LAGRANGIAN FORMALISM ON JET-GAUGE AND JET-DIFFEOMORPHISM GROUPS: TOWARDS A UNIFICATION OF GRAVITY WITH INTERNAL GAUGE INTERACTIONS [1]

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In this talk the description of gauge theories associated with internal symmetries is extended to the case in which the symmetry group is the space-time translation group (recovering Einstein's theory) using the standard jet-bundle formalism. We also reformulate these theories introducing the idea of jet-gauge and jet-diffeomorphism groups. Finally, we attempt to a simple, yet non-trivial, mixing of gravity and electromagnetism (or more general internal interaction) by turning to gauge symmetry a central extension of the Poincaré group.

1 Introduction

The formulation of gravity as a gauge theory appeared for the first time in Utiyama's works, following an structural scheme analogous to that of internal symmetries which had been introduced by Yang-Mills, and even earlier by Weyl in his attempts to unify electromagnetism and gravitation. After Utiyama's work, there have been many authors ([13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29], etc) who have tried to describe gravitation in the framework of gauge interactions. Nevertheless, there is some ambiguity about which external symmetry group is the correct one, i.e. Poincaré or just one of its subgroups. Another difference with respect to the internal case is that when dealing with space-time groups it is necessary to introduce new compensating fields k_ν^μ, usually known as tetrads or vierbeins, that allow the definition of a non-trivial metric $g_{\mu\nu}$ starting from the flat metric of Minkowski space-time $\eta_{\mu\nu}$.

The traditional formulation of gauge theories is developed by means of the Lagrangian formalism on the 1-jet bundle $J^1(E)$ of a bundle E on the Minkowski space-time. It is also assumed as a starting point, the hipothesis of invariance of the action associated with some matter fields under the rigid (global) symmetry Lie group of the free theory. But this procedure would not have a full sense if we desired to study pure radiation from the very begining, since we would not have any matter Lagrangian. Another problem of the usual treatment of gauge theories is related to the interpretation of the gauge potentials as connections, which as is well-known do not transform as a tensor.

In this paper we present gravitation as a consequence of gauging the space-time translation group $T(4)$ with very little deviation from the standard theory. It is worth noting that it is absolutely fair to consider non-trivial translational potentials $\mathcal{A}_\nu^\mu$ associated with the trivial action of the group $T(4)$ on the fibre, though they are redundant and the constraint $k_\nu^\mu = \delta_\nu^\mu + \mathcal{A}_\nu^\mu$ are required to maintain inaltered the total number of degrees of freedom of the theory. But the relevance of recovering Einstein's equations in the framework of the gauge theory of $T(4)$ is fundamental from the viewpoint of generalizing the conceptual and structural formulation of gauge gravitation theories. In this paper we also present a deep revision of the Minimal Coupling Principle, where the interpretation of gauge fields as connections and the requirement of the matter Lagrangians are removed. The main idea of this reformulation is the concept of "jet-diffeomorphism group" of a manifold M, $J^1(\mathcal{D}iff(M))$, with coordinates (ξ^μ, ξ_ν^μ). In this way, the compensating fields (tetrads or vierbeins) k_ν^μ can be interpreted as $k_\nu^\mu \equiv \xi_\nu^\mu$, that is, the compensating fields k_ν^μ can be considered as the 1-jets of the diffeomorphims ξ^μ of Minkowski space-time. As a consequence, both indices of k_ν^μ are space-time indices. We would like also to remark that we are keeping the name "gauge" for the case of the translation group because

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the general structure of the theory is similar to the structure of gauge theories for internal symmetries. But as was mentioned before, the additional compensating fields k_ν^μ are related to the coordinates of the bundle $J^1(\mathcal{D}iff(M))$ and we should properly speak of diffeomorphisms of M.

The version for internal symmetries arises with the idea of "jet-gauge group", $G^1(M) \equiv J^1(G(M))$ with coordinates $(\varphi^a, \varphi_\mu^b \equiv A_\mu^{(b)})$, where we denote by $G(M)$ the gauge (current or local) group. The gauge potentials $A_\mu^{(b)}$ are interpreted as the 1-jets of the Lie group parameters φ^b, departing from the traditional interpretation as connections. Moreover, in this framework the parameters of the Lie group (that is, the functions used to gauge the rigid Lie algebra in the standard formalism) acquire dynamical content.

The paper is organized as follows: Section 2 is devoted to the formulation of the variational principles on the jet bundle $J^1(E)$. In section 3 we revise the formulation of the gauge theory for internal symmetries and show its corresponding decription in terms of jet-gauge groups. Section 4 is devoted to the gauge theory of the space-time translation group $T(4)$ and the Lagrangian formalism on $J^1(\mathcal{D}iff(M))$ in order to describe gravity as a variational problem via the Modified Hamilton Principle on the bundle of the 1-jets of the 1-jets of the group of diffeomorphisms of Minkowski space-time $J^1(J^1(\mathcal{D}iff(M)))$. In section 5, we show an approach to the mixing between electromagnetism and gravity. Finally, we conclude in section 6 and incorporate an appendix about the Lagrangian formalism on $J^1(J^1(E))$.

2 Lagrangian Formalism on the Jet Bundle $J^1(E)$

Classical Field Theory is traditionally formulated on the *1-jet* bundle $J^1(E)$ of a vector bundle $E \xrightarrow{\pi} M$ on Minkowski space-time or any other space-time manifold (see e.g. [6, 44]). The sections of E, i.e. mappings φ from the base manifold M to the total space E such that $\pi \circ \varphi = I_M$, constitute the fields, and the 1-jet bundle generalizes to field theory the definition space for Lagrangians in ordinary Analytical Mechanics, that is, $T(R^3) \times R \to R$, where R is parametrized by the time t and $T(R^3)$ by $(q^i, \dot{q}^j)$. We give the formal construction of the jet bundle of a general bundle E, non-necessarily a vector bundle.

The four first greek letters $(\alpha, \beta, \gamma, \delta)$ will denote spinorial indices of the matter fields and the rest of the greek alphabet $(\epsilon, ..., \mu, \nu, ..., \tau, ...\omega = 0, 1, 2, 3)$ will be used to denote space-time indices. Let us consider a bundle E, $\pi : E \to M$, and name $\Gamma(E)$ the space of sections of E. We define the bundle of the *1-jets* of $\Gamma(E)$ as:

$$J^1(E) \equiv \frac{\Gamma(E) \times M}{\underset{\sim}{1}} \tag{1}$$

where the equivalence relation $\overset{1}{\sim}$ is defined as follows:

$$(\varphi, x) \sim (\varphi', x') \Longleftrightarrow \left\{ \begin{array}{rcl} x &=& x' \\ \varphi(x) &=& \varphi'(x) \\ \partial_\mu \varphi(x) &=& \partial_\mu \varphi'(x), \end{array} \right. \tag{2}$$

$\forall (\varphi, x), (\varphi', x') \in \Gamma(E) \times M$. The cartesian projection $(\varphi, x) \mapsto x$ defines a natural projection $\pi^1 : J^1(E) \to M$.

Locally, we can parametrize E by co-ordinates (x^μ, φ^α) defined on $\pi^{-1}(U)$, $U \subset M$. In fact, let us define the following functions on $\Gamma(\pi^{-1}(U)) \times U$:

$$\begin{array}{rcl} \varphi^\alpha : (\varphi, x) &\mapsto& \varphi^\alpha(\varphi(x)) \\ \varphi_\mu^\alpha : (\varphi, x) &\mapsto& \partial_\mu \varphi^\alpha(\varphi(x)). \end{array} \tag{3}$$

They are compatible with the equivalence relation $\overset{1}{\sim}$ and, therefore, go to the quotient defining the corresponding functions $(\varphi^\alpha, \varphi_\mu^\alpha)$.

The starting point of the geometric approach to the variational principles is the definition of the Lagrangian density as a real function $\mathcal{L} : J^1(E) \to R$ on the bundle $J^1(E)$ of a vector bundle E. Lagrangians depend locally on the arguments $(x^\mu, \varphi^\alpha, \varphi_\mu^\alpha)$, although usually, due to the Poincaré invariance, $\mathcal{L}$ will not depend explicitly on x^μ.

Let φ be a section of E. The *1-jet* extension (or prolongation) of φ, $j^1(\varphi) \equiv \overline{\varphi}$, is the only section of $J^1(E)$ such that j^1 is an injection of $\Gamma(E)$ into $\Gamma(J^1(E))$ and $\theta^\alpha \mid_{j^1(\varphi)(M)} = 0$ where θ^α are the *structure*

1-forms defined on $J^1(E)$ by

$$\theta^\alpha = d\varphi^\alpha - \varphi_\mu^\alpha dx^\mu. \tag{4}$$

Given an arbitrary vector field X on E, $X \in \Gamma(T(E)) \equiv \mathcal{X}(E)$,

$$X = X^\mu \frac{\partial}{\partial x^\mu} + X^\alpha \frac{\partial}{\partial \varphi^\alpha}, \tag{5}$$

acting on the space-time and on the fibre, its *1-jet* extension (or prolongation) through the injection j^1 is the only vector field $j^1(X) \equiv \bar{X}$ on $J^1(E)$ such that $\bar{X}$ projects on X, that is,

$$\bar{X} = X + \bar{X}_\mu^\alpha \frac{\partial}{\partial \varphi_\mu^\alpha}, \tag{6}$$

and it is an infinitesimal contact transformation, i.e.

$$L_{\bar{X}} \theta^\alpha = C_\beta^\alpha \theta^\beta, \tag{7}$$

where $L_{\bar{X}}$ is the Lie derivative with respect to the vector field $\bar{X}$, and C_β^α are constants. This second condition implies that

$$C_\beta^\alpha = \frac{\partial X^\alpha}{\partial \varphi^\beta} - \varphi_\mu^\alpha \frac{\partial X^\mu}{\partial \varphi^\beta}, \tag{8}$$

$$\bar{X}_\mu^\alpha = \frac{\partial X^\alpha}{\partial x^\mu} - \varphi_\nu^\alpha \frac{\partial X^\nu}{\partial x^\mu} + \left(\frac{\partial X^\alpha}{\partial \varphi^\beta} - \varphi_\nu^\alpha \frac{\partial X^\nu}{\partial \varphi^\beta} \right) \varphi_\mu^\beta. \tag{9}$$

Given a Lagrangian density $\mathcal{L}$, the Hamilton functional (the action) $\mathcal{S}$ is the mapping from $\Gamma(E)$ to R defined by

$$\mathcal{S} : \Gamma(E) \to R$$

$$\mathcal{S}(\varphi) = \int_{j^1(\varphi)(M)} \mathcal{L}(j^1(\varphi)) \pi^{1*} \omega, \tag{10}$$

$\forall \varphi \in \Gamma(E)$, where $\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is the volume 4-form on M and $\pi^{1*}\omega$ its pull-back to $J^1(E)$ (to be denoted just ω in the sequel).

The Ordinary Hamilton Principle states that the critical sections (trajectories) of the variational problem are the solutions of

$$(\delta \mathcal{S})_{j^1(\varphi)}(X) \equiv \int_{j^1(\varphi)(M)} L_{\bar{X}}(\mathcal{L}(j^1(\varphi))\omega) = 0,$$

where X is an arbitrary vector field on E.

