

# NONLINEAR MONOPOLE, REGULARITY CONDITIONS AND THE ELECTROMAGNETIC MASS IN EINSTEIN-BORN-INFELD THEORIES

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In this work a new asymptotically flat solution of the coupled Einstein-Born-Infeld equations for a static spherically symmetric space-time is obtained. When the intrinsic mass is zero the resulting spacetime is regular everywhere, in the sense given by B. Hoffmann and L. Infeld in 1937, and the Einstein-Born-Infeld theory leads to the identification of the gravitational with the electromagnetic mass.

## 1 Introduction and results:

The four dimensional solutions with spherical symmetry of the Einstein equations coupled to Born-Infeld fields have been well studied in the literature<sup>1-4</sup>. In particular, the electromagnetic field of the Born Infeld monopole, in contrast to Maxwell counterpart, contributes to the ADM mass of the system (it is, the four momentum of asymptotic flat manifolds). B. Hoffmann was the first who studied such static solutions in the context of the general relativity with the idea of to obtain a consistent particle-like model<sup>2</sup>. Unfortunately, these static Einstein-Born-Infeld (EBI) models generate conical singularities at the origin<sup>2-3</sup> that cannot be removed as in global monopoles or other non-localized defects of the spacetime<sup>5-6</sup>. With the existence of this type of singularities in the space-time of the monopole we can not identify the gravitational with the electromagnetic mass. In this work a *new* static spherically symmetric solution with Born-Infeld charge is obtained. The new metric, when the intrinsic mass of the system is zero, is *regular* everywhere in the sense that was given by B. Hoffmann and L. Infeld<sup>3</sup> in 1937 and the EBI theory leads to identification of the gravitational with the electromagnetic mass. This means that the metric, the electromagnetic field and their derivatives have not singularities and discontinuities in all the manifold. The fundamental feature of this solution is the lack of conical singularities at the origin. A distant observer will associate with this solution an electromagnetic mass that is a twice of the mass of the electromagnetic geon founded by M. Demianski<sup>4</sup> in 1986 . The energy-momentum tensor and the electric field are both regular with zero value at the origin and new parameters appear, given to the new metric surprising behaviours. The used convention<sup>7-8</sup> is the *spatial* of Landau and Lifshitz (1962), with signatures of the metric, Riemann and Einstein tensors all positives (+++).

The plan of this paper is as follows: in Section 2 we give a short introduction to the Born-Infeld theory: properties and principal features. In Section 3 the regularity condition as was given by B. Hoffmann and L. Infeld<sup>3</sup> in 1937 . Sections 4, 5, 6 and 7 are devoted to find the new solution and to analyze its properties. Finally, the conclusion and comments of the results are presented in section 8.

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## 2 The Born-Infeld theory:

The most significant non-linear theory of electrodynamics is, by excellence, the Born-Infeld theory<sup>1,9</sup>. Among its many special properties is an exact SO(2) electric-magnetic duality invariance. The Lagrangian density describing Born-Infeld theory (in arbitrary spacetime dimensions) is

$$\mathcal{L}_{BI} = \sqrt{-g}L_{BI} = \frac{b^2}{4\pi} \left\{ \sqrt{-g} - \sqrt{|\det(g_{\mu\nu} + b^{-1}F_{\mu\nu})|} \right\}, \quad (1)$$

where  $b$  is a fundamental parameter of the theory with field dimensions. In open superstring theory<sup>10</sup>, for example, loop calculations lead to this Lagrangian with  $b^{-1} = 2\pi\alpha'$  ( $\alpha' \equiv$  inverse of the string tension). In four spacetime dimensions the determinant in (1) may be expanded out to give

$$L_{BI} = \frac{b^2}{4\pi} \left\{ 1 - \sqrt{1 + \frac{1}{2}b^{-2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16}b^{-4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2} \right\}, \quad (2)$$

which coincides with the usual Maxwell Lagrangian in the weak field limit.

It is useful to define the second rank tensor  $P^{\mu\nu}$  by

$$P^{\mu\nu} = -\frac{1}{2} \frac{\partial L_{BI}}{\partial F_{\mu\nu}} = \frac{F^{\mu\nu} - \frac{1}{4}b^{-2}(F_{\rho\sigma}\tilde{F}^{\rho\sigma})\tilde{F}^{\mu\nu}}{\sqrt{1 + \frac{1}{2}b^{-2}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{16}b^{-4}(F_{\rho\sigma}\tilde{F}^{\rho\sigma})^2}} \quad (3)$$

(so that  $P^{\mu\nu} \approx F^{\mu\nu}$  for weak fields) satisfying the electromagnetic equations of motion

$$\nabla_{\mu}P^{\mu\nu} = 0 \quad (4)$$

which are highly non linear in  $F_{\mu\nu}$ . The energy-momentum tensor may be written as

$$T_{\mu\nu} = \frac{1}{4\pi} \left\{ \frac{F_{\mu}{}^{\lambda}F_{\nu\lambda} + b^2 [\mathbb{R} - 1 - \frac{1}{2}b^{-2}F_{\rho\sigma}F^{\rho\sigma}] g_{\mu\nu}}{\mathbb{R}} \right\} \quad (5)$$

$$\mathbb{R} \equiv \sqrt{1 + \frac{1}{2}b^{-2}F_{\rho\sigma}F^{\rho\sigma} - \frac{1}{16}b^{-4}(F_{\rho\sigma}\tilde{F}^{\rho\sigma})^2}.$$

