THE PAINLEVÉ ANALYSIS AND CONSTRUCTION OF SOLUTIONS FOR THE GENERALIZED HÉNON-HEILES SYSTEM

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The generalized Hénon–Heiles system with additional nonpolynomial term is considered. In two nonintegrable cases new two-parameter solutions have been obtained in terms of elliptic functions. These solutions generalize the known one-parameter solutions. The singularity analysis shows that it is possible that the obtained solutions are parts of three-parameter single-valued solutions.

1 Introduction

Beginning with the well-known investigations [1, 2], there was a renewal of interest in the question of integrability of galactic systems. The motion of a star in a time-independent, axially symmetric and gravitationally smooth galactic potential field was studied. Due to the symmetry these model was reduced to two-dimensional Hamiltonian systems. Numerical investigations of such systems with polynomial potentials allowed to detect the phenomenon of “dynamic chaos”. The best known Hamiltonian with this phenomenon is the Hénon–Heiles one.

Chaos is not only a mathematical property of some two-dimensional dynamical systems, but also a property of motion of some astronomical objects. The investigations of orbital evolution of various bodies of the Solar system at the long periods of time show various manifestations of the dynamic chaos, which is linked to the existence of orbital and spin-orbital resonances. The most impressive examples of chaotic motions are discovered in asteroid belts. The description of the domains of regular and chaotic motions by analytical methods is especially needed for the study of long-term orbital evolution of asteroids with the mean motions comparable to those of Jupiter.

2 The generalized hénon–heiles system

The generalized Hénon–Heiles system is described by the Hamiltonian

$$H=\frac{1}{2}\Big(x_t^2+y_t^2+\lambda_1x^2+\lambda_2y^2\Big)+x^2y-\frac{C}{3}\,y^3+\frac{\mu}{2x^2} \qquad (1)$$

and the corresponding system of the motion equations:

$$\left\{\begin{array}{l} x_{tt}=\,-\,\lambda_1x-2xy+\dfrac{\mu}{x^3},\\ y_{tt}=\,-\,\lambda_2y-x^2+Cy^2,\end{array}\right. \qquad (2)$$

where $x_{tt}\equiv\frac{d^2x}{dt^2}$ and $y_{tt}\equiv\frac{d^2y}{dt^2}$, λ_1, λ_2, μ and C are arbitrary numerical parameters. Note, that if $\lambda_2\neq 0$, then one can put $\lambda_2=sign(\lambda_2)$ without the loss of generality.

The generalized Hénon–Heiles system is a model, not only actively investigated by various mathematical methods (see [3] and references therein), but also widely used in astronomy and physics.

Due to the Painlevé analysis the following integrable cases have been found:

$$\begin{array}{lll} \text{(i)} & C=-1, & \lambda_1=\lambda_2, \\ \text{(ii)} & C=-6, & \lambda_1,\ \lambda_2 \text{ arbitrary}, \\ \text{(iii)} & C=-16, & \lambda_1=\lambda_2/16. \end{array}$$

In all above-mentioned cases system (2) is integrable at any value of μ. One of lines of investigation of the generalized Hénon–Heiles system is search of special solutions [4–9]. The general solutions in the analytic form are known [10–13] only in the integrable cases, in other cases not only four-, but even three-parameter exact solutions have yet to be found.

The function y, solution of system (2), satisfies fourth-order equation [6, 8, 14]:

$$y_{tttt}=(2C-8)y_{tt}y-(4\lambda_1+\lambda_2)y_{tt}+2(C+1)y_t^2+\frac{20C}{3}y^3+(4C\lambda_1-6\lambda_2)y^2-4\lambda_1\lambda_2 y-4H, \quad (3)$$

where H is the energy of the system. We note, that the energy H is not an arbitrary parameter, but a function of initial data: y_0, y_{0t}, y_{0tt} and y_{0ttt}. The form of this function depends on μ:

$$H=\frac{1}{2}(y_{0t}^2+y_0^2)-\frac{C}{3}y_0^3+\left(\frac{\lambda_1}{2}+y_0\right)(Cy_0^2-\lambda_2 y_0-y_{0tt})+\frac{(\lambda_2 y_{0t}+2Cy_0y_{0t}-y_{0ttt})^2+\mu}{2(Cy_0^2-\lambda_2 y_0-y_{0tt})}.$$