As is well known, this principle leads to the usual Euler-Lagrange motion equations

$$\frac{\partial \mathcal{L}}{\partial \varphi^\alpha} - \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial \varphi_\mu^\alpha} \right) = 0. \tag{11}$$

The variational calculus can by generalized through the Modified Hamilton Principle, which varies sections of $J^1(E)$ rather than sections of E. To this end a *modified Hamiltonian action* $\mathcal{S}^1 : \Gamma(J^1(E)) \to R$ is defined as follows:

$$\mathcal{S}^1(\varphi^1) = \int_{\varphi^1(M)} \Theta_{PC}, \tag{12}$$

where Θ_{PC}, the Poincaré-Cartan(-Hilbert) form is a $n \equiv \dim M$ form which generalizes that of Mechanics, $\frac{\partial L}{\partial \dot{q}^i}(dq^i - \dot{q}^i dt) + L dt = p_i dq^i - H dt$:

$$
\begin{aligned}
\Theta_{PC} &= \frac{\partial \mathcal{L}}{\partial \varphi_\mu^\alpha}(d\varphi^\alpha - \varphi_\nu^\alpha dx^\nu) \wedge \theta_\mu + \mathcal{L}\omega \\
&= \frac{\partial \mathcal{L}}{\partial \varphi_\mu^\alpha} d\varphi^\alpha \wedge \theta_\mu - \left(\frac{\partial \mathcal{L}}{\partial \varphi_\mu^\alpha} \varphi_\mu^\alpha - \mathcal{L} \right) \omega \\
&\equiv \pi_\alpha^\mu d\varphi^\alpha \wedge \theta_\mu - \mathcal{H}\omega,
\end{aligned}
\tag{13}
$$

where $\theta_\mu \equiv i_{\frac{\partial}{\partial x^\mu}} \omega$. In the regularity case, i.e. $det(\frac{\partial^2 \mathcal{L}}{\partial \varphi_\mu \partial \varphi_\nu}) \neq 0$, $\mathcal{H}$ is the covariant Hamiltonian and π_α^μ the n covariant momenta.

The *Modified Hamilton Principle* stablishes that the physical trajectories are those sections of $J^1(E)$, points in $\Gamma(J^1(E))$, where the derivative of the action (12) is zero:

$$
\begin{aligned}
(\delta \mathcal{S}')_{\varphi^1}(X^1) &\equiv \int_{\varphi^1(M)} L_{X^1} \Theta_{PC} = 0 , \quad \forall X^1 \in \mathcal{X}(J^1(E)) \\
&\Rightarrow \quad i_{X^1} d\Theta_{PC}|_{\varphi^1} = 0.
\end{aligned}
\tag{14}
$$

Equations (14) generalize the Euler-Lagrange ones (11) and, in the case of regularity, reproduce them and provide the covariant Hamilton equations[44]:

$$
\frac{\partial \mathcal{H}}{\partial \pi_\alpha^\mu} = \frac{\partial \varphi^\alpha}{\partial x^\mu} , \quad \frac{\partial \mathcal{H}}{\partial \varphi^\alpha} = -\frac{\partial \pi_\alpha^\mu}{\partial x^\mu} .
\tag{15}
$$

3   Gauge Theory of Internal Symmetries and Jet-Gauge Groups

Let us consider a matter Lagrangian density

$$
\mathcal{L}_{matt}(\varphi^\alpha, \varphi_\mu^\alpha)
\tag{16}
$$

and the corresponding action

$$
\mathcal{S} = \int \mathcal{L}_{matt}\, \omega .
\tag{17}
$$

Let us assume that $\mathcal{S}$ is invariant under a global (or rigid) Lie group of internal symmetry G, i.e.,

$$
L_{\bar{X}_{(a)}} (\mathcal{L}_{matt}\, \omega) = 0
\tag{18}
$$

where $\bar{X}_{(a)} \equiv j^1(X_{(a)})$ is the jet extension of the generator

$$
X_{(a)} = X_{(a)\beta}^\alpha \varphi^\beta \frac{\partial}{\partial \varphi^\alpha}
\tag{19}
$$

of G (whose components on $\frac{\partial}{\partial x^\mu}$ have been suppresed since we have supposed that G does not act on the base manifold M) satisfying the commutation relations

$$
(X_{(b)} X_{(a)} - X_{(a)} X_{(b)})_\beta^\alpha = C_{ab}^c X_{(c)\beta}^\alpha .
$$

We use the labels $(a), (b), ...$ to denote group indices. Needless to say that the term $\mathcal{L} L_{\bar{X}} \omega = 0$ since $div(X) = 0$.

According to the general expression for the jet extension and the assumption of the internal action of G, the invariance of the action under the rigid group G may be written simply as:

$$
\bar{X}_{(a)} \mathcal{L}_{matt}(\varphi^\alpha, \varphi_\mu^\alpha) = X_{(a)\beta}^\alpha \varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha}(\varphi^\alpha, \varphi_\mu^\alpha) + X_{(a)\beta}^\alpha \varphi_\mu^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi_\mu^\alpha}(\varphi^\alpha, \varphi_\mu^\alpha) = 0 .
\tag{20}
$$

Now we wish to study the invariance of the action under the ("current", "local" or) gauge group $G(M)$. The corresponding Lie algebra is the following tensor product:

$$
\mathcal{F}(M) \otimes \mathcal{G} \equiv \{f^{(a)} X_{(a)}\} ,
\tag{21}
$$

where $\mathcal{F}(M)$ is the multiplicative algebra of real analytic functions on M, and $\mathcal{G}$ is the Lie algebra of the Lie group G. Now the jet extension of the generators of this new algebra takes the form:

$$
\overline{f^{(a)} X_{(a)}} = f^{(a)} X_{(a)\beta}^\alpha \varphi^\beta \frac{\partial}{\partial \varphi^\alpha} + \left(f^{(a)} X_{(a)\beta}^\alpha \varphi_\mu^\beta + X_{(a)\beta}^\alpha \varphi^\beta \frac{\partial f^{(a)}}{\partial x^\mu} \right) \frac{\partial}{\partial \varphi_\mu^\alpha} .
\tag{22}
$$

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We see that the jet extension of a generator in (21) is no longer a tensor product but, rather, it acquires extra terms:

$$\overline{f^{(a)}X_{(a)}} = f^{(a)}\bar{X}_{(a)} + X_{(a)\beta}^{\alpha}\varphi^{\beta}\frac{\partial f^{(a)}}{\partial x^{\mu}}\frac{\partial}{\partial\varphi_{\mu}^{\alpha}}\,. \tag{23}$$

Therefore,

$$\overline{f^{(a)}X_{(a)}}\mathcal{L}_{matt} \neq 0$$

and, in order to restore the invariance, we have to introduce new compensating fields (usually known as Yang-Mills fields) $A_{\mu}^{(a)}$ with the following transformation law under $G(M)$:

$$X_{A_{\mu}^{(a)}} \equiv \delta A_{\mu}^{(a)} = f^{(b)}C_{bc}^{a}A_{\mu}^{(c)} + \frac{\partial f^{(a)}}{\partial x^{\mu}}\,, \tag{24}$$

where C_{bc}^{a} are the structure constants of the Lie algebra $\mathcal{G}$.

Then, we must consider the generators acting also on the new fields $A_{\mu}^{(a)}$:

$$f^{(a)}\mathcal{X}_{(a)} = f^{(a)}X_{(a)\beta}^{\alpha}\varphi^{\beta}\frac{\partial}{\partial\varphi^{\alpha}} + \left(f^{(b)}C_{bc}^{a}A_{\mu}^{(c)} + \frac{\partial f^{(a)}}{\partial x^{\mu}}\right)\frac{\partial}{\partial A_{\mu}^{(a)}}\,. \tag{25}$$

Now the situation is as follows: we have introduced new fields $A_{\mu}^{(a)}$ to modify the behaviour of the original Lagrangian of matter so that we have to find, on the one hand the expression for the new Lagrangian $\widehat{\mathcal{L}}_{matt}$ containing the matter fields and their interaction with the new compensating fields $A_{\mu}^{(a)}$ and, on the other, the free Lagrangian $\mathcal{L}_0$ corresponding to the new fields, which should depend on the new fields variables and their first derivatives, i.e. $A_{\mu}^{(a)}$, $A_{\nu,\sigma}^{(a)}$. Let us establish the required process under the aspect of a two-parts theorem to be referred to as Utiyama's Theorem (as suggested in mathematical literature) without aiming at a too formal presentation. Proofs of theorems will be presented elsewhere.

Theorem 1.A (The Minimal Coupling): *The new Lagrangian describing the matter fields as well as their interaction with the new compensating fields $A_{\nu}^{(a)}$,*

$$\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi_{\mu}^{\alpha}, A_{\nu}^{(a)}) \equiv \mathcal{L}_{matt}(\varphi^{\alpha}, \varphi_{\mu}^{\alpha} - A_{\mu}^{(a)}X_{(a)\beta}^{\alpha}\varphi^{\beta}), \tag{26}$$

is invariant under the current group $G(M)$ in the sense that

$$\overline{f^{(a)}\mathcal{X}_{(a)}}\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi_{\mu}^{\alpha}, A_{\nu}^{(a)}) = 0\,. \tag{27}$$

Thus, as **Corollary 1.A**, we conclude that the matter fields interact only with the new compensating fields and not with their derivatives and the actual interaction term appears in the Lagrangian through the combination

$$\phi_{\mu}^{\alpha} = \varphi_{\mu}^{\alpha} - A_{\mu}^{(a)}X_{(a)\beta}^{\alpha}\varphi^{\beta}, \tag{28}$$

usually known as covariant derivative of the matter fields (on this respect, see the end of this subsection).

The introduction of the new compensating fields naturally leads to considering a new action accounting also for the dynamics of these new fields with Lagrange density $\mathcal{L}_0$:

$$\mathcal{S}' = \int(\widehat{\mathcal{L}}_{matt}\omega + \mathcal{L}_0\omega). \tag{29}$$

Since we have already seen that $\int\widehat{\mathcal{L}}_{matt}\,\omega$ is invariant under $G(M)$, imposing the invariance of $\mathcal{S}'$ requires the invariance of $\int\mathcal{L}_0\,\omega$ itself. That is, the free Lagrangian $\mathcal{L}_0$, containing the new compensating fields and their first derivatives, must be invariant under the current group $G(M)$:

$$\overline{f^{(a)}\mathcal{X}_{(a)}}\mathcal{L}_0(A_{\mu}^{(c)}, A_{\nu,\sigma}^{(b)}) = 0\,. \tag{30}$$

In the proof of the following "theorem" we solve this system and obtain the structure of $\mathcal{L}_0$.

Theorem 1.B (Structure of the free Lagrangian $\mathcal{L}_0$): *The necessary condition for $\mathcal{L}_0$ to be invariant under the current group $G(M)$ is that $\mathcal{L}_0$ depends on the fields $A_\mu^{(a)}$ and their "derivatives" $A_{\mu,\nu}^{(a)}$ only through the specific combination:*

$$F_{\mu\nu}^{(a)} \equiv A_{\mu,\nu}^{(a)} - A_{\nu,\mu}^{(a)} - \frac{1}{2} C_{bc}^a (A_\mu^{(b)} A_\nu^{(c)} - A_\nu^{(b)} A_\mu^{(c)}) \,, \tag{31}$$

i.e.

$$\overline{f^{(a)} \mathcal{X}_{(a)}} \mathcal{L}_0 (A_\mu^{(c)}, A_{\nu,\sigma}^{(b)}) = 0 \implies \mathcal{L}_0 = \mathcal{L}_0 (F_{\mu\nu}^{(a)}) \,. \tag{32}$$

The expression (31) is traditionally called "curvature" of the "connection" $A_\mu^{(a)}$ (see again the end of this subsection).

It should be remarked that the actual dependence of $\mathcal{L}_0$ on the tensor F is not fixed and must be chosen with the help of extra criteria, for example the invariance under the rigid Poincaré group. In particular, to account for the standard Yang-Mills equations the Lagrangian must be of the form

$$\mathcal{L}_0 \sim \sum_{a=1}^{dimG} F_{\mu\nu}^{(a)} F_{\sigma\rho}^{(a)} \eta^{\sigma\mu} \eta^{\rho\nu} \,. \tag{33}$$

For space-time symmetries the ambiguity in the specific dependence of $\mathcal{L}_0$ will be patent.