Although it is by no means obvious, it may be verified that equations (3)-(5) are invariant under electric-magnetic rotations of duality  $F \longleftrightarrow *G$ . We can show that the SO(2) structure of the Born-Infeld theory is more easily seen in quaternionic form<sup>11-12</sup>

$$\begin{aligned} \frac{1}{R} (\sigma_0 + i\sigma_2\bar{\mathbb{P}}) L &= \mathbb{L} \\ \frac{\mathbb{R}}{(1 + \bar{\mathbb{P}}^2)} (\sigma_0 - i\sigma_2\bar{\mathbb{P}}) \mathbb{L} &= L \\ \bar{\mathbb{P}} &\equiv \frac{\mathbb{P}}{b}, \end{aligned}$$

where we defined

$$\begin{aligned} L &= F - i\sigma_2\tilde{F} \\ \mathbb{L} &= P - i\sigma_2\tilde{P} \end{aligned}$$

the pseudoscalar of the electromagnetic tensor  $F^{\mu\nu}$

$$\mathbb{P} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$$

and  $\sigma_0, \sigma_2$  the well know Pauli matrix.

In flat space, and for purely electric configurations, the Lagrangian (2) reduces to

$$L_{BI} = \frac{4\pi}{b^2} \left\{ 1 - \sqrt{1 - b^{-2}\vec{E}^2} \right\}$$

so there is an upper bound on the electric field strength  $\vec{E}$

$$|\vec{E}| \leq b. \quad (6)$$

### 3 The regularity condition

The new field theory initiated in 1934 by M. Born<sup>9</sup> introduces in the classical equations of the electromagnetic field a characteristic length  $r_0$  representing the radius of the elementary particle through the relation

$$r_0 = \sqrt{\frac{e}{b}},$$

where  $e$  is the elementary charge and  $b$  the fundamental field strength entering in a non-linear Lagrangian function. It was originally thought that the Lagrangian (1) was the simplest choice which would lead to a finite energy for an electric particle. This is, however, not the case. It is possible to find an infinite number of quite different action functions, each giving simple algebraic relations between the fields and each leading to a finite energy for an electric particle.

In 1937 B. Hoffmann and L. Infeld<sup>3</sup> introduce a regularity condition on the new field theory of M. Born<sup>9</sup> with the main idea of to solve the lack of uniqueness of the function action. They have already seen that the condition of regularity of the field gives the restriction in the spherically symmetric electrostatic case  $E_r = 0$  for  $r = 0$ .

In the general theory they applied the regularity condition not only to the  $F_{\mu\nu}$  field but also to the  $g_{\mu\nu}$  field. The regularity condition for the general theory was that:

*Only those solutions of the fields equations may have physical meaning for which space-time is everywhere regular and for which the  $F_{\mu\nu}$  and the  $g_{\mu\nu}$  fields and those of their derivatives which enter in the field equations and the conservation laws exist everywhere.*

In the general theory of the relativity the spherically symmetric solution of the purely gravitational field equations is given by the Schwarzschild line element

$$ds^2 = -Adt^2 + A^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

$$A \equiv 1 - \frac{2M}{r},$$

where  $(-2M)$  is a constant of integration  $M$  have having the significance of the gravitational mass of the body source of the field (we take the gravitational constant  $G = 1$ ). This line element has an essential singularity at  $r = 0$  and does not satisfy the regularity condition.

In the general relativity form of the original new field theory the requeriment that there be no infinities in the  $g_{\mu\nu}$  forces the identification of gravitational with electromagnetic mass. In<sup>3</sup> B. Hoffmann and L. Infeld have used for such identification the line element of the well known monopole solution studied by B. Hoffmann<sup>2</sup> in 1935

$$A \equiv 1 - \frac{8\pi}{r} \int_0^r [(r^4 + 1)^{1/2} - r^2] dr,$$

that is originated by an Einstein-Born-Infeld action as in equation (1). This line element approximates the Schwarzschild form for  $r$  greater than the electronic radius but avoid the infinities of that line element for  $r = 0$ . However is still a singularity of conical type at the pole. When  $r \rightarrow 0$  the above expression for  $A$ , gives

$$A \rightarrow (1 - 8\pi) \equiv \beta$$

so  $ds^2$  becomes

$$ds^2 = -\beta dt^2 + \beta^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) .$$

Thus the ratio of the circumference to the radius of a small circle having its centre at the pole is, in the limit,  $2\pi\beta$  and not  $2\pi$ . Therefore the origin (it is, at  $r = 0$ ) is a conical point and not regular. Note that, because the conical point, no coordinate can be introduced which will be non singular at  $r = 0$  and derivatives are actually undefined at this point.

This problem with the conical singularities at  $r = 0$ , that destroy the regularity condition, makes that in the reference<sup>3</sup> B. Hoffmann and L. Infeld change the action of the Born-Infeld form as in equation (1) for other non-linear Lagrangian of logarithmic type. The new logarithmic action does not presented such difficulties at  $r = 0$ , and makes that time ago many people changes the very nice form of the Einstein-Born-Infeld action (1) for others non-linear Lagrangians that solved the problem of the self-energy of the electron and the regularity condition given above.

