# QUANTIZATION of PHYSICAL MODELS and NON-COMMUTATIVE GEOMETRY

**P.A. Saponov**
*Institute for High Energy Physics, 142200, Protvino, Russia*

## 1. Introduction

In my talk I would like to give a simple introduction into some ideas of the non-commutative (NC) geometry. At present the NC geometry is a fairly wide and quickly developing branch of mathematics which has close connections to physics.

Loosely speaking, the main subject of the NC geometry is a "manifold" whose "coordinates" cannot be presented by the ordinary numbers but belong to a non-commutative algebra. Actually, such like objects are quite usual for physics — they appear in the quantization of physical systems.

Indeed, let us recall the standard scheme of canonical quantization (leaving apart the numerous subtleties and problems connected with some concrete systems). We begin with a real manifold $M$ called a configuration space. In simple cases its (local) coordinates $\{q_a\}$ represent the position of our system in the space. Then we pass to the cotangent bundle $T^*M$ — a phase space of the system, the fibre coordinates $\{p_a\}$ being the corresponding momenta. We are interested in the algebra of smooth real functions $\mathcal{A} = C^\infty(T^*M)$ on the cotangent bundle which is a commutative algebra of dynamical variables. The time evolution is defined by a Hamiltonian and a Poisson structure on $T^*M$. Due to the Darboux theorem, the Poisson structure an be locally brought to the canonical form

$$\{q_a, p_b\} = \delta_{ab}, \quad \{q_a, q_b\} = \{p_a, p_b\} = 0. \tag{1}$$

The canonical quantization consists in passing from the commutative algebra $\mathcal{A}$ to some non-commutative algebra $\mathcal{A}_\hbar$ generated by the elements $\hat{q}_a$ and $\hat{p}_a$ with the commutation relations

$$[\hat{q}_a, \hat{p}_b] = i\hbar\,\delta_{ab}, \quad [\hat{q}_a, \hat{q}_b] = [\hat{p}_a, \hat{p}_b] = 0. \tag{2}$$

From the mathematical point of view, this procedure can be interpreted as a transition from the commutative geometry to the NC one. Indeed, a theorem of algebraic geometry [Sh] tells that an affine variety $M$ is completely defined by the algebra $\mathcal{A}$ of regular (polynomial) functions on $M$ (the coordinates of a point is a particular example of such functions). Therefore, the non-commutative algebra $\mathcal{A}_\hbar$ (being sought of as an algebra of functions) defines some non-commutative "manifold". The quotation marks stand here in order to stress that the non-commutative (or quantum) "manifold" cannot be presented as a set of points in a space — a visual image is lost here.

The canonical quantization scheme described above needs a significant modification if the phase space cannot be covered by a single coordinate chart. In this case the canonical coordinates (1) cannot be fixed globally on the phase space. The typical and important example of such a situation is a *constrained system*, that is a system whose motion is somehow restricted. A simple example is given by a particle moving on the surface of a sphere.

In the most cases, the phase space of a constrained system is a submanifold $M \subset T^*V$, $V$ being an $n$-dimensional linear space, defined by the set of relations

$$\Phi_s(q, p) = 0, \quad s = 1, 2, \ldots, r, \tag{3}$$

which are called *the constraints*. The main idea of the Dirac quantization scheme for the constrained systems consists in a modification of the Poisson structure on $T^*V$ in such a way that it would be equivalent to its restriction on the surface of constraints (3). It allows us to work with the initial $(q, p)$ coordinates and set the constraints equal to zero in the strong sense [Di]. When quantizing the system, one should impose the commutation relations on $\hat{q}$ and $\hat{p}$ in accordance with these modified (or Dirac) brackets.

However, the Dirac brackets are often nonlinear in coordinates and the problems of ordering appears. The problem of the ordering intertwines with the fact that the quantum algebra of observables $\mathcal{A}_\hbar$ is an *associative* algebra (not a Lie algebra). Therefore, we must define the *product* $\hat{F}_1 \cdot \hat{F}_2$ (not only the commutator) of any two elements $\hat{F}_1$ and $\hat{F}_2$. But the canonical commutation relations fix only an *antisymmetrical* part of the product and we have to use some additional physical reasons in order to restore the whole quantum algebra and choose the correct ordering in the brackets. In practice the product of quantum operators for the constrained system can be defined only as a (formal) series in the Planck constant $\hbar$.