Euler-Lagrange equations

Let us consider the Lagrangian density $\mathcal{L}_{tot} \equiv \widehat{\mathcal{L}}_{matt} + \mathcal{L}_0$, where $\widehat{\mathcal{L}}_{matt}(\varphi^\alpha, \varphi_\mu^\alpha, A_\nu^{(a)}) \equiv \mathcal{L}_{matt}(\varphi^\alpha, \varphi_\mu^\alpha - A_\mu^{(a)} X_{(a)\beta}^\alpha \varphi^\beta)$. Then, the equations of motion are given by:

$$\frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}_{matt}}{\partial \varphi_\mu^\alpha} \right) - \frac{\partial \mathcal{L}_{matt}}{\partial \varphi^\alpha} + X_{(a)\alpha}^\beta A_\mu^{(a)} \frac{\partial \mathcal{L}_{matt}}{\partial \varphi_\mu^\beta} = 0 \tag{34}$$

$$\frac{d}{dx^\nu} \left(\frac{\partial \mathcal{L}_0}{\partial F_{\mu\nu}^{(a)}} \right) + C_{da}^b A_\nu^{(d)} \frac{\partial \mathcal{L}_0}{\partial F_{\nu\mu}^{(b)}} + \frac{1}{2} X_{(a)\beta}^\alpha \varphi^\beta \frac{\partial \mathcal{L}_{matt}}{\partial \varphi_\mu^\alpha} = 0 \,. \tag{35}$$

Remark: To finish the section corresponding to internal symmetries, we would like to note that, on sections,

$$F_{\mu\nu}^{(a)} = \partial_\nu A_\mu^{(a)} - \partial_\mu A_\nu^{(a)} - \frac{1}{2} (A_\mu^{(b)} A_\nu^{(c)} - A_\nu^{(b)} A_\mu^{(c)}). \tag{36}$$

Introducing the notation (spin connection)

$$\Gamma_{\mu\beta}^\alpha \equiv A_\mu^{(a)} X_{(a)\beta}^\alpha \,,$$

and taking into account the commutation relations

$$(X_{(b)} X_{(a)} - X_{(a)} X_{(b)})_\beta^\alpha = C_{ab}^c X_{(c)\beta}^\alpha \,,$$

the tensor $F_{\mu\nu}^{(a)}$ can be turned into a curvature tensor:

$$\begin{aligned} R_{\mu\nu\beta}^\alpha \equiv F_{\mu\nu}^{(a)} X_{(a)\beta}^\alpha &= \partial_\mu \Gamma_{\nu\beta}^\alpha - \partial_\nu \Gamma_{\mu\beta}^\alpha - \frac{1}{2} C_{bc}^a (A_\mu^{(b)} A_\nu^{(c)} (X_{(a)})_\beta^\alpha - A_\nu^{(b)} A_\mu^{(c)} (X_{(a)})_\beta^\alpha) \\ &= \partial_\mu \Gamma_{\nu\beta}^\alpha - \partial_\nu \Gamma_{\mu\beta}^\alpha - (\Gamma_{\mu\gamma}^\alpha \Gamma_{\nu\beta}^\gamma - \Gamma_{\nu\beta}^\alpha \Gamma_{\mu\beta}^\gamma) \,. \end{aligned} \tag{37}$$

Accordingly, the Minimal Coupling prescription can be easily reformulated by saying that the matter Lagrangian incorporating the interaction terms is obtained from the original one by replacing all derivatives of the matter fields with covariant derivatives, and that the Lagrangian for the connections must be a function of the curvature.

Now we would like to present a conceptual and structural revision of the standard formulation of gauge theories, in such a way that the gauge group acquires dynamical character, and associated natural structures,

which substitute the well-known spin connections, appear as physical fields (and these are tensorial objects) irrespective of the existence of fermionic matter.

Let G be a Lie group of internal symmetry, $G(M)$ the gauge group (also named current group), whose elements are the mappings from Minskowski space-time M into G (thus $G(M)$ may be viewed as the set of sections of a principal bundle with structural group G). Let $\{x^\mu\}$ be a coordinate system on the base manifold M and $\{\varphi^\alpha\}$ the corresponding coordinate system on the fibre G. Let's introduce the bundle of the 1-jets of the gauge group $G(M)$:

$$J^1(G(M)).$$

The formal definition of $J^1(G(M))$ is given through the relation of equivalence $\sim^1$ in the following way:

$$J^1(G(M)) \equiv G(M) \times M/\sim^1$$

such that

$$(\Phi, m) \sim^1 (\Phi', m')$$

for all $(\Phi, m),(\Phi', m')$ belonging to $G(M) \times M$, if and only if
1)

$$m = m' \ ,$$

2)

$$\Phi(m) = \Phi'(m) \ ,$$

3)

$$\partial_\mu \Phi(m) = \partial_\mu \Phi'(m).$$

This definition may easily be extended to the order r.

A coordinate system for $J^1(G(M))$ would be

$$\{x^\mu, \varphi^\alpha, \varphi_\mu^\alpha\},$$

where φ_μ^α is not necessarily the derivative of any section $\varphi^\alpha(x)$. In order to introduce "derivatives" of the fields in the theory, we introduce the bundle of the 1-jets of the 1-jets of the gauge group

$$J^1(J^1(G(M))),$$

defined in an analogous way as $J^1(G(M))$ (see the Appendix).

A suitable coordinate system for $J^1(J^1(G(M)))$ is, according to the appendinx devoted to the Lagrangian formalisn on $J^1(J^1(E))$,

$$\{x^\mu, \varphi^\alpha, \varphi_\mu^\alpha; \varphi_{,\nu}^\alpha, \varphi_{\mu,\nu}^\alpha\}.$$

According to the more usual field theory notation we have replaced the coordinates y and z of the appendix by φ^α and $\varphi_{,i}^\alpha$.

Once we have a certain coodinates system we can construct Lagrangian densities defined on $J^1(J^1(G(M)))$ and now we will explicitly proceed to study the abelian and non-abelian internal cases.

3.1 Abelian internal gauge interaction

The associated Lie group is $U(1)$ and the coordinates that we will use $\{\phi, \phi_\mu; \phi_{,\nu}, \phi_{\mu,\nu}\}$. Due to the invariance under Poincaré group the Lagrangian will not explicitly depend on x^μ. The more general scalar Lagrangian on $J^1(J^1(U(1)(M)))$ is given by the following combination with coefficients that will be determined later:

$$\mathcal{L}_{Abel}^{1,1} = a\phi_{\mu,\nu}\phi^{\mu,\nu} + b\phi_{\mu,\nu}\phi^{\nu,\mu} + c\phi_{,\mu}^\mu\phi_{,\nu}^\nu + e\phi\phi_{,\mu}^\mu + f\phi_\mu\phi^\mu + g\phi_{,\mu}\phi^{\cdot\mu}$$

$$+h\phi_\mu\phi^{\cdot\mu} + j\phi\phi \ .$$

Now we write the motion equations derived from the Modified Hamilton Principle:

$$\delta\phi_{\nu,\rho}: \quad \frac{\partial}{\partial\phi_{\nu,\rho}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}}\right)\left(\frac{\partial\phi}{\partial x^\mu} - \phi_{,\mu}\right) + \frac{\partial}{\partial\phi_{\nu,\rho}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{\sigma,\mu}}\right)\left(\frac{\partial\phi_\sigma}{\partial x^\mu} - \phi_{\sigma,\mu}\right) = 0 \ ,$$

$$\left(a\eta^{\nu\sigma}\eta^{\mu\rho} + b\eta^{\mu\sigma}\eta^{\nu\rho} + c\eta^{\nu\mu}\eta^{\sigma\rho}\right)\left(\frac{\partial\phi_\nu}{\partial x^\mu} - \phi_{\nu,\mu}\right) = 0$$

$$\delta\phi_{,\sigma}: \quad \frac{\partial}{\partial\phi_{,\sigma}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}}\right)\left(\frac{\partial\phi}{\partial x^\mu} - \phi_{,\mu}\right) + \frac{\partial}{\partial\phi_{,\sigma}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{\nu,\mu}}\right)\left(\frac{\partial\phi_\nu}{\partial x^\mu} - \phi_{\nu,\mu}\right) = 0$$

$$g\eta^{\sigma\mu}\left(\frac{\partial\phi}{\partial x^\mu} - \phi_{,\mu}\right) = 0 \implies \phi_{,\mu} = \partial_\mu\phi$$

$$\delta\phi_\nu: \quad -\frac{d}{dx^\mu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{\nu,\mu}}\right) + \frac{\partial\mathcal{L}^{1,1}}{\partial\phi_\nu} + \frac{\partial}{\partial\phi_\nu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}}\right)\left(\frac{\partial\phi}{\partial x^\mu} - \phi_{,\mu}\right)$$

$$+\frac{\partial}{\partial\phi_\nu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{\sigma,\mu}}\right)\left(\frac{\partial\phi_\sigma}{\partial x^\mu} - \phi_{\sigma,\mu}\right) = 0$$

$$-2a\partial_\mu\phi^{\nu,\mu} - 2b\partial_\mu\phi^{\mu,\nu} - 2c\eta^{\mu\nu}\partial_\mu\phi^\sigma_{,\sigma} - e\eta^{\mu\nu}\partial_\mu\phi + 2f\phi^\nu + h\phi^{,\nu} + h\eta^{\mu\nu}(\partial_\mu\phi - \phi_{,\mu}) = 0$$

$$\delta\phi: \quad -\frac{d}{dx^\mu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}}\right) + \frac{\partial\mathcal{L}^{1,1}}{\partial\phi} + \frac{\partial}{\partial\phi}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}}\right)\left(\frac{\partial\phi}{\partial x^\mu} - \phi_{,\mu}\right)$$

$$+\frac{\partial}{\partial\phi}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{\nu,\mu}}\right)\left(\frac{\partial\phi_\nu}{\partial x^\mu} - \phi_{\nu,\mu}\right) = 0$$

$$-2g\partial_\mu\phi^{,\mu} - h\partial_\mu\phi^\mu + e\phi^\mu_{,\mu} + e\eta^{\mu\nu}(\partial_\mu\phi_\nu - \phi_{\nu,\mu}) + 2j\phi = 0 \ .$$

Making the following choice of coefficients:
$2a = 1$
$2f = -m^2$
$2b = -1$
$h = e$
$c = 0$
$g = \frac{1}{2}$
$2j = -M^2$
we arrive at the following equations:

$$\delta\phi_{\sigma,\rho}: \quad \phi_{\mu,\nu} - \phi_{\nu,\mu} = \partial_\nu\phi_\mu - \partial_\mu\phi_\nu$$

$$\delta\phi_{,\sigma}: \quad \phi_{,\mu} = \partial_\mu\phi$$

$$\delta\phi_\nu: \quad -\partial_\mu\phi^{\nu,\mu} + \partial_\mu\phi^{\mu,\nu} - m^2\phi^\nu = 0 \implies \partial_\mu(\partial^\nu\phi^\mu - \partial^\mu\phi^\nu) - m^2\phi^\nu = 0$$

$$\implies [\eta_{\mu\nu}(\partial_\sigma\partial^\sigma + m^2) - \partial_\mu\partial_\nu]\phi^\mu = 0 \ ,$$

where we have used the two previous equations. This equation contains two variants: Maxwell equations ($m = 0$, the electromagnetic field being $A^\mu \equiv \phi^\mu$) and Proca equations ($m \neq 0$). If $m^2 \neq 0$ then, deriving with respect x^ν the last equation,

$$\eta_{\mu\nu}(\partial_\sigma\partial^\sigma + m^2)\partial^\nu\phi^\mu - \partial_\sigma\partial^\sigma\partial_\mu\phi^\mu = 0$$

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we get the transversality condition

$$\partial_\mu \phi^\mu = 0.$$

And finally,

$$\delta\phi: \quad \partial_\mu \partial^\mu \phi + M^2 \phi = 0 \ .$$

This is the Klein-Gordon equation for the scalar field ϕ. The role of these extra fields is quite analogous to that of the Goldstone bosons which appear in the Spontaneous Symmetry Breaking mechanism.