In this work we presented a new *exact* spherically symmetric solution of the Einstein-Born-Infeld equations. The metric, when the intrinsic mass of the system is zero, is *regular* everywhere in the sense that was given by B. Hoffmann and L. Infeld<sup>3</sup> in 1937, and the EBI theory leads to identification of the gravitational with the electromagnetic mass. In this manner we also show that more strong conditions are needed for to solve the problem of the lack of uniqueness of the function action.

## 4 Statement of the problem:

We propose the following line element for the static Born-Infeld monopole

$$ds^2 = -e^{2\Lambda} dt^2 + e^{2\Phi} dr^2 + e^{2F(r)} d\theta^2 + e^{2G(r)} \sin^2 \theta d\varphi^2 , \quad (7)$$

where the components of the metric tensor are

$$\begin{aligned} g_{tt} &= -e^{2\Lambda} & g^{tt} &= -e^{-2\Lambda} \\ g_{rr} &= e^{2\Phi} & g^{rr} &= e^{-2\Phi} \\ g_{\theta\theta} &= e^{2F} & g^{\theta\theta} &= e^{-2F} \\ g_{\varphi\varphi} &= \sin^2 \theta e^{2G} & g^{\varphi\varphi} &= \frac{e^{-2G}}{\sin^2 \theta} . \end{aligned} \quad (8)$$

For the obtention of the Einstein-Born-Infeld equations system we use the Cartan's structure equations method<sup>13</sup>, that is most powerful and direct where we work with differential forms and in a orthonormal frame (tetrad). The line element (7) in the 1-forms basis takes the following form

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 , \quad (9)$$

where the forms are

$$\begin{aligned} \omega^0 &= e^\Lambda dt & \Rightarrow & dt = e^{-\Lambda} \omega^0 \\ \omega^1 &= e^\Phi dr & \Rightarrow & dr = e^{-\Phi} \omega^1 \\ \omega^2 &= e^{F(r)} d\theta & \Rightarrow & d\theta = e^{-F(r)} \omega^2 \\ \omega^3 &= e^{G(r)} \sin \theta d\varphi & \Rightarrow & d\varphi = e^{-G(r)} (\sin \theta)^{-1} \omega^3 . \end{aligned} \quad (10)$$

Now, following the standard procedure of the structure equations (Appendix) for to obtain easily the components of the Riemann tensor, we can construct the Einstein equations

$$G^1_{\ 2} = -e^{-(F+G)} \frac{\cos \theta}{\sin \theta} \partial_r (G - F) \quad (11)$$

$$G^0_{\ 0} = e^{-2\Phi} \Psi - e^{-2F} \quad (12)$$

$$\Psi \equiv \left[ \partial_r \partial_r (F + G) - \partial_r \Phi \partial_r (F + G) + (\partial_r F)^2 + (\partial_r G)^2 + \partial_r F \partial_r G \right]$$

$$G^1_{\ 1} = e^{-2\Phi} [\partial_r \Lambda \partial_r (F + G) + \partial_r F \partial_r G] - e^{-2F} \quad (13)$$

$$G^2_{\ 2} = e^{-2\Phi} \left[ \partial_r \partial_r (\Lambda + G) - \partial_r \Phi \partial_r (\Lambda + G) + (\partial_r \Lambda)^2 + (\partial_r G)^2 + \partial_r \Lambda \partial_r G \right] \quad (14)$$

$$G^3_{\ 3} = e^{-2\Phi} \left[ \partial_r \partial_r (F + \Lambda) - \partial_r \Phi \partial_r (F + \Lambda) + (\partial_r \Lambda)^2 + (\partial_r F)^2 + \partial_r F \partial_r \Lambda \right] \quad (15)$$

$$G^1_{\ 3} = G^2_{\ 3} = G^0_{\ 3} = G^0_{\ 2} = G^0_{\ 1} = 0 . \quad (16)$$

In the tetrad defined by (10), the energy-momentum tensor of Born-Infeld takes a diagonal form, being its components the following

$$-T_{00} = T_{11} = \frac{b^2}{4\pi} \left( \frac{\mathbb{R} - 1}{\mathbb{R}} \right) \quad (17)$$

$$T_{22} = T_{33} = \frac{b^2}{4\pi} (1 - \mathbb{R}) , \quad (18)$$

where

$$\mathbb{R} \equiv \sqrt{1 - \left( \frac{F_{01}}{b} \right)^2} \quad (19)$$

of this manner, one can see from the Einstein equation (18) the characteristic property of the spherically symmetric space-times<sup>14</sup>

$$G^1_{\ 2} = -e^{-(F+G)} \frac{\cos \theta}{\sin \theta} \partial_r (G - F) = 0 \quad \Rightarrow \quad G = F . \quad (20)$$

Notice for that the interval be a spherically symmetric one, the functions  $F(r)$  and  $G(r)$  must be equal. As we saw in the precedent paragraph the components of the energy-momentum tensor of BI assures this condition in a natural form. Also it is interesting to see from eqs. (17) and (18) that the energy-momentum tensor of Born-Infeld has the same form as the energy-momentum tensor of an anisotropic fluid.

