} \equiv u_{[\lambda}S_{\alpha\beta]} + \frac{2}{3}(u_{(\lambda}S_{\alpha)\beta} - u_{(\lambda}S_{\beta)\alpha}) \ , \tag{6.5}$$

we have to conclude that the condition $S_{\alpha\beta} = 0$ is valid for the spin tensor of the matter source in the cosmological model considered. It can be understood in the sense that the mean value of the spin tensor is equal to zero under statistical averaging over all directions in the homogeneous and isotropic Universe. As a consequence of this fact in this section and in the sequential sections we shall simplify the equations of the theory by using the conditions (6.3) and $S_{\alpha\beta} = 0$. In case $S_{\alpha\beta} = 0$ dilaton-spin fluid becomes dilaton fluid.

For FRW metric (6.1) the continuity equation $\mathrm{d}(nu) = 0$ (d – the operator of exterior differentiation) yields the matter conservation law $na^3 = N = \text{const}$. As an equation of state of the dilaton fluid we choose the equation of state $p = \gamma\varepsilon$, $0 \le \gamma < 1$. Then integration of the energy conservation law (3.7) for FRW metric (6.1) yields

$$\varepsilon\, a^{3(1+\gamma)} = \mathcal{E}_\gamma = \text{const} \;, \qquad \mathcal{E}_\gamma > 0 \;. \tag{6.6}$$

Let us now decompose the field $\theta$-equation (4.23) into Riemannian and non-Riemannian parts using the formulae (2.14)–(2.19) and then transform the result to the component form. For this purpose let us substitute the decomposition (2.16) into the equation (4.23) and after this use the decomposition (2.19), the relation (6.4) being taken into account. We get the following results for the every term of the equation (4.23). The linear term or this equation decomposes as follows,

$$\begin{aligned} -\left(\overset{R}{R}{}^\alpha{}_\sigma - \frac{1}{2}\delta^\alpha_\sigma \overset{R}{R}\right)\eta_\alpha + \frac{2}{3}\left(\overset{R}{\nabla}_\alpha T^\alpha\right)\eta_\sigma - \frac{2}{3}\left(\overset{R}{\nabla}_\sigma T^\alpha\right)\eta_\alpha - \frac{1}{9}T_\alpha T^\alpha\eta_\sigma - \frac{2}{9}T_\sigma T^\alpha\eta_\alpha \\ +\frac{1}{12}T_\alpha Q^\alpha\eta_\sigma + \frac{1}{12}T_\sigma Q^\alpha\eta_\alpha + \frac{1}{12}Q_\sigma T^\alpha\eta_\alpha \\ +\frac{1}{4}\left(\overset{R}{\nabla}_\sigma Q^\alpha\right)\eta_\alpha - \frac{1}{4}\left(\overset{R}{\nabla}_\alpha Q^\alpha\right)\eta_\sigma - \frac{1}{64}Q_\alpha Q^\alpha\eta_\sigma - \frac{1}{32}Q_\sigma Q^\alpha\eta_\alpha \;. \end{aligned} \tag{6.7}$$

The other terms decompose as follows,

$$\begin{aligned} \varrho_1 \;:\;\; & \frac{2}{3}\left(\overset{R}{\nabla}_\sigma T^\alpha\right)\eta_\alpha - \frac{2}{3}\left(\overset{R}{\nabla}_\alpha T^\alpha\right)\eta_\sigma + \frac{2}{9}T_\sigma T^\alpha\eta_\alpha + \frac{1}{9}T_\alpha T^\alpha\eta_\sigma \\ & -\frac{1}{4}Q_\sigma T^\alpha\eta_\alpha - \frac{1}{12}T_\sigma Q^\alpha\eta_\alpha + \frac{1}{12}Q_\alpha T^\alpha\eta_\sigma \;, & (6.8)\\ \varrho_2 \;:\;\; & -\frac{4}{3}\left(\overset{R}{\nabla}_\sigma T^\alpha\right)\eta_\alpha + \frac{4}{3}\left(\overset{R}{\nabla}_\alpha T^\alpha\right)\eta_\sigma - \frac{4}{9}T_\sigma T^\alpha\eta_\alpha - \frac{2}{9}T_\alpha T^\alpha\eta_\sigma \\ & +\frac{1}{2}Q_\sigma T^\alpha\eta_\alpha + \frac{1}{6}T_\sigma Q^\alpha\eta_\alpha - \frac{1}{6}Q_\alpha T^\alpha\eta_\sigma \;, & (6.9)\\ \zeta \;:\;\; & -\left(\overset{R}{\nabla}_\alpha Q^\alpha\right)\eta_\sigma + \left(\overset{R}{\nabla}_\sigma Q^\alpha\right)\eta_\alpha + \frac{2}{3}T_\alpha Q^\alpha\eta_\sigma \\ & -\frac{2}{3}Q_\sigma T^\alpha\eta_\alpha + \frac{1}{8}Q_\alpha Q^\alpha\eta_\sigma - \frac{1}{2}Q_\sigma Q^\alpha\eta_\alpha \;, & (6.10)\\ \xi \;:\;\; & Q_\alpha Q^\alpha\eta_\sigma - 2Q_\sigma Q^\alpha\eta_\alpha \;. & (6.11) \end{aligned}$$

