ELASTICITY THEORY IN GR

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The basic equation of classical elasticity theory [1–4] which in linear approximation is

$$\sigma = \Lambda\epsilon,$$

relates 2nd rank deformation tensor ϵ to 2nd rank stress tensor σ by means of 4th rank elasticity tensor Λ, and equations of motion, relating gradients of stress with acceleration, though they look simple, have no simple relativistic or, moreover, general relativistic generalization, and it has taken more 80 years of attempts [6-36] to come to their exact GR equivalent [37].

Looking back one can see that the main problem was conceptual. The classical notion of deformation is based on purely geometrical comparison of shapes of small object at different times and places, what is easy to do in the absolute Newtonian space simply subtracting tensors describing shape. However, difference of tensors given at different space-time points becomes ambiguous in relativistic case and fully meaningless in GR. So, the classical 'tensor' reasoning can not be generalized to the relativistic case and elasticity theory in GR should be built from scratch starting from well-defined scalar quantities.

GR elasticity theory should, of course, become equivalent to classical one in nonrelativistic conditions and does so. However, the relation between GR and classical formulations, due to different logical basis, is not obvious and was not sufficiently clarified in [37]. The aim of present paper is to give short, logically simple formulation of elasticity theory in GR and compare it with the classical elasticity theory.

Equations of Motion and Elasticity Conditions

GR, as a physical theory, formulates basically equations of motion of any matter as a set of three equation systems [5]:
Einstein equations

$$R^{\alpha\beta} + \frac{1}{2}Rg^{\alpha\beta} = -8\pi T^{\alpha\beta}$$

specifying the gravitation influence of the matter, described by (density of) stress-energy tensor T, on the space metric g;
Bianchi identity

$$T^{\alpha\beta}_{;\beta} = 0$$

specifying the motion of matter in the space with given metric g; and state equations

$$f(T, ..., a, ...) = 0$$

specifying the dependence of stress-energy tensor of scalar parameters a, describing internal state of matter. Most common example of parameters a is temperature and state equation usually relates

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pressure to temperature. The concern of elasticity theory is to relate stress-energy tensor to scalars specifying relative positions of interacting particles, of which elastic matter is made.

The building of the elasticity theory in GR needs, first of all, the formulation of what matter is considered as 'elastic'. Its definition is not evident, especially since 'rigid' bodies do not exist in GR. Physically, the change of relative distances between particles may result in many other changes: in heating, in chemical transformations, in magnetization, etc. Matter is elastic, if all the changes are reversible and are uniquely related to stresses. Mathematically it means that stresses are partial derivatives of some potential W with respect to distances between neighboring particles in the matter or derivatives with respect to any parameters, of which these distances depend.

In nonrelativistic elasticity theory, one chooses the elements of the deformation tensor as such parameters, and elasticity condition can be written as

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}},$$

or, equivalently, as classical 15 symmetry conditions imposed on elasticity tensor

$$\Lambda_{ijkl} = \Lambda_{klij} = \Lambda_{klji},$$

which are automatically fulfilled, if

$$\Lambda_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}}.$$

In GR, since distances are scalar products depending on metric g, one may use derivatives with respect to g to formulate elasticity property. Let us split T , as usual, into density flow and stress part P

$$T^{\alpha\beta} = \rho U^\alpha U^\beta + P^{\alpha\beta}, \qquad (U \cdot U = -1, P \cdot U = 0),$$

where ρ is matter density, U is 4-velocity, and dot means scalar product using metric g. Matter is called elastic, if the elements of stress tensor are the derivatives of some scalar potential W with respect to metric g:

$$P^{\alpha\beta} = \frac{\partial W}{\partial g_{\alpha\beta}}.$$

Technically, similar formulation of elasticity conditions can be used in nonrelativistic case as well, though the derivatives of the stress tensor σ with respect to elements of Euclidean metric tensor would look out of style in a context of classical reasoning.

To finish the formulation of equations of motion, one have to specify the arguments of elastic potential W. If 3-dimensional matter is continuous and remain continuous during its motion, than the knowledge of time dependence of distances between 4 neighboring particles is sufficient for the calculation of time dependence of distances between all other neighboring particles. So, W at each space-time point x is a function of 6 distances between 4 particles of matter chosen near point x, or, equivalently, the function of 6 scalar products of vectors A connecting one of the particles with 3 other.

If particles are sufficiently close to each other, their relative velocities are small and choice of synchronization is inessential. For simplicity, one may define distances at the rest frame of matter at point x and place one of the particles at point x. Then the arguments of function W can be formally defined as follows.