## 5 Equations for the electromagnetic fields of Born-Infeld in the tetrad

The equations that describe the dynamic of the electromagnetic fields of Born-Infeld in a curved spacetime are

$$\nabla_a \mathbb{F}^{ab} = \nabla_a \left[ \frac{F^{ab}}{\mathbb{R}} + \frac{P}{b^2 \mathbb{R}} \tilde{F}^{ab} \right] = 0 \quad (\text{field equations}) , \quad (21)$$

$$\nabla_a \tilde{F}^{ab} = 0 \quad ( \text{Bianchi's identity} ) , \quad (22)$$

where

$$P \equiv -\frac{1}{4} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \quad (23)$$

$$S \equiv -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \quad (24)$$

$$\mathbb{R} \equiv \sqrt{1 - \frac{2S}{b^2} - \left(\frac{P}{b^2}\right)^2} . \quad (25)$$

The above equations can be solved explicitly giving the follow result

$$F_{01} = A(r) \quad (26)$$

$$\mathbb{F}_{01} = f e^{-2G} , \quad (27)$$

where  $f$  is a constant. We can see from equation (19) and (21) that

$$\mathbb{F}_{01} = \frac{F_{01}}{\sqrt{1 - (\bar{F}_{01})^2}} ,$$

where we obtain the following form for the electric field of the self-gravitating B-I monopole

$$F_{01} = \frac{b}{\sqrt{\left(\frac{b}{f} e^{2G}\right)^2 + 1}} \quad (28)$$

we can to associate<sup>1</sup>

$$f = br_0^2 \equiv Q \quad \Rightarrow \quad F_{01} = \frac{b}{\sqrt{\left(\frac{eG}{r_0}\right)^4 + 1}} . \quad (29)$$

Where  $r_0$  is a constant with units of longitude that in reference<sup>1</sup> was associated to the radius of the electron. Finally the components of the energy-momentum tensor of BI takes its explicit form reemplacing the  $F_{01}$  that we was found in equation (29) in expressions (17) and (18)

$$-T_{00} = T_{11} = \frac{b^2}{4\pi} \left( 1 - \sqrt{\left(\frac{r_0}{eG}\right)^4 + 1} \right) \quad (30)$$

$$T_{22} = T_{33} = \frac{b^2}{4\pi} \left( 1 - \frac{1}{\sqrt{\left(\frac{r_0}{eG}\right)^4 + 1}} \right) . \quad (31)$$

Expressions (11)–(16) together with (30)–(31) and (20) are the full set of Einstein equations in explicit form.

## 6 Reduction and solutions of the system of Einstein-Born-Infeld equations

Of the above expressions, we can see that  $G^0_0 = G^1_1$ , then

$$\partial_r \partial_r G + (\partial_r G)^2 - \partial_r G \partial_r (\Phi + \Lambda) = 0 . \quad (32)$$

In order to reduce the eq.(32) we will proceed as follow. First we make

$$\partial_r G \equiv \xi \quad (33)$$

with this change of variables, in the equation (32) we have first derivatives only

$$\partial_r \xi + \xi^2 - \xi \partial_r (\Phi + \Lambda) = 0 \quad (34)$$

dividing the above expression (34) by  $\xi$  and making the substitution

$$\chi \equiv \ln \xi \quad (35)$$

we have been obtained the following inhomogeneous equation

$$\partial_r \chi + e^\chi = \partial_r (\Phi + \Lambda) \quad (36)$$

the homogeneous part of the last equation is easy to integrate

$$\chi_h = -\ln r . \quad (37)$$

Now, of as usual, we make in eq. (36) the following substitution

$$\chi = \chi_h + \chi_p = -\ln r + \ln \mu = -\ln r + \ln(1 + \eta) , \quad (38)$$

then

$$\begin{aligned} \partial_r \ln(1 + \eta) + \frac{\eta}{r} &= \partial_r (\Phi + \Lambda) \Rightarrow \\ \partial_r [\ln(1 + \eta) + \mathcal{F}(r) - (\Phi + \Lambda)] &= 0 \\ \ln(1 + \eta) + \mathcal{F}(r) - (\Phi + \Lambda) &= cte = 0 , \end{aligned} \quad (39)$$

where  $\frac{d\mathcal{F}(r)}{dr} \equiv \frac{\eta(r)}{r}$ . The constant must be put equal to zero for to obtain the correct limit. Finally the form of the exponent  $G$  is

$$G = \ln r + \mathcal{F}(r) . \quad (40)$$

The next step is to put  $\Phi$  in function of  $\Lambda$  and  $G$  in the expression (13). After of tedious but straightforward computations and integrations, we obtain

$$e^{2\Lambda} = 1 + a_0 e^{-G} + e^{2G} \frac{2b^2}{3} - 2b^2 e^{-G} \int^{Y(r)} \sqrt{Y^4 + (r_0)^4} dY . \quad (41)$$

Where, we defined

$$Y(r) = e^G$$

and  $a(0)$  is an integration constant.