This formula is correct only if $x_0=Cy_0^2-\lambda_2 y_0-y_{0tt}\neq 0$. If $x_0=0$, that is possible only for $\mu=0$, then we can not express x_{0t} through y_0, y_{0t}, y_{0tt} and y_{0ttt}, so H is not a function of the initial data. If

$$y_{0tt}=Cy_0^2-\lambda_2 y_0 \qquad \text{and} \qquad y_{0ttt}=2Cy_0y_{0t}-\lambda_2 y_{0t},$$

then eq. (3) with an arbitrary H corresponds to system (2) with $\mu=0$, in opposite case:

$$y_{0tt}=Cy_0^2-\lambda_2 y_0 \qquad \text{and} \qquad y_{0ttt}\neq 2Cy_0y_{0t}-\lambda_2 y_{0t}$$

eq. (3) does not correspond to system (2).

To find a special solution of eq. (3) one can assume that y satisfies some more simple equation. For example, there exist solutions in terms of the Weierstrass elliptic functions, which satisfy the following equation:

$$y_t^2=\mathcal{A}y^3+\mathcal{B}y^2+\mathcal{C}y+\mathcal{D}, \quad (4)$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are some constants.

The following generalization of eq. (4):

$$y_t^2=\tilde{\mathcal{A}}y^3+\tilde{\mathcal{B}}y^{5/2}+\tilde{\mathcal{C}}y^2+\tilde{\mathcal{D}}y^{3/2}+\tilde{\mathcal{E}}y+\tilde{\mathcal{G}} \quad (5)$$

gives new one-parameter solutions in two nonintegrable cases [8]: $C=-16/5$ and $C=-4/3$ (λ_1 is an arbitrary number, $\lambda_2=1$). It is easy to show [8] that if $\tilde{\mathcal{B}}\neq 0$ or $\tilde{\mathcal{D}}\neq 0$ then $\tilde{\mathcal{G}}=0$, therefore, substitution $y=\varrho^2$ transforms eq. (4) into

$$\varrho_t^2=\frac{1}{4}\Big(\tilde{\mathcal{A}}\varrho^4+\tilde{\mathcal{B}}\varrho^3+\tilde{\mathcal{C}}\varrho^2+\tilde{\mathcal{D}}\varrho+\tilde{\mathcal{E}}\Big). \quad (6)$$

In [9] using the substitution $y\longrightarrow y-P_0$ a new parameter P_0 has been introduced and two-parameter solutions have been constructed for the above-mentioned values of C and a few values of λ_1 ($\lambda_2=1$). Due to Painlevé analysis local three-parameter solutions as the converging Laurent series have been found for an arbitrary λ_1 and $\lambda_2=1$ [19].

3 New solutions

Let us assume that solutions of eq. (3) in the neighborhood of singularity point t_0 tend to infinity as $y = c_\beta(t - t_0)^\beta$, where β, and c_β are some complex numbers. Of course, the real part of β has be less then zero. From this assumption it follows [15] that $\beta = -2$. The Laurent series of solutions of eq. (6) begin from term proportional to $(t - t_0)^{-1}$, so we seek solutions of eq. (3) as square polynomial: $y = P_2\varrho^2 + P_1\varrho + P_0$, where P_2, P_1 and P_0 are arbitrary numbers, ϱ is the general solution of eq. (6) with arbitrary coefficients $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{E}}$. Because of the function $\tilde{\varrho} = (\varrho - \frac{P_1}{2})/\sqrt{P_2}$ is a solution of eq. (6) as well, we can put $P_2 = 1$ and $P_1 = 0$ without loss the generality.