Let y be 3-dimensional time-independent coordinates co-moving with matter and labelling in a continuous way particles of matter. Trajectories of matter particles in space-time X are some functions of y and of some evolution parameter θ:

$$x = x(\theta, y).$$

Using these functions, one may define 3 4-vectors A_i in the direction of nearby particles as projections of vectors

$$B_i = \partial x/\partial y_i$$

on a subspace orthogonal to 'velocity' $v = \partial x/\partial \theta$:

$$A_i = B_i - v(B_i \cdot v)/(v \cdot v).$$

Then 6 scalar products

$$N_{ij} = A_i \cdot A_j$$

can be chosen as the arguments of potential W. To use this definition for the solution of equations of motion with a given initial state of matter in the observer frame, one needs the time derivative of B in that frame. Its calculation does not use the metric, goes in the same way as in the nonrelativistic case, and expresses the time derivative of space part $B^*$of vectors B in the observer frame through the Lie derivative with respect to space part of v:

$$\dot{B}^* = -\mathcal{L}_{v^*} B^* = -\frac{\partial B^*}{\partial x_i} v_i + \frac{\partial v^*}{\partial x_i} B_i.$$

If one sets $\theta = t$, this relation can be written as explicitly covariant equation with 4-dimensional Lie derivative

$$\mathcal{L}_v B = 0.$$

Now, the equations of motion of elastic matter in GR are complete and ready for use as soon as one chooses a specific elastic potential W. These equations do not use explicitly either deformation tensor, or some of its 4-dimensional generalization. It happens this way because potential W depends only on distances and locally (at small, microscopic scale) the change of distances due to motion of particles and due to change of metric are physically indistinguishable, and besides the elements of metric tensor no more parameters in potential W are needed.

Deformations

Though deformation tensor is not needed in GR formulation, its construction may be useful for better understanding of connection with classical elasticity theory, and in some special cases (when metric is constant, or almost constant, or relativistic effects are small).

Note, first of all, that 3x3 matrix of scalar products N_{ij} resembles much dilatation tensor, but is not correctly normalized: it does not turn into unit matrix in the unstressed state. However, if matter coordinates y are chosen so that at the moment, when the matter is unstressed (when potential W has minimal value), matrix N is a unit matrix, then its elements can be interpreted in the same way as elements of dilatation tensor and elements of its difference with a unit matrix, as elements of deformation tensor.

The problem is that matrix N is not a tensor in observer's space-time X. The elements of N are scalars in space X. Originally, matrix N transforms like a tensor with transformations of time-independent matter coordinates y and can be considered as a tensor in matter space (in 3-dimensional manifold) Y. But if the coordinates y are fixed by some relation to x in the unstressed state, or matrix N is somehow 'normalized' to 1 in unstressed state, it is not a tensor in space Y any more.

So, one may say that GR formulation of elasticity theory uses implicitly some 'unnormalized' dilatation tensor, which does not become unit matrix in unstressed state and which does not belong to the same space as stress tensor P does.

Still, the question remains whether the expression for 4-dimensional stress tensor P can be written as a product of two tensors similar to Λ, σ, but in space X. Earlier attempts to 'generalize' tensors Λ, σ to relativistic case were not too encouraging [29-36]. Now, after the GR version of elasticity theory is formulated, and one has matrix N and vectors A as building blocks, the construction of relativistic tensors Λ, σ becomes feasible, if one limits the task to some 'linear' elasticity.

Let $\bar{N}_{ij}$ be the value of N_{ij} for unstressed state of the matter and $\Delta_{ij} = N_{ij} - \bar{N}_{ij}$. Let stress tensor P be linear in Δ. Then potential W must be quadratic in Δ :

$$W = L^{ijkl}\Delta_{ij}\Delta_{kl},$$

where implied summation over Latin indexes goes from 1 to 3 and elements of matrix L (elasticity coefficients) are constants. Corresponding stress tensor has form

$$P^{\alpha\beta} = L^{ijkl}(\Delta_{ij}A^{\alpha\beta}_{kl} + \Delta_{kl}A^{\alpha\beta}_{ij}),$$

where tensor $A^{\alpha\beta}_{ij}$ is

$$A^{\alpha\beta}_{ij} = A^{\alpha}_{i}A^{\beta}_{j} + A^{\alpha}_{j}A^{\beta}_{i}.$$

The splitting of P into product $\Lambda\sigma$ is limited by the requirements that Λ should not depend on Δ , and that σ must vanish in the unstressed state, be dimensionless, be a tensor of second rank in X, and have spatial part that turns into nonrelativistic deformation tensor in nonrelativistic limit. The expressions, satisfying these requirements, are

$$\Lambda^{\alpha\beta\gamma\delta} = L^{ijkl}A^{\alpha}_{i}A^{\beta}_{j}A^{\gamma}_{k}A^{\delta}_{l},$$

$$\epsilon^{\alpha\beta} = \frac{1}{2}M^{mi}\Delta_{ij}M^{jn}A^{\alpha}_{m}A^{\beta}_{n},$$

where M is a matrix inverse to N.

The expression for relativistic deformation tensor due to presence of inverse matrices does not immediately suggest a simple geometrical interpretation.

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