Hitherto, we know that  $\mathcal{F}$  is an arbitrary function of the radial coordinate  $r$ , but for to be sure of it, we must to introduce the fuction  $\Lambda$  given for above equation, in the Einstein equations (14-15) and to verify that  $G_{22} = G_{33}$ . Successfully, this equality is verified and the functions  $\Lambda$ ,  $\Phi$  and  $G$  remains matematically determinate. In this manner the line element of our problem (7) takes the following form

$$ds^2 = -e^{2\Lambda} dt^2 + e^{2\mathcal{F}(r)} \left[ e^{-2\Lambda} (1 + r \partial_r \mathcal{F}(r))^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] . \quad (42)$$

## 6.1 Analysis of the function $\mathcal{F}(r)$ from the physical point of view

The function  $\mathcal{F}(r)$  must to have the behaviour in the form that the electric field of the configuration obey the following requirements for gives a regular solution in the sense that was given by B. Hoffmann and L. Infeld<sup>3</sup>

$$F_{01}|_{r=r_0} < b \quad (43)$$

$$F_{01}|_{r=0} = 0 \quad (44)$$

$$F_{01}|_{r \rightarrow \infty} = 0 \quad \text{asymptotically Coulomb} \quad (45)$$

the simplest function  $\mathcal{F}(r)$  that obey the above conditions, is of the type

$$e^{2\mathcal{F}(r)} = \left[ 1 - \left( \frac{r_0}{a|r|} \right)^n \right]^{2m}, \quad (46)$$

where  $a$  is an arbitrary constant, and the exponents  $n$  and  $m$  will obey the following relation

$$mn > 1 \quad (m, n \in \mathbb{N}) \quad (47)$$

with

$$0 < a < 1 \text{ or } -1 < a < 0$$

depending on  $m(n)$  is even or odd and

$$a \neq 0,$$

that put in sure a consistent regularization condition not only for the electric (magnetic) field but for the energy-momentum tensor (30) and (31) and the line element (42).

The analysis of the Riemann tensor indicate us that it is regular everywhere and its components goes faster than  $\frac{1}{r^3}$  when  $r \rightarrow \infty$ . With all this considerations, the metric solution to the problem is

$$ds^2 = -e^{2\Lambda} dt^2 + \left[ 1 - \left( \frac{r_0}{a|r|} \right)^n \right]^{2m} \left\{ e^{-2\Lambda} dr^2 \left[ \frac{1 - \left( \frac{r_0}{a|r|} \right)^n (mn-1)}{\left[ 1 - \left( \frac{r_0}{a|r|} \right)^n \right]} \right]^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right\} \quad (48)$$

and the electric field takes the form

$$F_{01} = \frac{b}{\sqrt{1 + \left[ 1 - \left( \frac{r_0}{a|r|} \right)^n \right]^{4m} \left( \frac{r}{r_0} \right)^4}}. \quad (49)$$

It is interesting to note that if we violating the condition (43) taken  $a = 1$  and  $F_{01}|_{r=r_0} = b$  (limit value for the electric field in BI theory) the energy momentum diverges automatically at  $r = r_0$ . Strictely, the regularity conditions for the energy-momentum tensor (without divergences and discontinuities in the neighborhood of  $r_0$ , physical radius of the spherical source of the non-linear electromagnetic field) are

$$T_{ab}|_{r=r_0} = \text{finite} \quad \Rightarrow \quad -1 < a < 0 \text{ or } 0 < a < 1$$

depending on parity of  $m, n$ ; and

$$T_{ab}|_{r=0} \rightarrow 0 \quad \Rightarrow \quad \mathbb{R} \rightarrow 1.$$

For the magnetic monopole case the line element is as expression (48) with the following obvious definition for the magnetic charge

$$br_0^2 \equiv Q_m.$$



The magnetic field takes the following form

$$\begin{aligned} F_{23} &= \frac{b}{\left[1 - \left(\frac{r_0}{a|r|}\right)^n\right]^{2m} \left(\frac{r}{r_0}\right)^2} \\ &= \frac{Q_m}{\left[1 - \left(\frac{r_0}{a|r|}\right)^n\right]^{2m} r^2} \end{aligned}$$

and the considerations about the regularity conditions on the energy momentum tensor is as the electric monopole case.

## 6.2 Interesting cases for particular values of $n$ and $m$

Because

$$\exp 2\mathcal{F}(r) = \left[1 - \left(\frac{r_0}{a|r|}\right)^n\right]^{2m}$$

is easy to see that for  $m = 0$

$$e^G = r$$

and we obtains the spherically symmetric line element of Hoffmann<sup>2</sup> and the electric field  $F_{01}$  and the energy-momentum tensor  $T_{ab}$  take the form of the well know EBI solution for the electromagnetic geon of Demiański<sup>4</sup>.

By other hand, in the *limit* when:  $a \rightarrow 1$ ,  $n \rightarrow 4$  and  $m \rightarrow \frac{1}{4}$  we have

$$F_{01} \rightarrow \frac{b}{\sqrt{1 + \left[1 - \left(\frac{r_0}{|r|}\right)^4\right] \left(\frac{r}{r_0}\right)^4}} = \frac{Q}{r^2},$$

where (as is usually taked)  $br_0^2 \equiv Q$ . How we see, we obtain as solution in the *limit* the Maxwellian linear field. Note that the values of  $a$  and the exponents  $m$  and  $n$  are restricted by conditions (47).