After gathering all expressions (6.7)–(6.11) together and substituting the relation (5.10) we receive the following results.

The terms with the derivatives of the dilatation field, like $\overset{R}{\nabla}_\alpha Q^\alpha$ and $\overset{R}{\nabla}_\sigma Q^\alpha$, and the same derivatives of the torsion trace $T^\alpha$ in a remarkable manner mutually compensate each other and vanish as a consequence of (5.10),

$$\frac{2}{3}(1-\varrho_1+2\varrho_2)\left(\eta_\sigma \overset{R}{\nabla}_\rho T^\rho - \eta_\rho \overset{R}{\nabla}_\sigma T^\rho\right) - \left(\frac{1}{4}+\zeta\right)\left(\eta_\sigma \overset{R}{\nabla}_\rho Q^\rho - \eta_\rho \overset{R}{\nabla}_\sigma Q^\rho\right) = 0 \;.$$

The terms with $\mathrm{d}\mathcal{Q}$ also vanish, as the equality $\mathrm{d}\mathcal{Q} = 0$ is valid identically for the FRW metric (6.1) that can be easy verified in the holonomic basis, when $\theta^\alpha = \mathrm{d}x^\alpha$,

$$\mathrm{d}\mathcal{Q} = \partial_\beta Q_\alpha \mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha = \partial_4 Q_4 \mathrm{d}x^4 \wedge \mathrm{d}x^4 = 0 \;. \tag{6.12}$$

This follows from the fact that for this metric one has $Q_4 = Q_4(t)$, $Q_i = 0$ ($i = 1, 2, 3$) as a consequence of (5.10) and the values of the trace torsion for the metric (6.1).

The remainder terms of the equation (4.23) after some algebra can be represented as follows,

$$\left(\overset{R}{R}{}^{\rho}{}_{\sigma} - \frac{1}{2}\delta^{\rho}_{\sigma}\overset{R}{R}\right)\eta_{\rho} + \Lambda\eta_{\sigma} + \alpha(2Q_{\sigma}Q^{\rho}\eta_{\rho} - Q_{\rho}Q^{\rho}\eta_{\sigma}) = \text{æ}\Sigma_{\sigma} \ , \tag{6.13}$$

$$\alpha = \frac{3\left(\frac{1}{4}+\zeta\right)^2}{4(1-\varrho_1+2\varrho_2)} + \xi - \frac{3}{64} \ . \tag{6.14}$$

Then we shall derive the Weyl 1-form $\mathcal{Q}$ algebraically as a consequence of (6.12) from the equation (5.17) via the right side of this equation,

$$Q^{\alpha} = \frac{\text{æ}}{2\lambda m^2} nJu^{\alpha} \ , \tag{6.15}$$

which is in accordance with the conditions (6.2) for the comoving system of reference.