In my talk I consider the quantization of the algebra of functions on a semisimple orbit of fhe coadjoint action of the general linear group $GL(n)$. For any connected Lie group $G$ such an orbit is a $G$-homogeneous symplectic submanifold [Ki] in the linear space $g^*$ dual to the Lie algebra $g$. Being symplectic manifold, the orbit can be treated as a phase space of a physical system with the symmetry group $G$. Moreover, any homogeneous symplectic manifold, whose symmetry group is a connected Lie group $G$, is locally isomorphic to some orbit of the coadjoint action of $G$ (or the coadjoint action of its central extension $G_1$ [Ki]). This fact shows the importance of the orbits and their universality in the problem of quantization of the physical systems whose symmetries form a Lie group $G$.

## 2. The Poisson brackets and orbits of a Lie group

In this section we recall some notions of the Hamiltonian formalism and the theory of coadjoint orbits of a Lie group (for detail see [Ar, Ki]).

### 2.1. The Hamiltonian mechanics

Consider a real manifold M and let $\mathcal{A} = C^\infty(M, \mathbb{R})$ be the set of smooth real functions on $M$. With respect to the pointwise multiplication, addition and multiplication by numbers the set $\mathcal{A}$ forms a commutative algebra. The elements of $\mathcal{A}$ are called *the dynamical variables*.

Suppose, that the manifold $M$ is endowed with a Poisson structure. This means that there exists a bilinear operation $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called a *Poisson bracket*, which satisfies the following requirements:

- $\{f_1, f_2\} = -\{f_2, f_1\}$;
- $\{f_1, f_2 f_3\} = \{f_1, f_2\} f_3 + \{f_1, f_3\} f_2$;
- $\{f_1, \{f_2, f_3\}\} + \text{cycle}(1, 2, 3) = 0$.

The time evolution of a dynamical variable $f$ is defined by a Hamiltonian $H \in \mathcal{A}$ in accordance with the formula

$$\dot{f} = \{f, H\}.$$

Denote by $z_a$ the set of (local) coordinates on $M$ and let

$$\{z_a, z_b\} = \omega_{ab}(z). \tag{4}$$

The requirements on the Poisson brackets listed above transform into the following properties of the tensor $\omega$

$$\begin{aligned} &\textbf{i)} \quad \omega_{ab}(z) = -\omega_{ba}(z) && \text{the skew-symmetry ,} \\ &\textbf{ii)} \quad \omega_{as}(z)\partial^s\omega_{bc}(z) + \text{cycle}(a,b,c) = 0 && \text{the Jacobi identity .} \end{aligned}$$

The Poisson bracket of two functions is now written in the explicit form

$$\{f,g\}(z) = \frac{\partial f}{\partial z_a}\,\omega_{ab}(z)\,\frac{\partial g}{\partial z_b}.$$

The Poisson bracket is called *non-degenerate* if

$$\{f,g\} \equiv 0 \quad \forall\, g \in \mathcal{A} \quad \Leftrightarrow \quad f \equiv 0. \tag{5}$$

This is equivalent to the invertibility of $\omega_{ab}$

$$\exists\, \omega^{ab}(z): \quad \omega^{ac}(z)\omega_{cb}(z) = \delta^a_b.$$

In this case $M$ is a symplectic manifold. The corresponding closed non-degenerate two form is given by

$$\Omega = \omega^{ab}(z)\, dz_a \wedge dz_b.$$

This is a common situation in mechanics — the manifold $M$ is often chosen to be the cotangent bundle of some configuration space $V$ and therefore is a symplectic manifold.