## 7 Analysis of the metric

We have the metric (42)

$$ds^2 = -e^{2\Lambda} dt^2 + e^{2\mathcal{F}(r)} \left[ e^{-2\Lambda} (1 + r \partial_r \mathcal{F}(r))^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right],$$

if we make the substitution

$$Y \equiv r e^{\mathcal{F}(r)}$$

and differentiating it

$$dY \equiv e^{\mathcal{F}(r)} (1 + r \partial_r \mathcal{F}(r)) dr$$

the interval (7) takes the form

$$ds^2 = -e^{2\Lambda} dt^2 + e^{-2\Lambda} dY^2 + Y^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

we can see that the metric (in particular the  $g_{tt}$  coefficient), in the new coordinate  $Y(r)$ , takes the similar form like a Demianski solution for the Born-Infeld monopole spacetime :

$$e^{2\Lambda} = 1 - \frac{2M}{Y} - \frac{2b^2 r_o^4}{3 \left( \sqrt{Y^4 + r_o^4} + Y^2 \right)} - \frac{4}{3} b^2 r_o^2 {}_2F_1 \left[ 1/4, 1/2, 5/4; - \left( \frac{Y}{r_o} \right)^4 \right],$$

here  $M$  is an integration constant, which can be interpreted as an intrinsic mass, and  ${}_2F_1$  is the Gauss hypergeometric function<sup>14</sup>. We have pass

$$g_{rr} \rightarrow g_{YY}, \quad g_{tt}(r) \rightarrow g_{tt}(Y) .$$

Specifically, for the form of the  $\mathcal{F}(r)$  given by (46),  $Y$  is

$$Y^2 \equiv \left[ 1 - \left( \frac{r_o}{a|r|} \right)^n \right]^{2m} r^2 .$$

Now, with the metric coefficients fixed to a asymptotically Minkowskian form, one can study the asymptotic behaviour of our solution. A regular, asymptotically flat solution with the electric field and energy-momentum tensor both regular, in the sense of B. Hoffmann and L. Infeld is when the exponent numbers of  $Y(r)$  take the following particular values:

$$n = 3 \quad \text{and} \quad m = 1 .$$

In this case, and for  $r \gg \frac{r_0}{a}$ , we have the following asymptotic behaviour for  $Y(r)$  and  $-g_{tt}$ , that does not depend on the  $a$  parameter

$$Y(r) \rightarrow r \quad \left( r \gg \frac{r_0}{a} \right)$$

$$e^{2\Lambda} \simeq 1 - \frac{2M}{r} - \frac{8b^2 r_o^4 K(1/2)}{3r_o r} + 2 \frac{b^2 r_o^4}{r^2} + \dots .$$

A distant observer will associate with this solution a total mass

$$M_{eff} = M + \frac{4b^2 r_o^4 K(1/2)}{3r_o}$$

and total charge

$$Q^2 = 2b^2 r_o^2 .$$

Notice that when the intrinsic mass  $M$  is zero the line element is regular everywhere, the Riemann tensor is also regular everywhere and hence the space-time is singularity free. The electromagnetic mass

$$M_{el} = \frac{4b^2 r_o^4 K(1/2)}{3r_o} \tag{50}$$

and the charge  $Q$  are the *twice* that the electromagnetic charge and mass of the Demianski solution<sup>4</sup> for the static electromagnetic geon. Notice that the  $M_{el}$  is necessarily positive, which was not the case in the Schwarzschild line element. The other important reason for to take the constant  $M = 0$  is that we must regard the quantity (let us to restore by one moment the gravitational constant  $G$ )

$$4\pi G \int_{Y(r=0)}^{Y(r)} T_0^0(Y) Y^2 dY$$

as the *gravitational mass* causing the field at coordinate distance  $r$  from the pole. In our case  $T_0^0$  is given by expression (30). This quantity is precisely (in gravitational units)  $M_{el}$  given by (50), the *total electromagnetic mass* within the sphere having its center at  $r = 0$  and coordinate  $r$ . We will take  $M = 0$  in the rest of the analysis.

On the other hand, the function  $Y(r)$  for the values of the  $m$  and  $n$  parameters given above has the following behaviour near of the origin

$$\begin{aligned} & \text{for } a < 0 && \text{when } r \rightarrow 0, Y(r) \rightarrow \infty, \\ & \text{for } a > 0 && \text{when } r \rightarrow 0, Y(r) \rightarrow -\infty. \end{aligned}$$

Notice that the case  $a > 0$  will be excluded because in any value  $r_0 \rightarrow Y(r_0) = 0$ , the electric field takes the limit value  $b$  and the condition (43) is violated. For  $M = 0$  and  $a < 0$ , expanding the hypergeometric function, we can see that the  $-g_{tt}$  coefficient has the following behaviour near the origin

$$e^{2\Lambda} \simeq 1 - \frac{8b^2 r_o^4 K(1/2)}{3r_o} r^2 \left(\frac{|a|}{r_0}\right)^3 + 2b^2 r_o^4 r^4 \left(\frac{|a|}{r_0}\right)^6 + \dots$$