Substituting $y = \varrho^2 + P_0$ in eq. (3), we obtain

$$\begin{aligned} \varrho_{tttt}\varrho = & -4\varrho_{ttt}\varrho_t - 3\varrho_{tt}^2 + 2(C-4)\varrho_{tt}\varrho^3 + (2P_0(C-4) - 4\lambda_1 - \lambda_2)\varrho_{tt}\varrho + \\ & + 2(3C-2)\varrho_t^2\varrho^2 + (2CP_0 - 4\lambda_1 - 8P_0 - \lambda_2)\varrho_t^2 + \frac{10}{3}C\varrho^6 + \\ & + (2C\lambda_1 + 10CP_0 - 3\lambda_2)\varrho^4 + 2(2\lambda_1 CP_0 + 5CP_0^2 - \lambda_1\lambda_2 - 3P_0\lambda_2)\varrho^2 + \\ & + \frac{10}{3}CP_0^3 + 2\lambda_1 CP_0^2 - 3P_0^2\lambda_2 - 2\lambda_1\lambda_2 P_0 - 2H. \end{aligned} \tag{7}$$

The function ϱ is a solution of eq. (6), hence, eq. (7) is equivalent to the following system:

$$\left\{ \begin{aligned} & (3\tilde{\mathcal{A}} + 4)(-3\tilde{\mathcal{A}} + 2C) = 0, \\ & \tilde{\mathcal{B}}(-21\tilde{\mathcal{A}} + 9C - 16) = 0, \\ & 96\tilde{\mathcal{A}}CP_0 - 240\tilde{\mathcal{A}}\tilde{\mathcal{C}} - 192\tilde{\mathcal{A}}\lambda_1 - 384\tilde{\mathcal{A}}P_0 - 48\tilde{\mathcal{A}}\lambda_2 - \\ & - 105\tilde{\mathcal{B}}^2 + 128\tilde{\mathcal{C}}C - 192\tilde{\mathcal{C}} + 128C\lambda_1 + 640CP_0 - 192\lambda_2 = 0, \\ & 40\tilde{\mathcal{B}}CP_0 - 90\tilde{\mathcal{A}}\tilde{\mathcal{D}} - 65\tilde{\mathcal{B}}\tilde{\mathcal{C}} - 80\tilde{\mathcal{B}}\lambda_1 - 160\tilde{\mathcal{B}}P_0 - 20\tilde{\mathcal{B}}\lambda_2 + 56C\tilde{\mathcal{D}} - 64\tilde{\mathcal{D}} = 0, \\ & 16\tilde{\mathcal{C}}CP_0 - 36\tilde{\mathcal{A}}\tilde{\mathcal{E}} - 21\tilde{\mathcal{B}}\tilde{\mathcal{D}} - 8\tilde{\mathcal{C}}^2 - 32\tilde{\mathcal{C}}\lambda_1 - 64\tilde{\mathcal{C}}P_0 - 8\lambda_2\tilde{\mathcal{C}} + 24C\tilde{\mathcal{E}} + \\ & + 64\lambda_1 CP_0 + 160CP_0^2 - 16\tilde{\mathcal{E}} - 32\lambda_1\lambda_2 - 96P_0\lambda_2 = 0, \\ & 10\tilde{\mathcal{B}}\tilde{\mathcal{E}} + (5\tilde{\mathcal{C}} + 8CP_0 - 16\lambda_1 - 32P_0 - 4\lambda_2)\tilde{\mathcal{D}} = 0, \\ & 384H = -48\tilde{\mathcal{C}}\tilde{\mathcal{E}} + 96C\tilde{\mathcal{E}}P_0 + 384C\lambda_1 P_0^2 + 640CP_0^3 - 9\tilde{\mathcal{D}}^2 - \\ & - 192\tilde{\mathcal{E}}\lambda_1 - 384\tilde{\mathcal{E}}P_0 - 48\tilde{\mathcal{E}}\lambda_2 - 384\lambda_1\lambda_2 P_0 - 576\lambda_2 P_0^2. \end{aligned} \right. \tag{8}$$

System (8) has been solved by computer algebra software REDUCE [16].

If $\tilde{\mathcal{B}} \neq 0$, then from two first equations of system (8) we obtain:

$$C = -\frac{4}{3} \quad \text{and} \quad \tilde{\mathcal{A}} = -\frac{4}{3} \qquad \text{or} \qquad C = -\frac{16}{5} \quad \text{and} \quad \tilde{\mathcal{A}} = -\frac{32}{15}.$$

If $\tilde{\mathcal{B}} = 0$ then solutions with $\tilde{\mathcal{D}} \neq 0$ are also possible at $C = -16$ and $C = -1$, but only in integrable cases, that is at $\lambda_1 = -16\lambda_2$ and $\lambda_1 = \lambda_2$ accordingly.