### 2.2. The Poisson-Lie brackets

The simplest case of (4) corresponds to the constant tensor $\omega$. We shall consider the next step in complexity when the manifold $M$ is a finite dimensional vector space and the tensor $\omega$ is a linear function in coordinates

$$\omega_{ab}(z) = C^s_{ab} z_s. \tag{6}$$

Taking into account the properties **i)** and **ii)** of $\omega$ we come to the following restrictions on the coefficients $C^c_{ab}$

$$C^c_{ab} = -C^c_{ba} \tag{7}$$

$$\sum_s C^s_{ab} C^r_{sc} + \text{cycle}(a,b,c) = 0. \tag{8}$$

As is well known, relations (7) and (8) means that the coefficients $C^c_{ab}$ form the set of structure constants of some Lie algebra $g$. The space $M$ can be identified with its dual space $g^*$. A Poisson bracket with the tensor $\omega$ given by (6) is called the *Poisson-Lie bracket.*

In what follows we shall consider the case $g = gl(n,\mathbb{R})$. Let us recall that the Lie algebra $gl(n,\mathbb{R})$ is generated by the $n^2$ elements $e_{ij}$, $1 \le i,j \le n$, subject to the following relations

$$[e_{ij}, e_{rs}] = \delta_{jr}\, e_{is} - \delta_{is} e_{rj}. \tag{9}$$

The elements $e_{ij}$ form a basis of the algebra and an arbitrary element $\boldsymbol{a} \in g$ is in one-to-one correspondence with the $n \times n$ matrix of its coefficients

$$\boldsymbol{a} \in g \quad \leftrightarrow \quad A = \|a_{ij}\| \in \text{Mat}_n(\mathbb{R}): \quad \boldsymbol{a} = \sum_{i,j} a_{ij} e_{ij}.$$

The dual space $g^*$ is by definition a space of linear functionals on $g$. If $g$ is finite dimensional, then $g^*$ can be described as a linear vector space endowed with a non-degenerated bilinear form $\langle\ ,\ \rangle : g^* \times g \to \mathbb{R}$. In our example we shall fix in $g^*$ the *dual* basis $\epsilon_{ij}$ defined by the following pairings

$$\langle \epsilon_{ij}, e_{rs} \rangle = \delta_{jr}\delta_{is}.$$

An arbitrary linear functional on $g$ is also represented by the matrix of its coefficients with respect to the basis $\epsilon_{ij}$

$$\boldsymbol{x} \in g^* \quad \leftrightarrow \quad X = \|x_{ij}\| \in \mathrm{Mat}_n(\mathbb{R}) : \quad \boldsymbol{x} = \sum_{i,j} x_{ij}\epsilon_{ij}.$$

Now the value of a linear functional $\boldsymbol{x} \in g^*$ on an arbitrary element $\boldsymbol{a} \in g$ can be written in the form

$$\langle \boldsymbol{x}, \boldsymbol{a} \rangle = \mathrm{Tr}(XA) \quad \boldsymbol{x} \in g^*,\ \boldsymbol{a} \in g. \tag{10}$$

The smooth functions on $g^*$ are those in the coordinates $x_{ij}$

$$f(\boldsymbol{x}) = f(x_{ij}), \quad \boldsymbol{x} = \sum_{i,j} x_{ij}\epsilon_{ij}.$$

The linear space $g^*$ is a Poisson manifold, the Poisson-Lie bracket is defined by the structure constants of (9)

$$\{x_{ij}, x_{rs}\} = \delta_{jr}x_{is} - \delta_{is}x_{rj}. \tag{11}$$

The subset of *linear* functions on $g^*$ forms a Lie algebra with respect to the above bracket and this algebra is isomorphic to the initial Lie algebra $gl(n, \mathbb{R})$.

The Poisson bracket (11) is degenerate. Consider the set of functions $p_k$ defined as follows

$$p_k(\boldsymbol{x}) = \mathrm{Tr}(X^k), \quad X = \|x_{ij}\|. \tag{12}$$

Since $\{x_{ij}, p_k(x)\} = 0$, the functions $p_k$ has zero Poisson bracket with any function $f \in C^\infty(g^*, \mathbb{R})$. In other words, the functions $p_k$ are central elements of the infinite dimensional Lie algebra $C^\infty(g^*, \mathbb{R})$ with respect to the Poisson-Lie bracket (11). In accordance with definition (5) this means that (11) is degenerate.