The metric (see figures) and the energy-momentum tensor remains *both* regulars at the origin (it is:  $g_{tt} \rightarrow -1, T_{\mu\nu} \rightarrow 0$  for  $r \rightarrow 0$ ). It is not very difficult to check that (for  $m = 1$  and  $n = 3$ ) the maximum of the electric field (see figures) is not in  $r = 0$ , but in the *physical border* of the spherical configuration source of the electromagnetic fields (this point is located around  $r_B = 2^{1/3} \frac{r_0}{|a|}$ ). It means that  $Y(r)$  maps correctly the internal structure of the source in the same form that the quasiglobal coordinate of the reference<sup>16</sup> for the global monopole in general relativity. The lack of the conical singularities at the origin is because the very well description of the manifold in the neighbourhood of  $r = 0$  given by  $Y(r)$

Because the metric is regular ( $g_{tt} = -1$ , at  $r = 0$  and at  $r = \infty$ ), its derivative must change sign. In the usual gravitational theory of general relativity the derivative of  $g_{tt}$  is proportional to the gravitational force which would act on a test particle in the Newtonian approximation. In Einstein-Born-Infeld theory with this new static solution, it is interesting to note that although this force is attractive for distances of the order  $r_0 \ll r$ , it is actually a repulsion for very small  $r$ . For  $r$  greater than  $r_0$ , the line element closely approximates to the Schwarzschild form. Thus the regularity condition shows that the electromagnetic and gravitational mass are the same and, as in the Newtonian theory, we now have the result that the attraction is zero in the center of the spherical configuration source of the electromagnetic field.

## 8 Conclusions

In this report a *new* exact solution of the Einstein-Born-Infeld equations for a static spherically symmetric monopole is presented. The general behaviour of the geometry, is strongly modified according to the value that takes  $r_0$  (Born-Infeld radius<sup>1, 9</sup>) and three new parameters:  $a$ ,  $m$  and  $n$ .

The fundamental feature of this solution is the lack of conical singularities at the origin when:  $-1 < a < 0$  or  $0 < a < 1$  (depends on parity of  $m$  and  $n$ ) and  $mn > 1$ . In particular, for  $m = 1$  and  $n = 3$ , with the parameter  $a$  in the range given above and the intrinsic mass of the system  $M$  is zero, the strong regularity conditions given by B. Hoffmann and L. Infeld in reference<sup>3</sup>, holds in all the spacetime. For the set of values for the parameters given above, the solution is asymptotically flat, free of singularities in the electric field, metric, energy-momentum tensor and their derivatives (with derivative values zero for  $r \rightarrow 0$ ); and the electromagnetic mass (ADM) of the system is a twice that the electromagnetic mass of other well known<sup>2, 4</sup> solutions for the Einstein-Born-Infeld monopole. The electromagnetic mass  $M_{el}$  asymptotically is necessarily positive, which was not the case in the Schwarzschild line element.

This solution have a surprising similitude with the metric for the global monopole in general relativity given in reference<sup>16-17</sup> in the sense that the physic of the problem have a correct description only by means of a new radial function  $Y(r)$ .

Because the metric is regular ( $g_{tt} = -1$ , at  $r = 0$  and at  $r = \infty$ ), its derivative (that is proportional to the the force in Newtonian approximation) must change sign. In Einstein-Born-Infeld theory with this new static solution, it is interesting to note that although this force is attractive for distances of the order  $r_0 \ll r$ , it is actually a repulsive for very small  $r$ .

With this new regular solution, we also show that more strong conditions are needed for to solve the problem of the lack of uniqueness of the function action in non-linear electrodynamics.

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## Appendix:

### Connections and curvature forms from the geometrical Cartan's formulation

The standard procedure of E. Cartan has its startpoint in the following equations

$$d\omega^\alpha = -\omega^\alpha{}_\beta \wedge \omega^\beta \quad (51)$$

$$\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\lambda \wedge \omega^\lambda{}_\beta, \quad (52)$$

these are denominated *the structure equations*. The procedure for to obtain the Einstein equations is by mean the following steps:

i. Making the exterior derivatives of  $\omega^\alpha$  we computing the connection 1-forms  $\omega^\alpha{}_\beta$ :

$$\begin{aligned} \omega^0{}_{1} &= \omega^1{}_{0} = e^{-\Phi} \partial_r \Lambda \omega^0 \\ \omega^2{}_{1} &= -\omega^1{}_{2} = e^{-\Phi} \partial_r F(r) \omega^2 \\ \omega^3{}_{1} &= -\omega^1{}_{3} = e^{-\Phi} \partial_r G(r) \omega^3 \\ \omega^3{}_{2} &= -\omega^2{}_{3} = \frac{\cos \theta}{\sin \theta} e^{-F(r)} \omega^3. \end{aligned} \quad (53)$$

ii. Making the exterior derivatives of  $\omega^\alpha{}_\beta$  we computing the curvature 2-forms  $\mathcal{R}^\alpha{}_\beta$ :