The obtained solutions of eq. (3) depend on two parameters: energy H, expressed through P_0, and parameter t_0, connected to homogeneity of time. Six solutions of system (8) correspond to each value of P_0. Two of them (with $\tilde{\mathcal{B}} = \tilde{\mathcal{D}} = 0$) generate solutions of eq. (4).

We will consider only solutions with $\tilde{\mathcal{B}} \neq 0$ or $\tilde{\mathcal{D}} \neq 0$. These solutions can be separated on pairs in such a way that solutions in one pair differ only in signs of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ (see Appendix). Basic properties of the obtained solution are considered in this section. In the next section we will analyse in detail solutions of system (8) for some values of λ_1 and λ_2.

If the right-hand side of eq. (6) is a polynomial with multiple roots, then ϱ and y can be expressed in terms of elementary functions. In opposite case y is the fourth-order elliptic functions, that is doubly-periodic and meromorphic [17, 18].

If $\varrho(t)$ is a solution of eq. (6), then the function $\xi(t) \equiv -\varrho(t)$ satisfies the following equation:

$$\xi_t^2 = \frac{1}{4}\Big(\tilde{\mathcal{A}}\xi^4 - \tilde{\mathcal{B}}\xi^3 + \tilde{\mathcal{C}}\xi^2 - \tilde{\mathcal{D}}\xi + \tilde{\mathcal{E}}\Big). \tag{6'}$$

It is evident that $y(t) = \varrho^2(t) + P_0 = \xi^2(t) + P_0$, so two solutions of system (8) correspond to one function $y(t)$. From eq. (6) we obtain a polynomial equation in y:

$$(y_t^2 - \tilde{\mathcal{A}}(y-P_0)^3 - \tilde{\mathcal{C}}(y-P_0)^2 - \tilde{\mathcal{E}}(y-P_0))^2 = (y-P_0)^3(\tilde{\mathcal{B}}(y-P_0)+\tilde{\mathcal{D}})^2. \tag{9}$$

The function $\varrho(t)$ can be expressed through the the Weierstrass elliptic function $\wp(t)$ [18, Ch. 5]:

$$\varrho(t-t_0) = \frac{a\wp(t-t_0)+b}{c\wp(t-t_0)+d} \qquad (ad-bc=1),$$

where t_0 is an arbitrary parameter. Periods of $\wp(t)$ and the constants a, b, c and d are determined by eq. (6). The function

$$y(t-t_0) = \left(\frac{a\wp(t-t_0)+b}{c\wp(t-t_0)+d}\right)^2 + P_0 \tag{10}$$

is the fourth-order elliptic functions. This function, as a solution of eq. (3), can have only the second-order poles, therefore, in the parallelogram of periods it can have only two poles with opposite residues. Solutions (10) differ from solutions of eq. (4), which are the second-order elliptic functions [18]. Note, that $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ can be zero at some relations between λ_1 and λ_2.

The function $x(t)$ satisfies the first equation of system (2) with

$$\begin{aligned}\mu = {} & \frac{8}{3}C^2{P_0}^5 + \left(2\lambda_1 C^2 - \frac{14}{3}\lambda_2 C\right)P_0^4 + \left(2\lambda_2^2 - \frac{10}{3}C\tilde{\mathcal{E}} - 4\lambda_1\lambda_2 C\right)P_0^3 + \big(2\lambda_1\lambda_2^2 - \\ & - 2\lambda_1 C\tilde{\mathcal{E}} - 4CH + 3\lambda_2\tilde{\mathcal{E}}\big)P_0^2 + \left(2\lambda_1\lambda_2\tilde{\mathcal{E}} + \tilde{\mathcal{E}}^2 + 4\lambda_2 H\right)P_0 + 2\tilde{\mathcal{E}}H + \frac{1}{2}\lambda_1\tilde{\mathcal{E}}^2 + \frac{9}{128}\tilde{\mathcal{D}}^2\tilde{\mathcal{E}}.\end{aligned} \tag{11}$$