After substituting (6.15) and (3.3) to the equation (6.13), the condition $S_{\alpha\beta}=0$ being used, we can represent the field equation (4.23) as an Einstein-like equation,

$$\overset{R}{R}_{\sigma\rho} - \frac{1}{2}g_{\sigma\rho}\overset{R}{R} = \text{æ}\Big((\varepsilon_{\rm e}+p_{\rm e})u_{\sigma}u_{\rho} + p_{\rm e}g_{\sigma\rho}\Big) \ , \tag{6.16}$$

where $\overset{R}{R}_{\sigma\rho}$, $\overset{R}{R}$ are a Ricci tensor and a curvature scalar of a Riemann space, respectively, $\varepsilon_{\rm e}$ and $p_{\rm e}$ are an energy density and a pressure of an effective perfect fluid:

$$\varepsilon_{\rm e} = \varepsilon + \varepsilon_{\rm v} - \mathcal{E}\left(\frac{n}{N}\right)^2 , \quad p_{\rm e} = p + p_{\rm v} - \mathcal{E}\left(\frac{n}{N}\right)^2 , \quad \mathcal{E} = \alpha\text{æ}\left(\frac{JN}{2\lambda m^2}\right)^2 ,$$

and $\varepsilon_{\rm v} = \Lambda/\text{æ}$ and $p_{\rm v} = -\Lambda/\text{æ}$ are an energy density and a pressure of a vacuum with the equation of state, $\varepsilon_{\rm v} = -p_{\rm v} > 0$.

The field equation (6.16) yields the modified Friedmann–Lemaître (FL) equation,

$$\left(\frac{\dot a}{a}\right)^2 + \frac{k}{a^2} = \frac{\text{æ}}{3a^6}\left(\varepsilon_{\rm v}a^6 + \mathcal{E}_{\gamma}a^{3(1-\gamma)} - \mathcal{E}\right) \ , \tag{6.17}$$

We put $k=0$ in (6.17) in accordance with the modern observational evidence [7, 9, 10], which shows that the Universe is spatially flat in cosmological scale.

The other component of the equation (6.16) has the form,

$$\frac{\ddot a}{a} = \frac{\text{æ}}{3a^6}\left[\varepsilon_{\rm v}a^6 - \frac{1}{2}(1+3\gamma)\mathcal{E}_{\gamma}a^{3(1-\gamma)} + 2\mathcal{E}\right]. \tag{6.18}$$

## 7. Scenario of evolution of the Universe with dilaton dark matter

Our hypothesis consists in preposition that the evolution of the Universe begins from the superrigid stage, when the equation of stage of dilaton fluid is $\gamma = 1$. In this case the equation (6.6) yields $\varepsilon\, a^6 = \mathcal{E}_1 = \text{const}$. The FL equation (6.17) reads,

$$\left(\frac{\dot a}{a}\right)^2 = \frac{\text{æ}}{3a^6}\left(\varepsilon_{\rm v}a^6 - \mathcal{E}_1 + \mathcal{E}\right) \ ,$$

and can be exactly integrated. The solution corresponding to the initial data $t=0$, $a=a_{\rm min}$ reads [14],

$$a = a_{\rm min}(\cosh\sqrt{3\Lambda}\,t)^{1/3} \ ,$$

$$a_{\rm min} = \left(\frac{\alpha\text{æ}^2}{\Lambda}\left(\frac{JN}{2\lambda m^2}\right)^2 - \frac{\text{æ}\mathcal{E}_1}{\Lambda}\right)^{1/6} \ .$$

This solution describes the inflation-like stage of the evolution of the Universe, which continues until the equation of state of the dilaton matter changes and becomes differ from the equation of state of the superrigid matter.

When the smooth jump of equation of state from $\gamma = 1$ to $\gamma = 1/3$ happens, the FL equation (6.17) describes the graceful exit from superrigid stage to the radiation stage of Universe evolution.

Let us consider the radiation dominated stage with $\gamma = 1/3$. In this case the equation (6.6) yields $\varepsilon\, a^4 = \mathcal{E}_{1/3} = \text{const}$. The modified Friedmann–Lemaître equation (6.17) for the expanding Universe ($\dot{a} > 0$) takes the form,

$$\dot{a} = \sqrt{\frac{æ\varepsilon_{\text{v}}}{3}} \cdot \frac{1}{a^2} \left( a^6 + \frac{\mathcal{E}_{1/3}}{\varepsilon_{\text{v}}} a^2 - \frac{\mathcal{E}}{\varepsilon_{\text{v}}} \right)^{1/2} . \tag{7.1}$$

When the condition $\dot{a} = 0$ holds, then the extremum of the scale factor realizes. In the case $a \ll 1$, when the value of $\mathcal{E}$ is positive ($\alpha > 0$) and sufficiently small in comparison with the value of $\mathcal{E}_{1/3}$, this extremum is approximately equal to

$$a_{m1} \approx \left( \frac{\mathcal{E}}{\mathcal{E}_{1/3}} \right)^{1/2} = a_1 \ll 1 .$$

This value realizes the minimum of the scale factor because from the equation (6.18) one can see that for this value the condition $\ddot{a} > 0$ holds.