With the additional choice $h = e = 0$, the Lagrangian $\mathcal{L}^{1,1}_{Abel}$ takes the form

$$\mathcal{L}^{1,1}_{U(1)} = \frac{1}{2}(\phi_{\mu,\nu}\phi^{\mu,\nu} - \phi_{\mu,\nu}\phi^{\nu,\mu}) - \frac{m^2}{2}\phi_\mu \phi^\mu + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - \frac{M^2}{2}\phi\phi.$$

3.2 Non-abelian internal gauge interaction

The coordinate system in this case is denoted by $(\phi^a, \phi^a_\mu; \phi^a_{,\nu}, \phi^a_{\mu,\nu})$. In analogy to the previous case, let us consider the following Lagrangian on $J^1(J^1(G(M)))$:

$$\mathcal{L}^{1,1}_{Non-Abel} = \mathcal{L}^{1,1}_{Abel} + \alpha C^a_{dc}\phi^d_\mu\phi^c_\nu\phi^{\mu,\nu}_a + \beta C^a_{bc}C_{de,a}\phi^b_\mu\phi^c_\nu\phi^{d\mu}\phi^{e\nu} \ ,$$

where

$$\mathcal{L}^{1,1}_{Abel} = \frac{1}{2}(\phi^a_{\mu,\nu}\phi^{\mu,\nu}_a - \phi^a_{\mu,\nu}\phi^{\nu,\mu}_a) - \frac{m^2}{2}\phi^a_\mu\phi^\mu_a + \frac{1}{2}\phi^a_{,\mu}\phi^{,\mu}_a - \frac{M^2}{2}\phi^a\phi_a$$

and C^a_{bc} are the structure constants. Using the Modified Hamilton Principle the motion equations read:

$$\delta\phi^a_{\nu,\rho}: \quad \frac{\partial}{\partial\phi^a_{\nu,\rho}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^b_{,\mu}}\right)\left(\frac{\partial\phi^b}{\partial x^\mu} - \phi^b_{,\mu}\right) + \frac{\partial}{\partial\phi^a_{\nu,\rho}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^b_{\sigma,\mu}}\right)\left(\frac{\partial\phi^b_\sigma}{\partial x^\mu} - \phi^b_{\sigma,\mu}\right) = 0$$

$$\partial_\nu \phi^a_\mu - \partial_\mu \phi^a_\nu = \phi^a_{\mu,\nu} - \phi^a_{\nu,\mu}$$

$$\delta\phi^a_{,\sigma}: \quad \frac{\partial}{\partial\phi^a_{,\sigma}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^b_{,\mu}}\right)\left(\frac{\partial\phi^b}{\partial x^\mu} - \phi^b_{,\mu}\right) + \frac{\partial}{\partial\phi^a_{,\sigma}}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^b_{\nu,\mu}}\right)\left(\frac{\partial\phi^b_\nu}{\partial x^\mu} - \phi^b_{\nu,\mu}\right) = 0$$

$$\eta^{\mu\nu}\delta_{ab}(\partial_\mu\phi^b - \phi^b_{,\mu}) = 0 \Longrightarrow \partial_\mu\phi^b = \phi^b_{,\mu}$$

$$\delta\phi^a_\nu: \quad -\frac{d}{dx^\mu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^a_{\nu,\mu}}\right) + \frac{\partial\mathcal{L}^{1,1}}{\partial\phi^a_\nu} + \frac{\partial}{\partial\phi^a_\nu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^b_{,\mu}}\right)\left(\frac{\partial\phi^b}{\partial x^\mu} - \phi^b_{,\mu}\right)$$

$$+\frac{\partial}{\partial\phi^a_\nu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi^b_{\sigma,\mu}}\right)\left(\frac{\partial\phi^b_\sigma}{\partial x^\mu} - \phi^b_{\sigma,\mu}\right) = 0$$

$$-\partial_\mu\phi^{\nu,\mu}_a + \partial_\mu\phi^{\mu,\nu}_a - \alpha C_{dc,a}\partial_\mu(\phi^{d\nu}\phi^{c\mu}) - m^2\phi^\nu_a + C^f_{dc}\alpha(\delta^d_a\phi^{\nu,\sigma}_f\phi^c_\sigma + \delta^c_a\phi^{\mu,\nu}_f\phi^d_\mu) + 4\beta C^g_{ac}C_{de,g}\phi^c_\sigma\phi^{d\nu}\phi^{e\sigma}$$

$$+C^f_{dc}\alpha(\delta_{fb}\eta^{\sigma\nu}\phi^{c\mu}\delta^d_a + \delta_{fb}\phi^{d\sigma}\eta^{\mu\nu}\delta^c_a)(\partial_\mu\phi^b_\sigma - \phi^b_{\sigma,\mu}) = 0 \ .$$

Using the equation $\partial_\nu\phi^a_\mu - \partial_\mu\phi^a_\nu = \phi^a_{\mu,\nu} - \phi^a_{\nu,\mu}$, it follows

$$C^f_{dc}\alpha(\delta_{fb}\eta^{\sigma\nu}\phi^{c\mu}\delta^d_a + \delta_{fb}\phi^{d\sigma}\eta^{\mu\nu}\delta^c_a)(\partial_\mu\phi^b_\sigma - \phi^b_{\sigma,\mu}) = 0$$

and, therefore, the third motion equation reads:

$$-\partial_\mu\partial^\mu\phi_a^\nu + \partial_\mu\partial^\nu\phi_a^\mu - m^2\phi_a^\nu - \alpha C_{dc,a}\partial_\mu(\phi^{d\nu}\phi^{c\mu}) + \alpha C_{ac}^b(\partial^\mu\phi_b^\nu - \partial^\nu\phi_b^\mu)\phi_\mu^c + 4\beta C_{ac}^g C_{de,g}\phi_\sigma^c\phi^{d\nu}\phi^{e\sigma} = 0.$$

Making the choice of coefficients: $\alpha = -1$ and $4\beta = 1$, we obtain

$$-\partial_\mu\partial^\mu\phi_a^\nu + \partial_\mu\partial^\nu\phi_a^\mu - m^2\phi_a^\nu + C_{dc,a}\partial_\mu(\phi^{d\nu}\phi^{c\mu}) - C_{ac}^b(\partial^\mu\phi_b^\nu - \partial^\nu\phi_b^\mu)\phi_\mu^c + C_{ac}^g C_{de,g}\phi_\sigma^c\phi^{d\nu}\phi^{e\sigma} = 0.$$

Taking $m = 0$, this equation is the Yang-Mills motion equation, if we identify $A_\mu^{(a)} \equiv \phi_\mu^a$ as the gauge fields.

Finally,

$$\delta\phi^a: \quad -\frac{d}{dx^\mu}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}^a}\right) + \frac{\partial\mathcal{L}^{1,1}}{\partial\phi^a} + \frac{\partial}{\partial\phi^a}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{,\mu}^b}\right)\left(\frac{\partial\phi^b}{\partial x^\mu} - \phi_{,\mu}^b\right)$$

$$+\frac{\partial}{\partial\phi^a}\left(\frac{\mathcal{L}^{1,1}}{\partial\phi_{\nu,\mu}^b}\right)\left(\frac{\partial\phi_\nu^b}{\partial x^\mu} - \phi_{\nu,\mu}^b\right) = 0$$

$$\partial_\mu\partial^\mu\phi^a + M^2\phi^a = 0 .$$

This is the Klein-Gordon equation for the scalar fields ϕ^a.

4 Gauge Theory of the Space-Time Translation Group $T(4)$ and Jet-Diffeomorphism Groups

Let us consider the matter action

$$S = \int \mathcal{L}_{matt}(\varphi^\alpha, \varphi_\mu^\alpha)d^4x , \tag{38}$$

where $\varphi_\mu^\alpha \equiv \frac{\partial\varphi^\alpha}{\partial x^\mu}$ on jet sections.
The general form of a generator of an arbitrary Lie algebra $\mathcal{G}$ (of a Lie group G) is:

$$X = X^\mu\frac{\partial}{\partial x^\mu} + X^\alpha\frac{\partial}{\partial\varphi^\alpha} \quad , \tag{39}$$

and its jet extension:

$$\overline{X} = X + \overline{X}_\mu^\alpha\frac{\partial}{\partial\varphi_\mu^\alpha} \quad , \tag{40}$$

where

$$\overline{X}_\mu^\alpha = \frac{\partial X^\alpha}{\partial x^\mu} + \frac{X^\alpha}{\partial\varphi^\beta}\varphi_\mu^\beta - \left(\frac{\partial X^\nu}{\partial x^\mu} + \frac{\partial X^\nu}{\partial\varphi^\beta}\varphi_\mu^\beta\right)\varphi_\nu^\alpha \quad . \tag{41}$$

The invariance condition of the action under the Lie group G implies the following invariance condition for the Lagrangian density:

$$\overline{X}\mathcal{L}(\varphi^\alpha, \varphi_\mu^\alpha) + \mathcal{L}(\varphi^\alpha, \varphi_\mu^\alpha)\partial_\mu X^\mu = 0 . \tag{42}$$

Let us take $G = T(4)$, the space-time translation group. The generators of the translation group are given by

$$X_{(a)} = X_{(a)}^{\mu} \frac{\partial}{\partial x^{\mu}} \ , \tag{43}$$

$$X_{(\nu)}^{\mu} = \delta_{\nu}^{\mu} \ , \tag{44}$$

where $(a) = (\nu)$ is the group index.

The rigid (global) translational invariance condition reads

$$\partial_{\mu} \mathcal{L} = 0, \tag{45}$$

that is, the Lagrangian density does not depend explicitly on x.

From the algebra point of view, making the parameters of the group depend on x is equivalent to multiply the generators by arbitrary functions of x :

$$X^{gauge} = X^{\mu}(x) \frac{\partial}{\partial x^{\mu}} \equiv f^{\mu}(x) \frac{\partial}{\partial x^{\mu}} \tag{46}$$

$$\overline{X}^{gauge} = f^{\mu}(x) \frac{\partial}{\partial x^{\mu}} - \partial_{\mu} f^{\nu}(x) \varphi_{\nu}^{\alpha} \frac{\partial}{\partial \varphi_{\mu}^{\alpha}} \ . \tag{47}$$

The invariance is broken and in order to restore it we introduce the compensating fields k_{μ}^{ν} (and their inverse q_{σ}^{μ}: $k_{\mu}^{\nu} q_{\sigma}^{\mu} = \delta_{\sigma}^{\nu}$), whose transformation law under the local $T(4)$ is given by:

$$X_{k_{\mu}^{\nu}} \equiv \delta k_{\mu}^{\nu} = k_{\mu}^{\sigma} \frac{\partial f^{\nu}(x)}{\partial x^{\sigma}} \quad . \tag{48}$$

We must consider the generators also acting on the new compensating fields:

$$\mathcal{X}^{gauge} = f^{\mu}(x) \frac{\partial}{\partial x^{\mu}} + k_{\mu}^{\sigma} \partial_{\sigma} f^{(\nu)}(x) \frac{\partial}{\partial k_{\mu}^{\nu}} \tag{49}$$

$$\overline{\mathcal{X}}^{gauge} = \mathcal{X}^{gauge} + \overline{X}_{\mu}^{\alpha} \frac{\partial}{\partial \varphi_{\mu}^{\alpha}} + \overline{X}_{k_{\epsilon,\sigma}^{\mu}} \frac{\partial}{\partial k_{\epsilon,\sigma}^{\mu}} \ , \tag{50}$$

where

$$\overline{X}_{\mu}^{\alpha} \equiv \varphi_{\nu}^{\alpha} \partial_{\mu} f^{\nu}(x) \tag{51}$$

$$\overline{X}_{k_{\epsilon,\sigma}^{\mu}} \equiv \frac{\partial X_{k_{\epsilon}^{\mu}}}{\partial x^{\sigma}} + \frac{\partial X_{k_{\epsilon}^{\mu}}}{\partial k_{\nu}^{\rho}} k_{\nu,\sigma}^{\rho} - \partial_{\sigma} X^{\rho} k_{\epsilon,\rho}^{\mu} = k_{\epsilon}^{\nu} \partial_{\sigma} \partial_{\nu} f^{\mu}(x) + k_{\epsilon,\sigma}^{\nu} \partial_{\nu} f^{\mu} - k_{\epsilon,\rho}^{\mu} \partial_{\sigma} f^{\rho}(x). \tag{52}$$

Note that the expression of $\overline{X}_{k_{\nu,\sigma}^{\mu}}$ arises from the general expresion of $\overline{X}_{\mu}^{\alpha}$.