The trajectory of the motion can be derived from the second equation of system (2). Substituting y_{tt}, we obtain:

$$x^2 = (C - \frac{3}{2}\tilde{\mathcal{A}})y^2 + (3\tilde{\mathcal{A}}P_0 - \tilde{\mathcal{C}} - 1)y - \frac{1}{4}(5\tilde{\mathcal{B}}y + 3\tilde{\mathcal{D}} - 5\tilde{\mathcal{B}}P_0)(y-P_0)^{1/2} - \frac{1}{2}(\tilde{\mathcal{E}} + 3\tilde{\mathcal{A}}P_0^2 - 2\tilde{\mathcal{C}}P_0).$$

Note that when the coefficients $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ take zero values we get simple algebraic trajectories. The full list of such trajectories is presented in [7]. The parameter P_0 is absent in these trajectory equations.

One value of the energy H can correspond to no more than three values of P_0 and, hence, no more than six different one-parameter solutions. Solutions (10) differ from solutions of eq. (4), which are the second-order elliptic functions [18].

4 Analysis of solutions in a particular case

4.1 The form of solutions

If $C = -16/5$ and $\lambda = 1/9$, then six solutions of (8) correspond to one value of P_0: two solutions with $\tilde{\mathcal{B}} = \tilde{\mathcal{D}} = 0$ and four solutions with nonzero $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$. It is simplicity itself that functions ϱ

and $-\varrho$ correspond to one function y, so, solutions of system (8) with opposite values of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ generate identical solutions of eq. (3). The solutions are the following[1]:

1. $\tilde{\mathcal{A}} = -\dfrac{32}{15}, \quad \tilde{\mathcal{B}} = 0, \quad \tilde{\mathcal{C}} = -\dfrac{32}{5}P_0 - 1, \quad \tilde{\mathcal{D}} = 0, \quad \tilde{\mathcal{E}} = -\dfrac{32}{5}P_0^2 - 2P_0;$

2. $\tilde{\mathcal{A}} = -\dfrac{4}{3}, \quad \tilde{\mathcal{B}} = 0, \quad \tilde{\mathcal{C}} = -4P_0 - \dfrac{17}{33}, \quad \tilde{\mathcal{D}} = 0, \quad \tilde{\mathcal{E}} = -4P_0^2 - \dfrac{34}{33}P_0 + \dfrac{20}{3267};$

3. $\tilde{\mathcal{A}} = -\dfrac{32}{15}, \quad \tilde{\mathcal{B}} = \pm\dfrac{8i\sqrt{15}}{45}, \quad \tilde{\mathcal{C}} = -\dfrac{32}{5}P_0 - \dfrac{4}{9},$

$\tilde{\mathcal{D}} = \pm\dfrac{4i\sqrt{15}}{9}P_0, \quad \tilde{\mathcal{E}} = -\dfrac{32}{5}P_0^2 - \dfrac{8}{9}P_0, \quad H = \dfrac{16}{15}P_0^3 - \dfrac{7}{72}P_0^2;$

4. $\tilde{\mathcal{A}} = -\dfrac{32}{15}, \quad \tilde{\mathcal{D}} = \pm\dfrac{\sqrt{65}\sqrt{561}}{11329956}(26928P_0 + 8125),$

$\tilde{\mathcal{B}} = \pm\dfrac{8}{8415}\sqrt{65}\sqrt{561}, \quad \tilde{\mathcal{E}} = -\dfrac{32}{5}P_0^2 - \dfrac{3496}{1683}P_0 - \dfrac{333125}{7553304},$

$\tilde{\mathcal{C}} = -\dfrac{32}{5}P_0 - \dfrac{1748}{1683}, \quad H = \dfrac{16}{15}P_0^3 + \dfrac{7291}{13464}P_0^2 + \dfrac{6426875}{181279296}P_0 + \dfrac{17551324375}{9762977765376}.$

If the right-hand side of eq. (6) is a polynomial with multiple roots, then the function y can be expressed in terms of elementary functions. For example, at $P_0 = 0$ substitution of solutions 3 into eq. (5) gives

$$y = -\frac{5}{3\left(1 - 3\sin\left(\frac{t-t_0}{3}\right)\right)^2},$$

where t_0 is an arbitrary constant.