$$\begin{aligned} \mathcal{R}^0{}_{1} &= e^{-2\Phi} \left( \partial_r \partial_r \Lambda - \partial_r \Phi \partial_r \Lambda + (\partial_r \Lambda)^2 \right) \omega^1 \wedge \omega^0 \\ \mathcal{R}^2{}_{1} &= e^{-2\Phi} \left( \partial_r \partial_r F - \partial_r \Phi \partial_r F + (\partial_r F)^2 \right) \omega^1 \wedge \omega^2 \\ \mathcal{R}^3{}_{2} &= e^{-(F+\Phi)} \partial_r (G - F) \frac{\cos \theta}{\sin \theta} \omega^1 \wedge \omega^3 + (e^{-2\Phi} \partial_r G \partial_r F - e^{-2F}) \omega^2 \wedge \omega^3 \\ \mathcal{R}^3{}_{1} &= e^{-2\Phi} \left( \partial_r \partial_r G - \partial_r \Phi \partial_r G + (\partial_r G)^2 \right) \omega^1 \wedge \omega^3 + \\ &\quad + e^{-(F+\Phi)} \partial_r (G - F) \frac{\cos \theta}{\sin \theta} \omega^2 \wedge \omega^3 \\ \mathcal{R}^0{}_{2} &= -e^{-2\Phi} \partial_r \Lambda \partial_r F \omega^0 \wedge \omega^2 \\ \mathcal{R}^0{}_{3} &= -e^{-2\Phi} \partial_r \Lambda \partial_r G \omega^0 \wedge \omega^3. \end{aligned} \quad (54)$$

iii. The components of the Riemann tensor are easily obtained from the well know geometrical relation of Cartan:

$$\mathcal{R}^\alpha{}_\beta = R^\alpha{}_{\beta\rho\sigma} \omega^\rho \wedge \omega^\sigma,$$

where we obtain explicitly

$$R^0{}_{110} = e^{-2\Phi} \left( \partial_r \partial_r \Lambda - \partial_r \Phi \partial_r \Lambda + (\partial_r \Lambda)^2 \right)$$

$$R^2{}_{112} = e^{-2\Phi} \left( \partial_r \partial_r F - \partial_r \Phi \partial_r F + (\partial_r F)^2 \right)$$

$$\begin{aligned}
R^3_{113} &= e^{-2\Phi} \left( \partial_r \partial_r G - \partial_r \Phi \partial_r G + (\partial_r G)^2 \right) \\
R^3_{213} &= e^{-(F+\Phi)} \partial_r (G - F) \frac{\cos \theta}{\sin \theta} \\
R^3_{123} &= e^{-(F+\Phi)} \partial_r (G - F) \frac{\cos \theta}{\sin \theta} \\
R^3_{223} &= e^{-2\Phi} \partial_r G \partial_r F - e^{-2F} \\
R^0_{330} &= e^{-2\Phi} \partial_r \Lambda \partial_r G, \\
R^0_{220} &= e^{-2\Phi} \partial_r \Lambda \partial_r F
\end{aligned} \tag{55}$$

from which we can construct the Einstein equations of the usual manner.

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Figure 1: electric field  $F_{10}$  of the EBI - monopole in function of  $r$ , for  $M = 0$ ,  $r_0 = 1$ ,  $m = 1$ ,  $n = 3$  and  $a = -0.9$

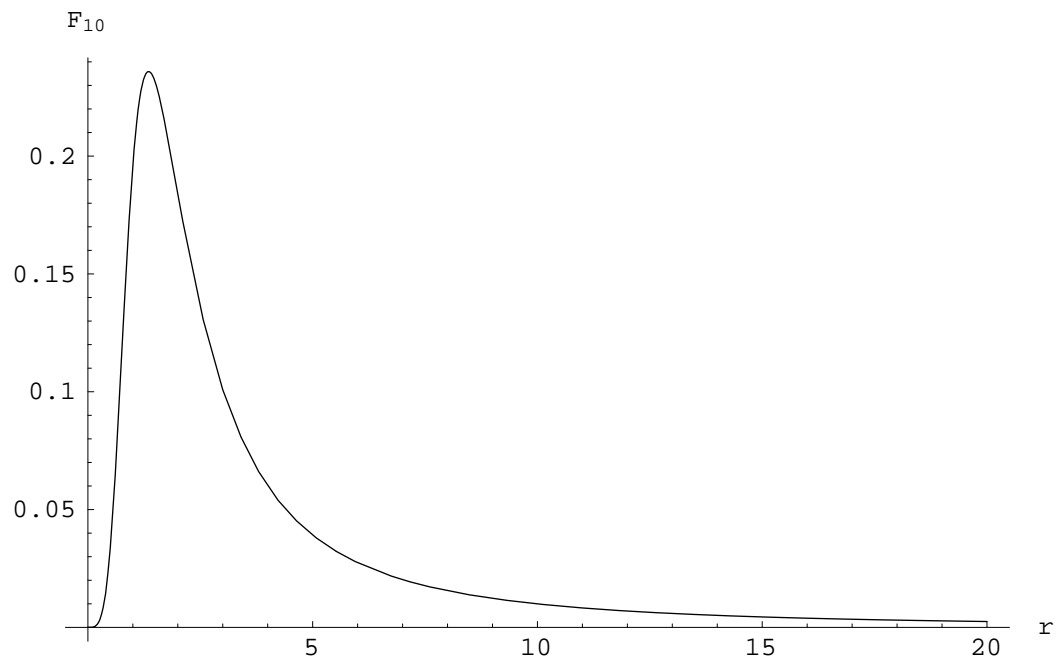


Figure 2: coefficient  $-g_{tt}$  of the EBI - monopole in function of  $r$ , for  $M = 0$ ,  $r_0 = 1$ ,  $m = 1$ ,  $n = 3$  and  $a = -0.9$