Theorem 2.A (The Minimal Coupling): The new Lagrangian describing the matter fields as well as their interaction with the compensating fields k_{μ}^{ν}

$$\widehat{L}_{matt}(\varphi^{\alpha}, \varphi_{\mu}^{\alpha}, k_{\mu}^{\nu}) \equiv \mathcal{L}_{matt}(\varphi^{\alpha}, k_{\mu}^{\nu} \varphi_{\nu}^{\alpha}) \tag{53}$$

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is invariant under the gauge group in the sense that

$$\overline{\mathcal{X}}^{gauge}\widehat{L}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, k^{\nu}_{\mu}) = 0 \quad . \tag{54}$$

Theorem 2.B (The matter action): The new action describing the matter fields as well as their interaction with the compensating fields k^{ν}_{μ}

$$\widehat{S}_{matt} = \int \Lambda \widehat{L}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, k^{\nu}_{\mu}) d^4 x \tag{55}$$

is invariant under the local $T(4)$, where

$$\Lambda \equiv det(q^{\nu}_{\mu}) \quad . \tag{56}$$

Theorem 3.A (The Lagrangian $\mathcal{L}_0$ for the compensating fields): The necessary condition for $\mathcal{L}_0$ to be invariant under the local $T(4)$ is that $\mathcal{L}_0$ depends on the fields k^{ν}_{μ} and their "derivatives" $k^{\nu}_{\mu,\sigma}$ only through the specific combination:

$$T^{\sigma}_{\mu\nu} = q^{\sigma}_{\rho}(k^{\rho}_{\nu,\theta}k^{\theta}_{\mu} - k^{\rho}_{\mu,\theta}k^{\theta}_{\nu}) \ . \tag{57}$$

Theorem 3.B (The action for the compensating fields): The new action describing the compensating fields k^{ν}_{μ}

$$\widehat{S}_0 = \int \widehat{\mathcal{L}}_0(k^{\nu}_{\mu}, k^{\nu}_{\mu,\sigma}) \equiv \int \Lambda \mathcal{L}_0(T^{\nu}_{\mu\sigma}) d^4 x \tag{58}$$

is invariant under the local $T(4)$.

Conclusion:

Starting with an invariant action under the rigid $T(4)$

$$S = \int \mathcal{L}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}) d^4 x \tag{59}$$

we have proved that the new action

$$\widehat{S} = \int \widehat{\mathcal{L}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, k^{\mu}_{\nu}, k^{\mu}_{\nu,\sigma}) d^4 x \tag{60}$$

is invariant under the local $T(4)$, where

$$\widehat{\mathcal{L}}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, k^{\mu}_{\nu}, k^{\mu}_{\nu,\sigma}) = \widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, k^{\mu}_{\nu}) + \widehat{\mathcal{L}}_0(k^{\mu}_{\nu}, k^{\mu}_{\nu,\sigma}) \tag{61}$$

with

$$\widehat{\mathcal{L}}_{matt}(\varphi^{\alpha}, \varphi^{\alpha}_{\mu}, k^{\nu}_{\mu}) \equiv \Lambda \mathcal{L}_{matt}(\varphi^{\alpha}, k^{\mu}_{\sigma}\varphi^{\alpha}_{\mu}) \ , \tag{62}$$

$$\widehat{\mathcal{L}}_0(k^{\mu}_{\nu}, k^{\mu}_{\nu,\sigma}) \equiv \Lambda \mathcal{L}_0(T^{\mu}_{\nu\sigma}) \ , \tag{63}$$

$$\Lambda = det(q_\mu^\nu) \ . \tag{64}$$

Remark: We can introduce in the theory gauge fields $\mathcal{A}_\nu^{(\mu)}$ associated with the action of the translations generators on the fibre. Nevertheless the matter fields components are inaltered under $T(4)$ and we can impose the constraint $k_\nu^\mu = \delta_\nu^\mu + \mathcal{A}_\nu^\mu$, so that the total number of degrees of freedom does not change.
Geometrical interpretation:
Using the compensating fields k_μ^ν we can construct the metric tensor

$$g_{\mu\nu} \equiv q_\mu^\rho q_\nu^\sigma \eta_{\rho\sigma} \tag{65}$$

with Levi-Civita connection

$$\Gamma_{\mu\nu}^{\sigma(L-C)} \equiv \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{66}$$

and the curvature tensor of this connection

$$R_{\mu\rho\nu}^{\sigma(L-C)} \equiv \partial_\rho \Gamma_{\mu\nu}^{\sigma(L-C)} - \partial_\nu \Gamma_{\mu\rho}^{\sigma(L-C)} + \Gamma_{\epsilon\rho}^{\sigma(L-C)}\Gamma_{\mu\nu}^{\epsilon(L-C)} - \Gamma_{\epsilon\mu}^{\sigma(L-C)}\Gamma_{\mu\rho}^{\epsilon(L-C)} \ . \tag{67}$$

We can decompose the Levi-Civita connection in two parts:

$$\Gamma_{\mu\nu}^{\sigma(L-C)} = \Gamma_{\mu\nu}^\sigma - K_{\mu\nu}^\sigma \ , \tag{68}$$

where

$$\Gamma_{\mu\nu}^\sigma \equiv k_\rho^\sigma q_{\mu,\nu}^\rho = -q_\mu^\rho k_{\rho,\nu}^\sigma \tag{69}$$

is not a Levi-Civita connection but a Cartan connection. The other term is known as *contortion* and is given by

$$K_{\mu\nu}^\sigma \equiv \frac{1}{2}(\mathcal{T}_\mu{}^\sigma{}_\nu + \mathcal{T}_\nu{}^\sigma{}_\mu - \mathcal{T}_{\mu\nu}^\sigma) \tag{70}$$

with

$$\mathcal{T}_{\mu\nu}^\sigma = k_\rho^\sigma q_\mu^\epsilon q_\nu^\xi T_{\epsilon\xi}^\rho = \Gamma_{\mu\nu}^\sigma - \Gamma_{\nu\mu}^\sigma. \tag{71}$$

$\mathcal{T}_{\mu\nu}^\sigma$ can be interpreted as the torsion of the connection $\Gamma_{\mu\nu}^\sigma$.
Now we can see that the Hilbert-Einstein Lagrangian density, which depends on the scalar curvature of the Levi-Civita connection, is equivalent to a certain choice of Lagrangian $\mathcal{L}_0^{T(4)}$ of the compensating fields k_μ^ν in the gauge theory of the translation group:

$$\mathcal{L}_{Hilbert-Einstein} \equiv \Lambda R^{(L-C)} \equiv \Lambda g^{\rho\nu} R_{\rho\mu\nu}^{\mu(L-C)} = \Lambda \mathcal{L}_0^{T(4)} + divergence \ , \tag{72}$$

where

$$\mathcal{L}_0^{T(4)} = T_{\nu\sigma}^\mu T_{\epsilon\theta}^\rho \left(\frac{1}{4}\eta^{\epsilon\nu}\eta^{\sigma\theta}\eta_{\mu\rho} + \frac{1}{2}\eta_\mu^\theta \eta^{\nu\epsilon}\eta_\rho^\sigma - \eta_\mu^\sigma \eta_\rho^\theta \eta^{\nu\epsilon} \right) \ , \tag{73}$$

which turns out to be the Lagrangian of the teleparallel description of gravity. Therefore we finally regain Einstein equations:

$$R_{\mu\nu}^{(L-C)} - \frac{1}{2}g_{\mu\nu}R^{(L-C)} = 0 \quad . \tag{74}$$

Now we would like to present the corresponding jet-diffeomorphism formalism. We are allowed to define the bundle of the 1-jets of the group of diffeomorphisms

$$J^1(\mathcal{D}iff(M)) \equiv \frac{\mathcal{D}iff(M) \times M}{\sim^1} \quad , \tag{75}$$

where the equivalence relation is defined as follows:

$$1) \quad m = m' \ ,$$

$$2) \quad \xi^\mu(m) = \xi'^\mu(m) \ ,$$

$$3) \quad \partial_\nu \xi^\mu(m) = \partial_\nu \xi'^\mu(m) \ ,$$

$\forall (\xi^\mu, m), (\xi'^\mu, m') \in \mathcal{D}iff(M) \times M.$

A coordinate system for $J^1(\mathcal{D}iff(M))$ is $\{\xi^\mu, \xi_\nu^\mu\}$. Note that, as usual in jet theory, ξ_ν^μ parametrize transformations in the tangent bundle of the base manifold M, $T(M)$, which are not the derivative of the diffeomorphism ξ^μ of M, except for jet extensions. The quantization of gravity requires the elimination of the diffeomorphims constraints, so this parametrization will be very relevant at quantizing because the jets ξ_ν^μ are not simple changes of coordinates, i.e., they are not "gauge" in the strict sense of leaving the solution manifold invariant pointwise. At this point we would like to remark that the structure just defined, $J^1(\mathcal{D}iff(M))$, can be identified with the frame bundle with the Minkowski space-time as base manifold. The general definition of the frame bundle with an arbitrary n-dimensional base manifold M is given by the following total bundle space

$$F(M) = \{(m, a_1, a_2, ..., a_n)/m \in M, (a_1, a_2, ..., a_n) \, basis \, of \, T_m(M)\} \ , \tag{76}$$

where $T_m(M)$ is the tangent space of M in the point m.

Note that the structure group of this bundle is the linear general group $GL(n)$. This fact is the reason to take into account the gauge theory of this group among the possible gauge theories for gravity, as can be seen in the literature.

It is also worth noting that given a group, the 1-jets of this group also has group structure in general.

In order to allow in the theory the presence of the derivatives of the previous coordinates $\{\xi^\mu, \xi_\nu^\mu\}$, we define the bundle of the 1-jets of the 1-jets of the group of diffeomorphisms

$$J^1(J^1(\mathcal{D}iff(M))),$$

with associated coordinate system

$$\{\xi^\mu, \xi_\nu^\mu; \xi_{,\sigma}^\mu, \xi_{\nu,\sigma}^\mu\}. \tag{77}$$

where in general $\xi_{,\sigma}^\mu \neq \partial_\sigma \xi^\mu$ and $\xi_{\nu,\sigma}^\mu \neq \partial_\sigma \xi_\nu^\mu$. In this framework the compensating fields k_ν^μ, which were introduced in the gauge theory of the space-time translation group in the previous section, may be now viewed as the jets of the diffeomorphisms of Minkowski space-time ξ_ν^μ:

$$\xi_\nu^\mu \equiv k_\nu^\mu.$$

Then the Minkowski metric $\eta_{\mu\nu}$ can be transformed with $\xi_\nu^\mu \equiv k_\nu^\mu$ and a metric $g_{\mu\nu}$ can be defined and thanks to the fact that the gauge theory of the translation group yields Einstein's theory, the same result follows in this construction.

5 Mixing of Electromagnetism and Gravity

The simplest procedure to approach the non-trivial mixing between electromagnetism and gravity is to consider the central extension of Poincaré group as a rigid symmetry, $\mathcal{P}$, by the group $U(1)$:

$$G = \mathcal{P} \widetilde{\otimes} U(1) \equiv \widetilde{\mathcal{P}} \,. \tag{78}$$

Then the commutator of boosts and translations is modified in such a form that

$$[K^i, P_j] = \delta^i_j \left(\frac{1}{c^2} P_0 + \Xi \right) , \tag{79}$$

where Ξ is the generator of $U(1)$. The group index (a) now runs over $\{(\mu)$ translations, $(\nu\sigma)$ Lorentz, $(elec)$ $U(1)\}$. As a report on central extensions of Lie groups with trivial cohomology, i.e. pseudo-extensions, and the role that they play in representation theory, we refer the reader to Ref. [50] and references there in.

