ANALYTICITY WITHOUT POLYNOMIAL BOUNDEDNESS

Yu.S. Vernov[1]
Institute for Nuclear Research, Russian Academy of Sciences, Moscow, Russia
M.N. Mnatsakanova[2]
Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, Russia

1 Introduction

The proof of analytical properties of scattering amplitudes is one of the most remarkable achievements of axiomatic approach in quantum field theory.Dispersion relations (DR) for elastic scattering amplitude were derived in papers of Gell-Mann, Goldberger, Thirring, Miyazawa, Nambu and Oehme [1–4]. They were rigorously proved in works of Bogoliubov, Oehme, Symanzik, Bremermann, Tailor and Lehmann [5–9]. The detailed proof of DR was made in the book of Bogoliubov, Medvedev, Polivanov [10].

In deriving analytical properties of scattering amplitudes the standard assumption is the supposition of polynomial boundedness of them, that is there exists n such that

$$\frac{F(E)}{E^n} \to 0, \qquad E \to \infty.$$

Consideration of analytical properties of scattering amplitudes under more general condition was done in [11]. The main idea of such a proof implies the use of a damping function introduced in [12].

In our report we show that a classical proof of analyticity given in the book of Bogoliubov and Shirkov [13] can be extended on amplitudes, for which the condition of polynomial boundedness is substituted by a weaker condition. Namely, we assume the absence of exponential growth of scattering amplitude (precisely the validity of the inequality (5)). It is very interesting that just the same condition was obtained by Jaffe in the paper [13], where he introduced more general class of generalized functions than tempered distribution.

In spite of the fact that polynomial boundedness of scattering amplitudes is the consequence of very week conditions [15, 16] it is important to weaken as much as possible the conditions under which analyticity in question can be proved. First of all it is significant for various extensions of standard theory, e.g. for noncommutative quantum field theory [17]–[19], (for a review, see [20]).

Let us concentrate our efforts on the simplest and the most important case of forward elastic scattering of two spin-free particles with masses m (meson) and M (nucleon). In reality the same analytical properties are valid for meson-nucleon scattering after the averaging over spin. Thus we can omit the complications related with spins of particles. For simplicity we also consider meson as neutral particle, that is our results are valid directly for $(\pi^0 - N)$-scattering or for the sum of $(\pi^+ - N)$- and $(\pi^- - N)$-scattering amplitudes.

[1]E-mail address: vernov@ms2.inr.ac.ru
[2]E-mail address: mnatsak@theory.sinp.msu.ru

2 Forward $(\pi - N)$–scattering

We consider the scattering of two neutral particles with masses m and M.

It should be noted that we admit the condition of local commutativity:

$$[j(x), j(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0, \tag{1}$$

$j(x)$ is the current of interacting fields.

In accordance with the standard LSZ reduction formulas (see, e.g. [21]) scattering amplitude is:

$$F(E, \vec{q}) = \int d^4x \, e^{i(E x_0 - \vec{q}\vec{x})} \, \tau(x_0) \, F(x), \tag{2}$$

where

$$F(x) = < M \left| \left[j\left(\frac{x}{2}\right), j\left(-\frac{x}{2}\right) \right] \right| M >,$$

$\tau(x_0) = 1, \; x_0 \geq 0, \; \tau(x_0) = 0, \; x_0 < 0.$

We omit in eq. (2) the factor, which is irrelevant to analytical properties of $F(E, \vec{q})$. Eq. (2) is written in the reference frame, in which particle with the mass M is at rest. E and $\vec{q}$ are energy and momentum of the particle with mass m respectively.

Let us write $\vec{q}$ in the form: $\vec{q} = \vec{e}\,|\vec{q}|, \; |\vec{e}| = 1$. Then

$$F(E, \vec{q}) = \int d^4x \, e^{i(E x_0 - \vec{e}\vec{x}\sqrt{E^2 - m^2})} \, \tau(x_0) \, F(x), \tag{3}$$

In order to exclude singularity of $\sqrt{E^2 - m^2}$ we substitute $F(E, |\vec{q}|, \vec{e})$ by

$$\frac{1}{2}\left(F(E, |\vec{q}|, \vec{e}) + F(E, |\vec{q}|, -\vec{e}) \equiv F(E).\right.$$

This is a standard step. Thus

$$F(E) = \int e^{i E x_0} \cos(\vec{e}\vec{x}\sqrt{E^2 - m^2}) \, \tau(x_0) \, F(x) \, d^4x. \tag{4}$$

The direct extension $F(E)$ on complex E is impossible ([13], chapter 10) as

$$Im \sqrt{E^2 - m^2} > Im \, E.$$

To overcome this obstacle we following Bogolyubov and Shirkov substitute $F(E)$ by regularized amplitude $F_\varepsilon(E)$:

$$F_\varepsilon(E) =$$

$$\int e^{i E x_0} \cos(\vec{e}\vec{x}\sqrt{E^2 - m^2}) \, \tau(x_0) \, F_\varepsilon(x) \, d^4x. \tag{4'}$$

$$F_\varepsilon(x) = \exp(-\varepsilon({x_0}^2 + |\vec{x}|^2)) \, F(x)$$

$F_\varepsilon(E)$ is an analytical function in an upper half-plane as integral in the latter equation converges.