In the limiting case $t \to 0$, $a \to a_1$ the equation (7.1) can be integrated with the solution,

$$a^2 + a_1^2 \ln \frac{a}{a_1} = a_1^2 + 2\sqrt{\frac{æ\mathcal{E}_{1/3}}{3}}\, t .$$

This solution demonstrates the correct behavior $a \sim \sqrt{t}$ of the scale factor of the Friedmann radiation dominated stage under small (but not infinitesimal) time.

When the radiation energy density becomes sufficiently small in comparison with the energy density of matter, the matter dominated stage begins with $\gamma = 2/3$. Then (6.6) yields $\varepsilon a^5 = \mathcal{E}_{2/3} = \text{const}$ and the modified FL equation (6.17) for the case $\dot{a} > 0$ takes the form,

$$\dot{a} = \sqrt{\frac{æ\varepsilon_{\text{v}}}{3}} \cdot \frac{1}{a^2} \left( a^6 + \frac{\mathcal{E}_{2/3}}{\varepsilon_{\text{v}}} a - \frac{\mathcal{E}}{\varepsilon_{\text{v}}} \right)^{1/2} . \tag{7.2}$$

If the value of $\mathcal{E}$ is sufficiently small in comparison with the value of $\mathcal{E}_{2/3}$, the minimum of the scale factor (when $\dot{a} = 0$ holds) in the case $a \ll 1$ is approximately equal to the value

$$a_{m2} \approx \frac{\mathcal{E}}{\mathcal{E}_{2/3}} = a_2 \ll 1 . \tag{7.3}$$

The equation (6.18) in case $\gamma = 2/3$ reads,

$$\frac{\ddot{a}}{a} = \frac{æ}{3a^6} \left( \varepsilon_{\text{v}} a^6 - \frac{3}{2} a \mathcal{E}_{2/3} + 2\mathcal{E} \right) , \tag{7.4}$$

Using this equation, one can easily verify that the value (7.3) of the scale factor corresponds to the condition $\ddot{a} > 0$.

In the limiting case $t \to 0$, $a \to a_2$ the equation (7.2) can be integrated with the solution,

$$a^{5/2} + \frac{5}{6} a_2 a^{3/2} = \frac{11}{6} a_2^{5/2} + \frac{5}{2}\sqrt{\frac{æ\mathcal{E}_{2/3}}{3}}\, t .$$

This solution demonstrates the correct behavior $a \sim t^{2/5}$ of the scale factor of the Friedmann matter dominated stage under small (but not infinitesimal) time.

In the limiting case $t \to \infty$, $a \to \infty$ the equation (7.2) has the de-Sitter-like solution with $\ddot{a} > 0$,

$$a = C \exp\left(\frac{\Lambda}{3}\,t\right), \qquad C > 0\,,$$

where $C$ is an arbitrary positive constant. Therefore an accelerating stage of the evolution of the Universe is predicted.

By equating the right side of the equation (7.4) to zero one can see that two points of inflection of the scale factor plot exist [14]. The first one has very small value, but the value of the second one,

$$a_{infl} \approx \left(\frac{3æ\mathcal{E}_{2/3}}{2\Lambda}\right)^{1/5},$$

corresponds to the modern era. This is the point, when the Friedmann expansion with deceleration has been replaced by the expansion with acceleration, which is in accordance with the modern observational data [8], [9].

The last stage of expansion is the dust stage with $\gamma = 0$. In this case the equation (6.6) yields $\varepsilon a^3 = \mathcal{E}_0 = \text{const}$, where $\mathcal{E}_0$ is the total mass-energy of the dilaton matter of the Universe.