According to [40] the Lagrangian for the free compensating fields should be a general function of the form

$$\mathcal{L}_0 = \mathcal{L}_0(\mathcal{F}^{(\sigma)}_{\mu\nu}, \mathcal{F}^{(\sigma\rho)}_{\mu\nu}, \mathcal{F}^{(elec)}_{\mu\nu}) , \tag{80}$$

where the presence of the mixing constant

$$C^{elec}_{\mu,\sigma\rho} \equiv -\kappa(\eta_{\rho\mu}\delta^0_\sigma - \eta_{\sigma\mu}\delta^0_\rho) \tag{81}$$

in the electromagnetic field strength, due to the central pseudo-extension, is to be remarked, κ being the coupling constant of the mixing.

Let us suppose that the Lagrangian for the free compensating fields has the form:

$$\mathcal{L}_0 = \frac{1}{2}\Lambda(\mathcal{F}^{(ele)}_{\mu\nu}\mathcal{F}^{(ele)\mu\nu} + \mathcal{F}^{(\mu\nu)}_{\mu\nu}) , \tag{82}$$

where $\mathcal{F}^{(ele)\mu\nu} \equiv \mathcal{F}^{(ele)}_{\sigma\rho}\eta^{\sigma\mu}\eta^{\rho\nu}$, and write the curvatures of the compensating fields in the following way:

$$\mathcal{F}^{(a)}_{\mu\nu} \equiv k^\sigma_\mu k^\rho_\nu F^{(a)}_{\sigma\rho} \,,$$

where

$$F^{(a)}_{\sigma\rho} \equiv A^{(a)}_{\sigma,\rho} - A^{(a)}_{\rho,\sigma} + \frac{1}{2}\widetilde{C^a_{bc}}(A^{(b)}_\sigma A^{(c)}_\rho - A^{(b)}_\rho A^{(c)}_\sigma) \,,$$

$$A^{(a)}_\mu \equiv q^\nu_\mu \mathcal{A}^{(a)}_\nu \,.$$

Here, (a) runs over the entire group $\widetilde{\mathcal{P}}$ and $\widetilde{C^a_{bc}}$ denotes its structure constants including the mixing constant (81). Explicitly:

$$
\begin{aligned}
F^{(\epsilon\rho)}_{\mu\nu} &= A^{(\epsilon\rho)}_{\mu,\nu} - A^{(\epsilon\rho)}_{\nu,\mu} - \eta_{\theta\sigma}(A^{(\epsilon\theta)}_\mu A^{(\sigma\rho)}_\nu - A^{(\epsilon\theta)}_\nu A^{(\sigma\rho)}_\mu) \,, \\
F^{(ele)}_{\mu\nu} &= A^{(ele)}_{\mu,\nu} - A^{(\epsilon\gamma)}_{\nu,\mu} - \frac{1}{2}C^{ele}_{\epsilon,\theta\rho}(A^{(\epsilon)}_\mu A^{(\theta\rho)}_\nu - A^{(\epsilon)}_\nu A^{(\theta\rho)}_\mu) \,,
\end{aligned}
$$

with $A^{(\epsilon\rho)}_\mu \equiv q^\nu_\mu \mathcal{A}^{(\epsilon\rho)}_\nu$ and $A^{(\epsilon)}_\theta \equiv q^\nu_\theta \mathcal{A}^{(\epsilon)}_\nu = q^\nu_\theta(k^\epsilon_\nu - \delta^\epsilon_\nu) = \delta^\epsilon_\theta - q^\epsilon_\theta$.

We have not considered $F^{(\sigma)}_{\mu\nu}$, corresponding to the subgroup of translations, because the Lorentz curvature is enough to describe pure gravity in vacuum.

The Euler-Lagrange motion equations read:

$$(1): \quad \frac{\partial \mathcal{L}_0}{\partial A^{(\nu\rho)}_\mu} - \frac{d}{dx^\sigma}\left(\frac{\partial \mathcal{L}_0}{\partial A^{(\nu\rho)}_{\mu,\sigma}} \right) = 0 \quad \Rightarrow \tag{83}$$

$$C^{ele}_{\sigma,\epsilon\theta} A^{(\sigma)}_{\nu} F^{(ele)\mu\nu} + k^{\mu}_{\rho} T^{\rho}_{\epsilon\theta} - k^{\mu}_{\theta} T^{\rho}_{\epsilon\rho} + k^{\mu}_{\epsilon} T^{\rho}_{\theta\rho} + (k^{\mu}_{\rho} k^{\nu}_{\theta} - k^{\mu}_{\theta} k^{\nu}_{\rho}) A^{(\rho}_{\epsilon)\nu} - (k^{\mu}_{\epsilon} k^{\nu}_{\rho} + k^{\mu}_{\rho} k^{\nu}_{\epsilon}) A^{(\rho}_{\theta)\nu} = 0 \,,$$

where

$$F^{(ele)\mu\nu} = F^{(ele)}_{\rho\lambda} g^{\rho\mu} g^{\lambda\nu} \,,$$

$$g^{\rho\mu} = k^{\rho}_{\sigma} k^{\mu}_{\theta} \eta^{\sigma\theta} \,,$$

$$A^{(\mu}_{\nu)\sigma} \equiv \eta_{\nu\rho} A^{(\mu\rho)}_{\sigma} \,,$$

$$T^{\rho}_{\epsilon\theta} = q^{\rho}_{\mu} (k^{\mu}_{\epsilon,\tau} k^{\tau}_{\theta} - k^{\mu}_{\theta,\tau} k^{\tau}_{\epsilon}) \,;$$

$$(2): \quad \frac{\partial \mathcal{L}_0}{\partial A^{(ele)}_{\mu}} - \frac{d}{dx^{\sigma}} \left(\frac{\partial \mathcal{L}_0}{\partial A^{(ele)}_{\mu,\sigma}} \right) = 0 \quad \Rightarrow \tag{84}$$

$$\frac{d}{dx^{\sigma}} (\Lambda F^{(ele)\mu\sigma}) = 0 \,;$$

$$(3): \quad \frac{\partial \mathcal{L}_0}{\partial k^{\mu}_{\nu}} - \frac{d}{dx^{\sigma}} \left(\frac{\partial \mathcal{L}_0}{\partial k^{\mu}_{\nu,\sigma}} \right) = 0 \quad \Rightarrow \tag{85}$$

$$F^{(\nu\sigma)}_{\mu\sigma} - \frac{1}{2} \delta^{\nu}_{\mu} F^{(\sigma\lambda)}_{\sigma\lambda} = \mathcal{T}^{\nu}_{\mu} \,,$$

where

$$\mathcal{T}^{\nu}_{\mu} \equiv \mathcal{T}^{\nu(mix)}_{\mu} + \mathcal{T}^{\nu(ele)}_{\mu} \,.$$

The tensor

$$\mathcal{T}^{\nu(ele)}_{\mu} \equiv -F^{(ele)\nu}_{\sigma} F^{(ele)\sigma}_{\mu} + \frac{1}{2} \delta^{\nu}_{\mu} F^{(ele)}_{\sigma\lambda} F^{(ele)\sigma\lambda}$$

defines the energy-momentum tensor corresponding to the electromagnetic field (with $F^{(ele)\nu}_{\sigma} = g^{\lambda\nu} F^{(ele)}_{\sigma\lambda}$), and

$$\mathcal{T}^{\nu(mix)}_{\mu} = \frac{1}{2} \delta^{\sigma}_{\mu} C^{ele}_{\sigma,\theta\epsilon} q^{\nu}_{\rho} F^{(ele)\rho\tau} A^{(\theta\epsilon)}_{\tau}$$

is a direct consequence of the mixing, and it is quite particular, since in the absence of matter and considering the mixture between electromagnetism and gravity, we would have a non-trivial energy-momentum tensor.

Equation (83) gives the form of the Lorentz potentials in terms of k^{ν}_{μ}, q^{σ}_{ρ} and their derivatives. Note that for the gauge theory of Poincaré group in vacuum the Euler-Lagrange equation for the Lorentz potentials reads:

$$k^{\mu}_{\rho} T^{\rho}_{\epsilon\theta} - k^{\mu}_{\theta} T^{\rho}_{\epsilon\rho} + k^{\mu}_{\epsilon} T^{\rho}_{\theta\rho} + (k^{\mu}_{\rho} k^{\nu}_{\theta} - k^{\mu}_{\theta} k^{\nu}_{\rho}) A^{(\rho}_{\epsilon)\nu} - (k^{\mu}_{\epsilon} k^{\nu}_{\rho} + k^{\mu}_{\rho} k^{\nu}_{\epsilon}) A^{(\rho}_{\theta)\nu} = 0 \,,$$

whose solution is given in equation (107). In [13] the Lorentz potentials, in the case in which the existence of matter is also considered, are explicitly written (under some assumptions) in terms of the tetrad fields and their derivatives, and using similar steps one could obtain the explicit form of $A^{(\mu\nu)}_{\sigma}$ to the first order in κ, but for the moment we shall not use this explicit result.

Equation (84) can be rewritten in the form of Maxwell equations in a gravitational field, but containing an electromagnetic current $J^{\mu(mix)}$ made of gravitational potentials:

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$$\partial_\sigma[\Lambda g^{\nu\mu} g^{\epsilon\sigma}(A^{(ele)}_{\nu,\epsilon} - A^{(ele)}_{\epsilon,\nu})] = J^{\mu(mix)} \, ,$$

with

$$J^{\mu(mix)} \equiv \frac{1}{2} C^{ele}_{\rho,\theta\xi} \partial_\sigma[\Lambda g^{\nu\mu} g^{\epsilon\sigma}(A^{(\rho)}_\nu A^{(\theta\xi)}_\epsilon - A^{(\rho)}_\epsilon A^{(\theta\xi)}_\nu)] \, .$$

Note that if we do not consider the mixing between electromagnetism and gravity, the previous motion equations reduce to those of the case of gauging Poincaré group in vacuum.

In order to study the solution of the equations with mixture a good aproximation will be working at first order in the mixing coupling constant κ since it is expected that its value is about 10^{-8} times the electron mass [41].

The form of these equations is rather complicated, so that we shall restrict our attention here to the equation for the electromagnetic field:

$$\partial_\sigma(\Lambda F^{(ele)\mu\sigma}) = 0 \, .$$

Let us write the electromagnetic field of the theory with mixture to the first order in κ:

$$A^{(ele)}_\mu = A^{(ele)0}_\mu + \kappa A^{(ele)1}_\mu \, ,$$

and, for simplicity, the rest of the compensating fields will be considered to zero order in κ.

Order by order in κ, the electromagnetic motion equation leads to:

Zero order:

$$\partial_\sigma[\Lambda g^{\mu\epsilon} g^{\sigma\theta}(A^{(ele)0}_{\epsilon,\theta} - A^{(ele)0}_{\theta,\epsilon})] = 0 \, , \tag{86}$$

which shows that $A^{(ele)0}_\mu$ is the usual electromagnetic potential in the presence of a gravitational field, without mixing.

First order:

$$\partial_\sigma\{\Lambda g^{\mu\epsilon} g^{\sigma\theta}[A^{(ele)1}_{\epsilon,\theta} - A^{(ele)1}_{\theta,\epsilon} - \frac{1}{2} M^{ele}_{\rho,\nu\xi}(A^{(\rho)}_\epsilon A^{(\nu\xi)}_\theta - A^\rho_\theta A^{(\nu\xi)}_\epsilon)]\} = 0 \, , \tag{87}$$

$$M^{ele}_{\rho,\nu\xi} \equiv -\frac{1}{\kappa} C^{ele}_{\rho,\nu\xi} \, .$$

Here the Lorentz potentials $A^{(\mu\nu)}_\sigma$ coincide with the Lorentz potentials of the case of gauging the Poincaré group in vacuum, $\mathcal{A}^{Poinc.}_{(\sigma\rho)\mu} = \frac{1}{2} T_{\mu\sigma\rho} + \frac{1}{2}(T_{\sigma\rho\mu} - T_{\rho\sigma\mu})$, since we are working to the first order in κ.