The main problem is to prove the existence of analytical function $F(E) = \lim_{\varepsilon \to 0} F_\varepsilon(E)$. To this end let us use the analytical properties of $F_\varepsilon(E)$. Our goal is to represent $F_\varepsilon(E)$ at complex E as integral over real axe only and then go to $\varepsilon = 0$. But it is impossible to do this directly as $F_\varepsilon(E) \not\to 0$ at $E \to \infty$. So first we have to construct such a function. Usually to this end the polynomial boundedness of $F(E)$ is used. Here we use the weaker bound. Namely we suppose the existence of the following inequality

$$|\, F(E) \,| < \exp \frac{E}{(\ln E)^{1+\alpha}}, \qquad E \to \infty, \tag{5}$$

where $\alpha > 0$ can be arbitrary small. Evidently inequality (5) is fulfilled also for $F_\varepsilon(E)$. Condition similar with the inequality (5) is valid also at $E \to -\infty$ as

$$F(-E + i\,0) = F^*(E + i\,0),\; F_\varepsilon(-E + i\,0) = F_\varepsilon^*(E + i\,0). \tag{6}$$

Eq. (6) is a standard crossing symmetry condition.

Evidently, function

$$\psi_\varepsilon(E) = F_\varepsilon(E)\,\xi(E), \tag{7}$$

where

$$\xi(E) = \exp\frac{\sqrt{m^2 - E^2}}{\ln^\beta(-\sqrt{m^2 - E^2})}, \qquad \alpha > \beta,$$

satisfies the necessary condition

$$\psi_\varepsilon(E) \to 0, \qquad E \to \pm\infty. \tag{8}$$

Indeed, it is easy to see that at $|E| \to \infty$

$$\xi(E) \cong \exp\frac{i\,E}{\left(\ln E - \frac{i\,\pi}{2}\right)^\beta}. \tag{7'}$$

For any φ, $0 < \varphi < \pi$

$$|\xi(E)| < \exp\frac{-|E|\,\sin\varphi}{(\ln E)^\beta}.$$

At $\varphi = 0$

$$|\xi(E)| < \exp\frac{-\pi\,\beta\,|E|}{2\,(\ln E)^{1+\beta}}.$$

It is easy to see that $\xi(E)$ is an analytical function in the whole E-plane with cuts (m, ∞), $(-\infty, -m)$ satisfying the conditions

$$\xi(-E + i\,0) = \xi^*(E + i\,0) = \xi(E - i\,0) \tag{9}$$

Using the Cauchy formula we see that

$$\psi_\varepsilon(E) = \frac{1}{2\,\pi\,i}\int_C \frac{\psi_\varepsilon(E')\,d\,E'}{E' - E}, \quad Im\,E > 0. \tag{10}$$

Contour C consists of interval $(-R, R)$ and semicircle in an upper half-plane.

Now let us demonstrate that owing to cond.(1)

$$\psi_\varepsilon(R\,e^{i\,\varphi}) \to 0 \quad \text{if} \quad R \to \infty, \quad 0 < \varphi < \pi. \tag{8'}$$

Indeed, if $|E| \to \infty$, then

$$Im\sqrt{E^2 - m^2} \cong Im\,E - Im\,\frac{m^2}{2\,E}$$

Thus

$$\left|e^{i\,E\,x_0}\,\cos(\vec{e}\vec{x}\,\sqrt{E^2 - m^2})\right| \le e^{-Im\,E\,(x_0 - |\vec{x}|)}\cdot e^{\frac{m^2\,|\vec{x}|\,\sin\varphi}{R}}. \tag{11}$$