From (11) we obtain the following values of μ:

$$1. \qquad \mu = 0,$$

$$2. \qquad \mu = \frac{160}{1089}P_0^3 + \frac{680}{11979}P_0^2 - \frac{800}{1185921}P_0 - \frac{7000}{1056655611},$$

$$3-4. \quad \mu = \frac{4}{3}P_0^4 + \frac{5}{54}P_0^3 + \frac{50}{729}P_0^2,$$

$$5-6. \quad \mu = -\frac{52}{561}P_0^4 - \frac{81640}{944163}P_0^3 - \frac{4458460825}{152546527584}P_0^2 - \\ - \frac{539878421875}{128367902961936}P_0 - \frac{728473377734375}{6703885364284145664}.$$

4.2 Motion trajectories

When the solutions of system (8) are found the trajectory of the motion can be derived from the second equation of system (2). Substituting y_{tt} into the last equation, we obtain:

$$x^2 = (C - \frac{3}{2}\tilde{\mathcal{A}})y^2 + (3\tilde{\mathcal{A}}P_0 - \tilde{\mathcal{C}} - 1)y - \frac{1}{4}(5\tilde{\mathcal{B}}y + 3\tilde{\mathcal{D}} - 5\tilde{\mathcal{B}}P_0)(y - P_0)^{1/2} - \frac{1}{2}(\tilde{\mathcal{E}} + 3\tilde{\mathcal{A}}P_0^2 - 2\tilde{\mathcal{C}}P_0).$$

Note that when the coefficients $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ take zero values we get simple algebraic trajectories. The full list of such trajectories is presented in [7]. The parameter P_0 is absent in these trajectory equations.

[1]One parameter solutions ($P_0 = 0$) corresponding to these values of C and λ have been considered in detail in our previous papers [8, 19].

Let us consider the equations of the motion trajectories at $C=-16/5$ and $\lambda=1/9$. In the case of the solutions with $\tilde{\mathcal{B}}=\tilde{\mathcal{D}}=0$ the trajectory equation can be reduced either to $x^2=0$ (solution 1), or to

$$x^2+\frac{6}{5}\left(y+\frac{20}{99}\right)^2=\frac{50}{1089}.$$

In the last case (solution 2) the motion trajectory is an ellipse. Note, however, that the real motion does not necessarily affect the whole ellipse: it depends on two arbitrary parameters, the energy constant H can be considered as one of them.

In the case of solutions 3 the trajectory equation is the following:

$$\left(x^2+\frac{5}{9}y\right)^2+\frac{5}{27}(y-P_0)(2y+P_0)^2=0.$$

If $P_0=0$ (see (9)), the equation for one of the trajectory branches entirely coincides with the equation obtained in [8]. The condition $y<0$ is always required for the existence of real motion along these trajectories. Formula (9) describes precisely such a solution. For solutions 4 the trajectory equation has the same form as for solutions 3.

5 Global three-parameter solutions

Let us consider the possibility to generalize the obtained two-parameter elliptic solutions. Due to the Painlevé analysis it has been shown [19] that in the both above-mentioned nonintegrable cases there exist three-parameter local solutions in the form of converging Laurent series. One of parameters determines the singularity point location, other parameters determine coefficients of the obtained Laurent series. For some values of these parameters these Laurent series solutions coincide with the Laurent series of the obtained elliptic solutions. Of course, solutions, which are single-valued in the neighborhood of one singularity point, can be multivalued in the neighborhood of another singularity point. So we can only assume that global three-parameter solutions are single-valued. If we assume this and moreover that these solutions are elliptic functions (or some degenerations of them), then we can use a new method, which have been proposed in [20].

The classical theorem, which was established by Briot and Bouquet [21], proves that if the general solution of the autonomous ODE is single-valued, then this solution is either an elliptic function, or a rational function of $e^{\gamma x}$, γ being some constant, or a rational function of x. Note that the third case is a degeneracy of the second one, which in its turn is a degeneracy of the first one. At the same time, there exist elementary functions, for example, the function $f(t)=t+\sin(t)$, which are not solutions of any first order polynomial ODE.

It has been proved [22] that if the general solution of the autonomous ODE is single-valued, then the necessary form of this ODE is

$$\sum_{k=0}^{m}\sum_{j=0}^{2m-2k} h_{jk}\, y^j {y'}^k=0, \qquad h_{0m}=1, \tag{8}$$

in which m is a positive integer number and h_{jk} are constants.