Note that a simple "mathematical" solution for $A^{(ele)1}_{\mu,\nu}$ would be

$$A^{(ele)1}_{\mu,\nu} = \frac{1}{2} M^{ele}_{\rho,\epsilon\theta} A^{(\rho)}_\mu A^{(\epsilon\theta)}_\nu = \eta_{\sigma\rho} A^{(\rho)}_\mu A^{(0\sigma)}_\nu = \eta_{ij} A^{(j)}_\mu A^{(0i)}_\nu \, , \tag{88}$$

where the last equality follows from the antisymmetry of the Lorentz potentials $A^{(\mu\nu)}_\sigma = -A^{(\nu\mu)}_\sigma$, and the latin index runs from 1 to 3. Just having a look at this result, we conclude that the electromagnetic field of the mixture theory between electromagnetism and gravity in the context of gauging the central (pseudo-)extension of Poincaré group by $U(1)$, contains an additional contribution made of pure gravitational potentials and this could be the origin of some electromagnetic force associated with (very massive) rotating systems, as $A^{(0i)}_\nu$ is somehow associated with "Coriolis-like forces".

5.1 Particle in the presence of the mixed electromagnetic and gravitational fields

Now we would like to consider some situations of physical interest in order to proceed further in the understanding and consequences of the mixing between electromagnetism and gravity. A natural example is the "geodesic" motion.

We shall apply the Minimal Coupling Prescription (developed in the previous sections) to the Lagrangian of a particle of mass m, momentum $p_\mu(= m u_\mu = m\frac{dx_\mu}{d\tau})$ and charge e:

$$\mathcal{L}_{matt} = \frac{1}{2m} p_\mu p_\nu \eta^{\mu\nu}. \tag{89}$$

Now we have to replace p_μ by $k_\mu^\nu(p_\mu - eA_\mu^{(ele)})$. Note that we are assuming that the particle does not have spin, that is why the term associated with the Lorentz subgroup does not contribute to the "covariant derivative". The modified Lagrangian reads:

$$\widehat{\mathcal{L}}_{matt} = \frac{1}{2m} k_\nu^\mu(p_\mu - eA_\mu^{(ele)}) k_\sigma^\rho(p_\rho - eA_\rho^{(ele)}) \eta^{\nu\sigma} = \frac{1}{2m} g^{\mu\nu}(p_\mu - eA_\mu^{(ele)})(p_\nu - eA_\nu^{(ele)}) =$$

$$\frac{m}{2} u^\mu u^\nu g_{\mu\nu} - e u^\mu A^{(ele)\nu} g_{\mu\nu} + \frac{e^2}{2m} A^{(ele)\mu} A^{(ele)\nu} g_{\mu\nu} \, ,$$

although in what follows we shall neglect the misleading term $\frac{e^2}{2m} A^{(ele)\mu} A^{(ele)\nu} g_{\mu\nu}$, which, by the way, does not appear when working directly with the Poincaré-Cartan form instead of with the Lagrangian.

We also consider the Lagrangian for the free compensating fields $\mathcal{L}_0(\mathcal{F}_{\mu\nu}^{(\sigma)}, \mathcal{F}_{\mu\nu}^{(\epsilon\theta)}, \mathcal{F}_{\mu\nu}^{(ele)})$ and the action $S = \int \mathcal{L}_{Tot} d^4x$ with $\mathcal{L}_{Tot} \equiv \widehat{L}_{matt} + \Lambda \mathcal{L}_0(\mathcal{F}_{\mu\nu}^{(\sigma)}, \mathcal{F}_{\mu\nu}^{(\epsilon\theta)}, \mathcal{F}_{\mu\nu}^{(ele)})$, invariant under the local group $\widetilde{\mathcal{P}}(M)$.

As regards the interaction between a particle and a field, in general, it is required to distinguish between the coordinates y^σ where the fields are evaluated and the coordinates x^σ for the particle. In $\widehat{\mathcal{L}}_{matt}$ the fields are evaluated at the position of the particle, where the interaction occurs, but in $\mathcal{L}_0$ the fields are evaluated at y^σ.

Let us write the Euler-Lagrange motion equations:

The equation for the particle,

$$\frac{\partial \mathcal{L}_{Tot}}{\partial x^\sigma} - \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}_{Tot}}{\partial u^\sigma}\right) = 0 \Longrightarrow \frac{\partial \widehat{\mathcal{L}}_{matt}}{\partial x^\sigma} - \frac{d}{d\tau}\left(\frac{\partial \widehat{\mathcal{L}}_{matt}}{\partial u^\sigma}\right) = 0 \, ,$$

results in

$$g_{\mu\sigma}\frac{du^\mu}{d\tau} = u^\mu u^\nu(-\partial_\nu g_{\mu\sigma} + \frac{1}{2}\partial_\sigma g_{\mu\nu}) - \frac{e}{m} u^\mu(\partial_\sigma A_\mu^{(ele)} - \partial_\mu A_\sigma^{(ele)}) \, . \tag{90}$$

Note that the Levi-Civita connection $\Gamma_{\mu\nu}^{(L-C)\epsilon}$ associated with the metric $g_{\mu\nu}$ is defined as usual, that is:

$$\Gamma_{\mu\nu}^{(L-C)\epsilon} = \frac{1}{2} g^{\epsilon\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \, ,$$

and

$$\Gamma_{\mu\nu,\epsilon}^{(L-C)} \equiv \Gamma_{\mu\nu}^{(L-C)\epsilon} g_{\epsilon\theta} = \frac{1}{2}(\partial_\mu g_{\nu\theta} + \partial_\nu g_{\mu\theta} - \partial_\theta g_{\mu\nu}) \, .$$

On the other hand,

$$u^\mu u^\nu(-\partial_\nu g_{\mu\sigma} + \frac{1}{2}\partial_\sigma g_{\mu\nu}) = -\frac{1}{2}u^\mu u^\nu(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma}) + \frac{1}{2}u^\mu u^\nu \partial_\sigma g_{\mu\nu} = -u^\mu u^\nu \Gamma_{\mu\nu,\sigma}^{(L-C)}.$$

Then, we arrive at:

$$g_{\mu\sigma}\frac{du^\mu}{d\tau} = -u^\mu u^\nu \Gamma_{\mu\nu,\sigma}^{(L-C)} - \frac{e}{m} u^\mu(\partial_\sigma A_\mu^{(ele)} - \partial_\mu A_\sigma^{(ele)}) \, . \tag{91}$$

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The last equation would be the usual motion equation for a particle in the presence of both gravitational and electromagnetic fields, although the difference manisfests itself in that the fields satisfy motion equations with mixing terms.

Decomposing the electromagnetic field to the first order in κ,

$$A_\mu^{(ele)} = A_\mu^{(ele)0} + \kappa A_\mu^{(ele)1} ,$$

the motion equation for the particle reads:

$$g_{\mu\sigma} \frac{du^\mu}{d\tau} = -u^\mu u^\nu \Gamma_{\mu\nu,\sigma}^{(L-C)} - \frac{e}{m} u^\mu (\partial_\sigma A_\mu^{(ele)0} - \partial_\mu A_\sigma^{(ele)0}) - \frac{\kappa e}{m} u^\mu (\partial_\sigma A_\mu^{(ele)1} - \partial_\mu A_\sigma^{(ele)1}) . \tag{92}$$

This equation is compatible with that obtained in Ref.[41] by group-theoretical methods, although, there, the results were given up to a lower order in the group law, making the exact identification of terms by no means direct. Putting the approximate solution (88) into (92), the general comments after (88) do apply here.

As far as the field equations are concerned, and assuming that the Lagrangian density for the free compensating fields has the form $\Lambda \mathcal{L}_0 \sim \Lambda(\mathcal{F}_{\mu\nu}^{(ele)} \mathcal{F}^{(ele)\mu\nu} + \mathcal{F}_{\mu\nu}^{(\mu\nu)})$, the motion equations for the fields are similar to those of the previous section except for additional terms due to the presence of the particle:

1) $C_{\sigma,\epsilon\theta}^{(ele)} A_\nu^{(\sigma)} F^{(ele)\mu\nu} + k_\rho^\mu T_{\epsilon\theta}^\rho - k_\theta^\mu T_{\epsilon\rho}^\rho + k_\epsilon^\mu T_{\theta\rho}^\rho + (k_\rho^\mu k_\theta^\nu - k_\theta^\mu k_\rho^\nu A_{\epsilon)\nu}^{(\rho} - (k_\epsilon^\mu k_\rho^\nu + k_\rho^\mu k_\epsilon^\nu) A_{\theta)\nu}^{(\rho} = 0 ,$

2) $\partial_\sigma(\Lambda F^{(ele)\mu\sigma}) = e u^\mu ,$

3) $F_{\mu\sigma}^{(\nu\sigma)} - \frac{1}{2} \delta_\mu^\nu F_{\sigma\lambda}^{(\sigma\lambda)} = \mathcal{T}_{\mu(Tot)}^\nu,$

with

$$\mathcal{T}_{\mu(Tot)}^\nu \equiv \mathcal{T}_\mu^{\nu(mix)} + \mathcal{T}_\mu^{\nu(ele)} + \mathcal{T}_\mu^{\nu(matt)}$$

defining the total energy-momentum tensor, where

$$\mathcal{T}_\mu^{\nu(mix)} = -k_\mu^\rho \frac{\partial \widehat{\mathcal{L}}_{matt}}{\partial k_\nu^\rho}$$

$$\mathcal{T}_\mu^{\nu(ele)} = -F_\sigma^{(ele)\nu} F_\mu^{(ele)\sigma} + \frac{1}{2} \delta_\mu^\nu F_{\sigma\lambda}^{(ele)} F^{(ele)\sigma\lambda}$$

$$\mathcal{T}_\mu^{\nu(matt)} = \frac{1}{2} \delta_\mu^\sigma C_{\sigma,\theta\epsilon}^{ele} q_\rho^\nu F^{(ele)\rho\tau} A_\tau^{(\theta\epsilon)}.$$

Similar comments to those made for equations (83), (84) and (85) also apply here.

Finally, we consider these equations to the first order in κ. In equation 1), the electromagnetic tensor is already accompanied by the constant $C_{\sigma,\epsilon\theta}^{(ele)}$, which is proportional to κ, so that we must replace $F^{(ele)\mu\nu}$ with the unmixed electromagnetic tensor in order to obtain an equation for the Lorentz potential to first order in κ.

With respect to equation 2) we have:

Zero order:

$$\partial_\sigma[\Lambda g^{\mu\epsilon} g^{\sigma\theta} (A_{\epsilon,\theta}^{(ele)0} - A_{\theta,\epsilon}^{(ele)0})] = e u^\mu . \tag{93}$$

$A_\mu^{(ele)0}$ is the usual electromagnetic field without mixing.

First order:

$$\partial_\sigma\{\Lambda g^{\mu\epsilon} g^{\sigma\theta} [A_{\epsilon,\theta}^{(ele)1} - A_{\theta,\epsilon}^{(ele)1} - \frac{1}{2} M_{\rho,\nu\xi}^{ele} (A_\epsilon^{(\rho)} A_\theta^{(\nu\xi)} - A_\theta^{(\rho)} A_\epsilon^{(\nu\xi)})]\} = 0. \tag{94}$$

As in equation (87), up to first order in κ, the Lorentz potentials coincide with the Lorentz potentials of the gauge theory of Poincaré group in vacuum.

6   Outlook

The possibility of formulating the Lagrangian version of the already proposed revision of the Higgs-Kibble mechanism, in a group approach to the quantization of Yang-Mills theories [45], to provide mass to the massive vector bosons, using only the group parameters without extra fields, is also expected to be carried out in the near future.

Related to the unification between gravity and the internal gauge interactions it is also known that a semidirect action of the group of diffeomorphisms on the gauge group

$$\mathcal{D}iff(M) \otimes^s G(M)$$

exists and that given a gauge group $G(M)$ it is possible to find a formal group composition law for $G(M)$. Therefore we can also find a group composition law for that semidirect product of groups, what implies a non-trivial mixture between gravity and the internal interactions. This new mixing just proposed here is quite different from the one described in this work and deserves further study.