Owing to the factor $\exp(-\varepsilon\,|\vec{x}|^2)$ integral over $\vec{x}$ converges when $x_0 \to \infty$, so the integration is really taken over some finite interval. The first factor in (11) is less than unity as $x_0 > |\vec{x}|$. The second factor tends to unity at any fixed ε if $R \to \infty$. Thus the growing factor in exponential in eq. (4′) disappears at $|E| \to \infty$. So we can conclude that cond. (8′) follows from the cond. (8). Actually in order to prove that condition (8′) is the consequence of condition (8), it is sufficient to assume that $\psi_\varepsilon\,(R\,e^{i\,\varphi})$ grows more slowly than any exponent and use the Phragmen-Lindelöf theorem (see e.g. [22]). Thus we can put $R=\infty$ in eq. (10). So

$$\psi_\varepsilon\,(E) = \frac{1}{2\,\pi\,i}\int\limits_{-\infty}^{\infty}\frac{\psi_\varepsilon\,(E')\,d\,E'}{E'-E}, \qquad Im\,E>0. \tag{12}$$

Eq. (12) is valid at any fixed ε. Now let us go to $\varepsilon=0$. First let us consider the interval (m,∞). We can go to the limit $\varepsilon=0$ without any problem as in this interval $\lim_{\varepsilon\to0} F_\varepsilon\,(E') = F\,(E')$. The interval $(-\infty,-m)$ can be treated similarly in accordance with eq. (6).

Let us show that the remained interval can be considered as well as usually. To consider this interval let us construct the analytical function in the lower half-plane and then prove that this function is an analytical continuation of $F_\varepsilon\,(E)$. To this end we use the function

$$\tilde F\,(E,\vec q) = \int d^4x\,e^{i\,(E\,x_0-\vec q\,\vec x)}\,\tau\,(-x_0)\,F\,(-x). \tag{13}$$

Then we substitute $\tilde F\,(E,\vec q)$ by

$$\tilde F_\varepsilon\,(E) = \int e^{i\,E\,x_0}\cos\,(\vec e\vec x\sqrt{E^2-m^2})\tau\,(-x_0)\,F_\varepsilon\,(-x)\,d^4\,x, \tag{13′}$$

$$F_\varepsilon\,(-x) = \exp(-\varepsilon({x_0}^2+|\vec x|^2))\,F\,(-x).$$

Evidently

$$\tilde F_\varepsilon\,(E-i\,0) = F_\varepsilon\,(-E+i\,0) = F^*_\varepsilon\,(E+i\,0) \tag{14}$$

The last equality in (14) is eq. (6), to prove the first one it is sufficient to replace x by $-x$. We can put $\varepsilon=0$ in eq. (14).

$$\tilde F\,(E-i\,0) = F^*\,(E+i\,0) \tag{14′}$$

The function

$$\tilde\psi_\varepsilon\,(E) = \tilde F_\varepsilon\,(E)\,\xi\,(E) \tag{15}$$

is an analytical function in a lower half-plane and

$$\tilde\psi_\varepsilon\,(E)\to0,\;\;E\to\pm\,\infty.$$

We use the same arguments as for proof of analyticity of $\psi_\varepsilon\,(E)$ in an upper half-plane. Thus

$$\frac{1}{2\,\pi\,i}\int_{\tilde C}\frac{\tilde\psi_\varepsilon\,(E')\,d\,E'}{E'-E} = 0, \quad Im\,E>0. \tag{16}$$

$\tilde C$ consists of interval $(R,-R)$ and semicircle in a lower half-plane. Let us sum expressions (10) and (16). Using eq. (14) and taking into account that integral over semicircle in a lower half-plane tends to zero if $R\to\infty$ on the same reason as corresponding integral in an upper half-plane, we obtain that

$$\psi_\varepsilon\,(E) = \frac{1}{\pi}\int\limits_{m}^{\infty}\frac{Im\,\psi_\varepsilon\,(E')\,d\,E'}{E'-E} + \frac{1}{\pi}\int\limits_{-\infty}^{-m}\frac{Im\,\psi_\varepsilon\,(E')\,d\,E'}{E'-E} +$$

$$\frac{1}{2\,\pi\,i}\int\limits_{-m}^{m}\frac{\left(\psi_\varepsilon\,(E')-\tilde\psi_\varepsilon\,(E')\right)\,d\,E'}{E'-E}, \qquad Im\,E>0. \tag{17}$$

In the two first terms in (17) we can go to $\lim \varepsilon = 0$. To consider the remained integral let us obtain in the physical domain the expression for $F(E,\vec{q}) - \tilde{F}(E,\vec{q})$ suitable to extension for nonphysical $E(-m < E < m)$. From the definitions (3) and (13) it follows that

$$F(E,\vec{q}) - \tilde{F}(E,\vec{q}) = F_+(E,\vec{q}) - F_-(E,\vec{q}), \tag{18}$$

where

$$F_\pm(E,\vec{q}) = \int d^4x\, e^{i(E x_0 - \vec{q}\vec{x})} F_\pm(x). \tag{19}$$

$$F_+(x) =< M\left|j\left(\frac{x}{2}\right) j\left(-\frac{x}{2}\right)\right| M >,$$

$$F_-(x) =< M\left|j\left(-\frac{x}{2}\right) j\left(\frac{x}{2}\right)\right| M > . \tag{20}$$