The modified Friedmann–Lemaître equation (6.17) for the expanding Universe ($\dot{a} > 0$) takes the form,

$$\dot{a} = \sqrt{\frac{æ\varepsilon_{\mathrm{v}}}{3}} \cdot \frac{1}{a^2}\left(a^6 + \frac{\mathcal{E}_0}{\varepsilon_{\mathrm{v}}}a^3 - \frac{\mathcal{E}}{\varepsilon_{\mathrm{v}}}\right)^{1/2}. \tag{7.5}$$

When the condition $\dot{a} = 0$ holds, then the minimum of the scale factor realizes,

$$a_{m3}^3 = -\frac{\mathcal{E}_0}{2\varepsilon_{\mathrm{v}}} + \frac{1}{2}\sqrt{\frac{\mathcal{E}_0^2}{\varepsilon_{\mathrm{v}}^2} + \frac{4\mathcal{E}}{\varepsilon_{\mathrm{v}}}}\,.$$

If the value $\mathcal{E}_0$ is very large, then one has

$$a_{m3} \approx \left(\frac{\mathcal{E}}{\mathcal{E}_0}\right)^{1/3} = a_0 \ll 1\,.$$

In this case the equation (7.5) can be exactly integrated with the solution,

$$a^3 + \frac{\mathcal{E}_0}{2\varepsilon_{\mathrm{v}}} + \sqrt{a^6 + \frac{\mathcal{E}_0}{\varepsilon_{\mathrm{v}}}a^3 - \frac{\mathcal{E}}{\varepsilon_{\mathrm{v}}}} = \mathcal{C}e^{\sqrt{3\Lambda}\,t}\,, \tag{7.6}$$

where $C$ is an arbitrary constant. For initial conditions $t = 0$, $a = a_{m3}$, the value of this constant is

$$\mathcal{C} = \frac{1}{2}\sqrt{\frac{\mathcal{E}_0^2}{\varepsilon_{\mathrm{v}}^2} + \frac{4\mathcal{E}}{\varepsilon_{\mathrm{v}}}}\,,$$

and the solution (7.6) takes the form

$$a = \left(\frac{æ\mathcal{E}_0}{2\Lambda}\right)^{\frac{1}{3}}\left(\sqrt{1 + \frac{4\Lambda\mathcal{E}}{æ\mathcal{E}_0^2}}\,\coth(\sqrt{3\Lambda}\,t) - 1\right)^{\frac{1}{3}}.$$

Another form of this solution is

$$a = \left( \frac{æ\mathcal{E}_0}{2\Lambda} \left(\coth(\sqrt{3\Lambda}\, t) - 1\right) \right.$$
$$\left. + a_{m3}^3 \coth(\sqrt{3\Lambda}\, t) \right)^{1/3} . \qquad (7.7)$$

If one puts in (7.7) $a_{m3} = 0$ then the cosmological monotonic model of $M_1$ type universe [25] appears which after of the point of inflection asymptotically turns into the empty de Sitter universe under $t \to \infty$.

## 8. Conclusions

Various non-standard cosmological theories lead to various modifications of the Friedmann–Lemaître equation.

In the metric-affine gravitation (MAG) one receives the modified FL equation similar to (6.17), but without a cosmological term [20]. After analyzing this equation the authors of [20] conclude that "purely dilational matter amplifies gravitational attraction. In particular, it accelerates rather then retards the possible collapse of a system." In this cosmological theory the analog of our constant $\alpha$ is negative that "corresponds to an additional effective *attractive* force dominating during the very early stages of evolution" of the Universe. Recently in [26] the SN Ia supernovae data were analyzed within this non-standard cosmological model with the cosmological term added.

In [12] the similar modified FL equation (also without a cosmological term) was received within the framework of the Einstein–Proca–matter system appearing from the Weyl–Cartan geometrical approach to the gravitational theory. For the pressure-free dust case $\gamma = 0$ it was shown by numerical methods that this equation has both singular and nonsingular solutions.

In [27] within the framework of the version of D-brain cosmology on the boundary of anti-de Sitter space the modified FL equation similar to (6.17) appears with $\mathcal{E} \sim Q_{4+1}^2$, where $Q_{4+1}$ corresponds to an "electric charge" in (4+1)-dimensional sense.

In our theory the system of equations (6.17)–(6.18) describe the nonsingular model of evolution of the Universe starting from an inflation-like stage (for the superrigid equation of state), passing radiation dominated and matter dominated decelerating stages and turning into the post-Friedmann accelerating era.

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