Appendix. Lagrangian Formalism on $J^1(J^1(E))$

The space $J^1(J^1(E))$ is the bundle of the 1-jets associated with the fibre space $(J^1(E), M, \pi^1)$. The coordinate system for $J^1(J^1(E))$ will be defined by

$$(x^\mu, y^\alpha, y^\alpha_\mu; z^\alpha_{,\nu}, z^\alpha_{\mu,\nu}) \ . \tag{95}$$

A Lagrangian density of type $\mathcal{L}^{1,1}$ is an application of $J^1(J^1(E))$ on R.

The 1-jet prolongation of φ^1, $j^1(\varphi^1) \equiv \bar{\varphi}^1$ is the only section of $(J^1(J^1(E)), M, \pi)$ such that j^1 is an injection of $\Gamma(J^1(E))$ into $\Gamma(J^1(J^1(E)))$ and

$$\theta^{1,1\alpha}|_{\bar{\varphi}^1} = 0 \tag{96}$$

$$\theta^{1,1\alpha}_\mu|_{\bar{\varphi}^1} = 0 \ , \tag{97}$$

where the structure forms $\theta^{1,1}$ are now

$$\theta^{1,1\alpha} = dy^\alpha - z^\alpha_{,\nu}dx^\nu \tag{98}$$

$$\theta^{1,1\alpha}_\mu = dy^\alpha_\mu - z^\alpha_{\mu,\nu}dx^\nu \ . \tag{99}$$

The jet extension (prolongation) condition requires

$$z^\alpha_{,\nu} = \partial_\nu y^\alpha \tag{100}$$

$$z^\alpha_{\mu,\nu} = \partial_\nu y^\alpha_\mu \ . \tag{101}$$

Given a vector field on $J^1(E)$, $X^1 \in \Gamma(T(J^1(E)))$, its 1-jet extension (or prolongation) through the injection j^1 is the only field $j^1(X^1) \equiv \bar{X}^1$ such that it is an infinitesimal contact transformation, i.e.,

$$L_{\bar{X}^1}\theta^{1,1\alpha} = A^{\alpha;}_{;\beta}\theta^{1,1\beta} + A^{\alpha;\nu}_{;\beta}\theta^{1,1\beta}_\nu \tag{102}$$

$$L_{\bar{X}^1}\theta^{1,1\alpha}_\mu = A^{\alpha;}_{\mu;\beta}\theta^{1,1\beta} + A^{\alpha;\nu}_{\mu;\beta}\theta^{1,1\beta}_\nu \ , \tag{103}$$

where the 1-jet extension of X^1 is determined from the general form of $X^{1,1}$,

$$X^{1,1} = X^\mu \frac{\partial}{\partial x^\mu} + X^\alpha \frac{\partial}{\partial y^\alpha} + X^\alpha_\mu \frac{\partial}{\partial y^\alpha_\mu} + X^\alpha_{,\mu} \frac{\partial}{\partial z^\alpha_{,\mu}} + X^\alpha_{\mu,\nu} \frac{\partial}{\partial z^\alpha_{\mu,\nu}} \ . \tag{104}$$

Hence

$$A^{\alpha;}_{;\beta} = \frac{\partial X^\alpha}{\partial y^\beta} - z^\alpha_{,\sigma} \frac{\partial X^\sigma}{\partial y^\beta} \tag{105}$$

$$A^{\alpha;}_{\mu;\beta} = \frac{\partial X^\alpha_\mu}{\partial y^\beta} - z^\alpha_{\mu,\sigma} \frac{\partial X^\sigma}{\partial y^\beta} \tag{106}$$

$$A^{\alpha;\nu}_{;\beta} = \frac{\partial X^\alpha}{\partial y^\beta_\nu} - z^\alpha_{,\sigma} \frac{\partial X^\sigma}{\partial y^\beta_\nu} \tag{107}$$

$$A^{\alpha;\nu}_{\mu;\beta} = \frac{\partial X^\alpha_\mu}{\partial y^\beta_\nu} - z^\alpha_{\mu,\sigma} \frac{\partial X^\sigma}{\partial y^\beta_\nu} \tag{108}$$

and the components $\bar{X}^\alpha_{,\mu}$, $\bar{X}^\alpha_{\nu,\mu}$ read:

$$\bar{X}^\alpha_{,\mu} = \frac{\partial X^\alpha}{\partial x^\mu} - z^\alpha_{,\sigma} \frac{\partial X^\sigma}{\partial x^\mu} + A^{\alpha;}_{;\beta} z^\beta_{,\mu} + A^{\alpha;\sigma}_{;\beta} z^\beta_{\sigma,\mu} \tag{109}$$

$$\bar{X}^\alpha_{\nu,\mu} = \frac{\partial X^\alpha_\nu}{\partial x^\mu} - z^\alpha_{\nu,\sigma} \frac{\partial X^\sigma}{\partial x^\mu} + A^{\alpha;}_{\nu;\beta} z^\beta_{,\mu} + A^{\alpha;\sigma}_{\nu;\beta} z^\beta_{\sigma,\mu} \ . \tag{110}$$

Given a Lagrangian density $\mathcal{L}^{1,1}$, the Hamilton functional $\mathcal{S}^{1,1}$ is the application of $\Gamma(J^1(E))$ on R defined by

$$\mathcal{S}^{1,1}(\varphi^1) = \int_{j^1(\varphi)(M)} \mathcal{L}^{1,1}(j^1(\varphi^1))\omega \tag{111}$$

$\forall \varphi^1 \in \Gamma(J^1(E))$.

The Hamilton Principle states that the trajectories of the variational problem are the solutions of

$$\int_{j^1(\varphi^1)(M)} L_{\bar{X}^1}(\mathcal{L}^{1,1}(j^1(\varphi^1))\omega)) = 0 \tag{112}$$

$\forall X^1 \in \Gamma(T(J^1(E)))$.

This principle leads to a set of Euler-Lagrange equations of the form

$$\frac{\partial \mathcal{L}^{1,1}}{\partial y^\alpha} - \frac{d}{dx^\mu}\left(\frac{\partial \mathcal{L}^{1,1}}{\partial z^\alpha_{,\mu}}\right) = 0 \tag{113}$$

$$\frac{\partial \mathcal{L}^{1,1}}{\partial y^\alpha_\nu} - \frac{d}{dx^\mu}\left(\frac{\partial \mathcal{L}^{1,1}}{\partial z^\alpha_{\nu,\mu}}\right) = 0 \ . \tag{114}$$

Given a $\mathcal{L}^{1,1}$, the Poincaré -Cartan form is given by

$$\Theta^{1,1}_{PC} = \mathcal{L}^{1,1}\omega + \theta^{1,1\alpha} \wedge \Omega_\alpha + \theta^{1,1\alpha}_\mu \wedge \Omega^\mu_\alpha \ , \tag{115}$$

where

$$\Omega_\alpha \equiv \frac{\partial \mathcal{L}^{1,1}}{\partial z^\alpha_{,\nu}} \theta_\nu \tag{116}$$

$$\Omega^\mu_\alpha \equiv \frac{\partial \mathcal{L}^{1,1}}{\partial z^\alpha_{\mu,\nu}} \theta_\nu \tag{117}$$

$$\theta_\mu \equiv (-)^\mu dx^0 \wedge ... \wedge d\hat{x}^\mu \wedge ... \wedge dx^3 \ . \tag{118}$$

Given $\Theta_{PC}^{1,1}$ on $J^1(J^1(E))$ the modified Hamilton functional $\mathcal{S}'^{1,1}$ is the application of $\Gamma(J^1(J^1(E)))$ on R defined by

$$\mathcal{S}'^{1,1}(\varphi^{1,1}) \equiv \int_{\varphi^{1,1}(M)} \Theta_{PC}^{1,1} \tag{119}$$

$\forall \varphi^{1,1} \in \Gamma(J^1(J^1(E)))$
    The Modified Hamilton Principle states that a cross section is critical if and only if

$$(\delta \mathcal{S}'^{1,1})_{\varphi^{1,1}}(X^{1,1}) = \int_{\varphi^{1,1}(M)} L_{X^{1,1}}\Theta_{PC}^{1,1} = 0 \tag{120}$$

$\forall X^{1,1} \in \Gamma(T(J^1(J^1(E))))$
    The critical sections are thus the solutions of

$$i_{X^{1,1}} d\Theta_{PC}^{1,1}|_{\varphi^{1,1}} = 0 \tag{121}$$

from where the system of equations which come from the coefficients of $X_{\sigma,\nu}^{\beta}$ and $X_{,\nu}^{\beta}$ read respectively:

$$\frac{\partial}{\partial z_{\sigma,\nu}^{\beta}} \frac{\partial \mathcal{L}^{1,1}}{\partial z_{,\mu}^{\alpha}} \left(\frac{\partial y^{\alpha}}{\partial x^{\mu}} - z_{\mu}^{\alpha} \right) + \frac{\partial}{\partial z_{\sigma,\nu}^{\beta}} \frac{\partial \mathcal{L}^{1,1}}{\partial z_{\rho,\mu}^{\alpha}} \left(\frac{\partial y_{\rho}^{\alpha}}{\partial x^{\mu}} - z_{\rho,\mu}^{\alpha} \right) = 0 \tag{122}$$

$$\frac{\partial}{\partial z_{,\nu}^{\beta}} \frac{\partial \mathcal{L}^{1,1}}{\partial z_{,\mu}^{\alpha}} \left(\frac{\partial y^{\alpha}}{\partial x^{\mu}} - z_{\mu}^{\alpha} \right) + \frac{\partial}{\partial z_{,\nu}^{\beta}} \frac{\partial \mathcal{L}^{1,1}}{\partial z_{\rho,\mu}^{\alpha}} \left(\frac{\partial y_{\rho}^{\alpha}}{\partial x^{\mu}} - z_{\rho,\mu}^{\alpha} \right) = 0 \ . \tag{123}$$

When this system admits the trivial solution (regularity condition for $\mathcal{L}^{1,1}$), the 1-jet prolongation condition is satisfied, and then the remaining equations (from the coefficients of X_{ν}^{α} and X^{α} respectively)

$$-\frac{d}{dx^{\mu}} \left(\frac{\partial \mathcal{L}^{1,1}}{\partial z_{\nu,\mu}^{\alpha}} \right) + \frac{\partial \mathcal{L}^{1,1}}{\partial y_{\nu}^{\alpha}} + \frac{\partial}{\partial y_{\nu}^{\alpha}} \left(\frac{\partial \mathcal{L}^{1,1}}{\partial z_{,\mu}^{\beta}} \right) \left(\frac{\partial y^{\beta}}{\partial x^{\mu}} - z_{,\mu}^{\beta} \right) + \frac{\partial}{\partial y_{\nu}^{\alpha}} \left(\frac{\partial \mathcal{L}^{1,1}}{\partial z_{\sigma,\mu}^{\beta}} \right) \left(\frac{\partial y_{\sigma}^{\beta}}{\partial x^{\mu}} - z_{\sigma,\mu}^{\beta} \right) = 0 \tag{124}$$

$$-\frac{d}{dx^{\mu}} \left(\frac{\partial \mathcal{L}^{1,1}}{\partial z_{,\mu}^{\alpha}} \right) + \frac{\partial \mathcal{L}^{1,1}}{\partial y^{\alpha}} + \frac{\partial}{\partial y^{\alpha}} \left(\frac{\partial \mathcal{L}^{1,1}}{\partial z_{,\mu}^{\beta}} \right) \left(\frac{\partial y^{\beta}}{\partial x^{\mu}} - z_{,\mu}^{\beta} \right) + \frac{\partial}{\partial y^{\alpha}} \left(\frac{\partial \mathcal{L}^{1,1}}{\partial z_{\sigma,\mu}^{\beta}} \right) \left(\frac{\partial y_{\sigma}^{\beta}}{\partial x^{\mu}} - z_{\sigma,\mu}^{\beta} \right) = 0 \tag{125}$$

reproduce the Euler-Lagrange equations of the Ordinary Hamilton Principle and thus both principles are equivalent.

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