Let us suppose that vectors $|p,n>$ form the complete system of basis vectors and so

$$< M\left|j\left(\frac{x}{2}\right) j\left(-\frac{x}{2}\right)\right| M >=$$

$$\sum_n \sum_{p_n^0} \int d^3p < M\left|j\left(\frac{x}{2}\right)\right| p,n >< p,n\left|j\left(-\frac{x}{2}\right)\right| M >, \tag{21}$$

p is a momentum, p_n^0 is the energy of the state $|p,n>$, n denotes all other quantum numbers.

Using the equality

$$< p^{'}\,|j\,(x)|\,p >= e^{i(p^{'}-p)\,a}\; < p^{'}\,|j\,(x-a)\,|\,p >,$$

where $|\,p\;>$ and $|\,p^{'}\;>$ are eigenvectors of operator p, we see that owing to eqs. (20), (21)

$$F_\pm(E,\vec{q}) = \sum_n \sum_{p_n^0} |< M\,|j\,(0)|\,p,n >|^2 \cdot \delta\,(p_n^0 - M \mp E), \tag{22}$$

$\vec{p} = \mp\vec{q}$.
Thus $F_\pm(E,\vec{q}) \neq 0$ only if

$$\sqrt{M_n^2 + \vec{q}^2} = M \pm E, \quad p_n^0 = \sqrt{M_n^2 + \vec{q}^2}. \tag{23}$$

Let us suppose that as in case of $\pi - N$-scattering $M_n \geq M + m$, excluding one M particle intermediate state. We can extend expression (23) on the E belonging to the interval $(-m, m)$. $F_\pm(E,\vec{q}) \neq 0$ in this interval only if $M_n = M$ and $E = \mp m^2/2\,M$.

Thus we see that in the integral under consideration

$$\lim_{\varepsilon=0} (\psi_\varepsilon(E) - \tilde{\psi}_\varepsilon(E)) = 0,$$

excluding 2 points: $\pm\frac{m^2}{2\,M}$. In order to vanish integral over the interval $(-m, m)$ it is sufficient to substitute $\psi_\varepsilon(E)$ and $\tilde{\psi}_\varepsilon(E)$ by functions:

$$\Phi_\varepsilon(E) = \left(E^2 - \frac{m^4}{4\,M^2}\right)\psi_\varepsilon(E),$$

$$\tilde{\Phi}_\varepsilon(E) = \left(E^2 - \frac{m^4}{4\,M^2}\right)\tilde{\psi}_\varepsilon(E).$$

Representing $\Phi_\varepsilon(E)$ by the expression analogous to (17) we see that in accordance with this representation there exists $\lim\limits_{\varepsilon=0} \Phi_\varepsilon(E) = \Phi(E)$. Function $\Phi(E)$ (and so function $\left(E^2 - \frac{m^4}{4M^2}\right) F(E)$) is an analytical function in the whole E-plane excluding cuts $(-\infty, -m)$, (m, ∞). Moreover $\tilde{\Phi}(E) = \lim\limits_{\varepsilon=0} \tilde{\Phi}_\varepsilon(E)$ is an analytical continuation of $\Phi(E)$ and $F(E)$ is an analytical function in the same domain excluding points $\pm \frac{m^2}{2M}$, where it has poles.

Thus we come to the following expression for $F(E)$:

$$F(E)\,\xi(E) = \frac{2}{\pi} \int\limits_m^\infty \frac{E'\, Im\left(F(E')\,\xi(E')\right) dE'}{E'^2 - E^2} + \text{pole terms}, \quad Im\, E > 0. \tag{24}$$

Eq. (24) gives us the necessary continuation of $F(E)$ into the upper-half plane. Passing to the limit $Im\, E = 0$ in eq. (17) and using eqs. (6) and (9), we come to the relation

$$Re\,(F(E)\,\xi(E)) = \frac{2}{\pi} \int\limits_m^\infty \frac{E'\, Im\left(F(E')\,\xi(E')\right) dE'}{E'^2 - E^2} + \text{pole terms}. \tag{25}$$

Expression (25) is the generalization of standard dispersion relations for amplitudes, which grow more rapidly than any polynomial.

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