R. Conte and M. Musette have proposed the method which allows to find elliptic solutions [20]. This method is based on the Painlevé test and uses the Laurent series expansion. Rather than to substitute eq. (8) in some nonintegrable system, they substitute the Laurent series of unknown special solutions in the eq. (8) and obtain a system, which is linear in h_{jk} and nonlinear in parameters which are included in the Laurent coefficients. There are a few computer algebra algorithms which allow to obtain this system from given the Laurent series. Moreover it is possible to exclude all h_{jk} from this system and obtain a nonlinear system in parameters of nonintegrable system and free parameters from the Laurent series.

6 Conclusions

Two-parameter elliptic solutions for the Hénon–Heiles system with $C = -16/5$ and $C = -4/3$ and arbitrary λ_1 and λ_2 have been found. There are no obstacles to exist three-parameter single-valued solutions, so, the probability of finding of exact three-parameter solutions, which generalize the obtained solutions, is high.

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APPENDIX

For an arbitrary λ we obtain the same structure of solutions as in consedered particular case. Solutions with $\tilde{\mathcal{B}} = \tilde{\mathcal{D}} = 0$ give solutions of eq. (4), other solutions of system (8) can be separated on pairs such as each pair of solutions corresponds to one two-parameter function $y = \varrho^2 + P_0$, where ϱ satisfies eq. (6) with the following values of coefficients:

$$
\begin{aligned}
&C = -\frac{16}{5},\\
&\tilde{\mathcal{A}} = -\frac{32}{15},\\
&\tilde{\mathcal{B}} = -\frac{\sqrt{1122}(1120\lambda_1 + 41888P_0 + 65S_q + 6195\lambda_2)\sqrt{F_1(\lambda_1,\lambda_2,P_0)}}{29373960(3600\lambda_1^2 - 1120\lambda_1 P_0 - 2425\lambda_1\lambda_2 - 20944P_0^2 - 6195\lambda_2 P_0 + 225\lambda_2^2)},\\
&\tilde{\mathcal{C}} = -\frac{240}{187}\lambda_1 - \frac{32}{5}P_0 + \frac{4}{1309}S_q - \frac{112}{187}\lambda_2,\\
&\tilde{\mathcal{D}} = \frac{\sqrt{1122}}{5874792}\sqrt{F_1(\lambda_1,\lambda_2,P_0)},\\
&\tilde{\mathcal{E}} = \frac{88320}{244783}\lambda_1^2 - \frac{480}{187}\lambda_1 P_0 + \frac{885}{244783}\lambda_1 S_q - \frac{153375}{244783}\lambda_1\lambda_2 -\\
&\quad - \frac{32}{5}P_0^2 + \frac{8}{1309}P_0 S_q - \frac{224}{187}\lambda_2 P_0 - \frac{685}{3916528}\lambda_2 S_q + \frac{168855}{3916528}\lambda_2^2,\\
&H = -\frac{11516270}{45774421}\lambda_1^3 + \frac{8740}{34969}\lambda_1^2 P_0 - \frac{3296515}{2563367576}\lambda_1^2 S_q + \frac{50336425}{183097684}\lambda_1^2\lambda_2 +\\
&\quad + \frac{258}{187}\lambda_1 P_0^2 - \frac{8209}{1958264}\lambda_1 P_0 S_q + \frac{76915}{279752}\lambda_1\lambda_2 P_0 + \frac{12202395}{82027762432}\lambda_1\lambda_2 S_q -\\
&\quad - \frac{131879855}{11718251776}\lambda_1\lambda_2^2 - \frac{43}{13090}P_0^2 S_q + \frac{103}{1496}\lambda_2 P_0^2 + \frac{8881}{31332224}\lambda_2 P_0 S_q -\\
&\quad - \frac{71205}{4476032}\lambda_2^2 P_0 - \frac{12990165}{1312444198912}\lambda_2^2 S_q - \frac{168661575}{187492028416}\lambda_2^3 + \frac{16}{15}P_0^3,
\end{aligned}
$$

$$
\begin{aligned}
&C = -\frac{4}{3},\\
&\tilde{\mathcal{A}} = -\frac{4}{3},\\
&\tilde{\mathcal{B}} = \frac{\sqrt{330}(952\lambda_1 - 616P_0 + 13R_q - 945\lambda_2)\sqrt{F_2(\lambda_1,\lambda_2,P_0)}}{38115(432\lambda_1^2 + 952\lambda_1 P_0 - 291\lambda_1\lambda_2 - 308P_0^2 - 945P_0\lambda_2 + 27\lambda_2^2)},\\
&\tilde{\mathcal{C}} = -\frac{4}{33}\lambda_1 - 4P_0 - \frac{1}{66}R_q - \frac{31}{22}\lambda_2,\\
&\tilde{\mathcal{D}} = \frac{\sqrt{330}}{7623}\sqrt{F_2(\lambda_1,\lambda_2,P_0)},\\
&\tilde{\mathcal{E}} = \frac{3394}{363}\lambda_1^2 + \frac{54}{11}\lambda_1 P_0 - \frac{1123}{10164}\lambda_1 R_q - \frac{5897}{484}\lambda_1\lambda_2 -\\
&\quad - \frac{17}{3}P_0^2 - \frac{31}{308}P_0 R_q - \frac{349}{44}\lambda_2 P_0 + \frac{1223}{27104}\lambda_2 R_q + \frac{13005}{3872}\lambda_2^2,\\
&H = -\frac{552922}{83853}\lambda_1^3 - \frac{29801}{2541}\lambda_1^2 P_0 + \frac{173605}{2347884}\lambda_1^2 R_q + \frac{778033}{74536}\lambda_1^2\lambda_2\lambda_2 - \frac{185}{66}\lambda_1 P_0^2 +\\
&\quad + \frac{3001}{20328}\lambda_1 P_0 R_q + \frac{104959}{6776}\lambda_1\lambda_2 P_0 - \frac{695609}{12522048}\lambda_1\lambda_2 R_q - \frac{2990049}{596288}\lambda_1\lambda_2^2 + \frac{89}{1232}P_0^2 R_q +\\
&\quad + \frac{5}{2}P_0^3 + \frac{865}{176}\lambda_2 P_0^2 - \frac{3065}{54208}\lambda_2 P_0 R_q - \frac{225909}{54208}\lambda_2^2 P_0 + \frac{2733}{260876}\lambda_2^2 R_q + \frac{57699}{74536}\lambda_2^3\,,
\end{aligned}
$$

where

$$F_1(\lambda_1, \lambda_2, P_0) \equiv 39474176000\lambda_1^3 + 122782105600\lambda_1^2 P_0 - 104358400\lambda_1^2 S_q -$$
$$- 17822336000\lambda_1^2\lambda_2 + 210552545280\lambda_1 P_0^2 - 680261120\lambda_1 P_0 S_q - 10941145600\lambda_1\lambda_2 P_0 -$$
$$- 41066800\lambda_1\lambda_2 S_q + 8305290000\lambda_1\lambda_2^2 - 501315584 P_0^2 S_q - 65797670400\lambda_2 P_0^2 +$$
$$+ 55920480 P_0 S_q + 1611640800\lambda_2^2 P_0 + 2884725\lambda_2^2 S_q - 468507375\lambda_2^3,$$
$$S_q \equiv \pm\sqrt{35(2048\lambda_1^2 - 1280\lambda_1\lambda_2 + 387\lambda_2^2)},$$
$$F_2(\lambda_1, \lambda_2, P_0) \equiv 2099776\lambda_1^3 - 497728\lambda_1^2 P_0 - 20008\lambda_1^2 R_q - 4911144\lambda_1^2\lambda_2 + 948640\lambda_1 P_0^2 +$$
$$+ 19096\lambda_1 P_0 R_q + 1458072\lambda_1\lambda_2 P_0 + 37173\lambda_1\lambda_2 R_q + 3943233\lambda_1\lambda_2^2 + 6776 P_0^2 R_q -$$
$$- 711480\lambda_2 P_0^2 - 9240\lambda_2 P_0 R_q - 615384\lambda_2^2 P_0 - 13581\lambda_2^2 R_q - 1006425\lambda_2^3,$$
$$R_q \equiv \pm\sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}.$$