### 2.3. The orbits of the coadjoint representation

For a Lie group $G$ there exists an important representation in the linear space of its Lie algebra $g$ which is called the *adjoint representation*. In our example the Lie group $GL(n, \mathbb{R})$, corresponding to $g = gl(n, \mathbb{R})$, can be identified with the group of invertible real $n \times n$ matrices. By definition the adjoint representation reads

$$\mathrm{Ad}_M(\boldsymbol{a}) = \sum_{ij} (MAM^{-1})_{ij} e_{ij}, \quad M \in G, \quad \boldsymbol{a} = \sum_{ij} a_{ij}e_{ij} \in g,\ A = \|a_{ij}\|.$$

With the help of a non-degenerate bilinear form $\langle\, , \rangle$ one can define the *coadjoint* representation $\mathrm{Ad}^*_M$ of $G$ in the dual space $g^*$. By definition

$$\langle \mathrm{Ad}^*_M(\boldsymbol{x}), \boldsymbol{a} \rangle = \langle \boldsymbol{x}, \mathrm{Ad}_{M^{-1}}(\boldsymbol{a}) \rangle$$

Using (10) we can find the coordinate matrix $X'$ of the element $\mathrm{Ad}^*_M(\boldsymbol{x})$

$$\mathrm{Tr}(X'A) = \mathrm{Tr}(XM^{-1}AM) = \mathrm{Tr}(MXM^{-1}A).$$

Since $A$ is an arbitrary matrix, we come to the following form of the *coadjoint* action of $G$ on $g^*$

$$\mathrm{Ad}^*_M(\boldsymbol{x}) = \boldsymbol{x}' \quad \Rightarrow \quad X' = MXM^{-1}. \tag{13}$$

From now on, we shall identify the space $g^* = gl(n,\mathbb{R})^*$ with the space $\mathrm{Mat}_n(\mathbb{R})$ of the $n \times n$ real matrices.

Let us fix a matrix $X \in g^*$ and consider the subset $\mathcal{O}_X \subset g^*$ formed by the following matrices

$$\mathcal{O}_X = \left\{ MXM^{-1} \mid M \in G \right\}. \tag{14}$$

The set $\mathcal{O}_X$ is called the *orbit* of the coadjoint representation (action) of $G$ passing through $X$.

Let us list some properties of the orbits.

**a)** Any orbit $\mathcal{O}_X$ is a $G$-homogeneous set. This means that $G$ acts transitively on $\mathcal{O}_X$: any two points $X_1, X_2 \in \mathcal{O}_X$ can be mapped into each other by some element of $G$. Indeed, if $X_1 = M_1 X M_1^{-1}$, $X_2 = M_2 X M_2^{-1}$ then

$$X_1 = (M_1 M_2^{-1}) X_2 (M_1 M_2^{-1})^{-1}.$$

Therefore, any point of an orbit can be taken as its representative element.

**b)** Any orbit is a symplectic $G$-invariant manifold [Ki]. This theorem comprises several important statements.

b1) An orbit $\mathcal{O}_X$ is not simply a set, it is a submanifold in $g^*$. Below we explain how one can define an orbit by a set of polynomial equations in coordinates of $g^*$.

b2) Being restricted to the algebra of functions on $\mathcal{O}_X$, the Poisson bracket (11) is non-degenerated.

b3) The action of $G$ on $\mathcal{O}_X$ is Poisson. This means the following. The coadjoint action of $G$ on the space $g^*$ defined by (13) is extended to the functions $f$ on $g^*$ in the standard way

$$f \xrightarrow{M} f_M : \quad f_M(\boldsymbol{x}) = f(\mathrm{Ad}^*_{M^{-1}}(\boldsymbol{x})) \quad M \in G,\ \boldsymbol{x} \in g^*.$$

This action is Poisson if it commutes with the calculation of the Poisson bracket

$$\{f_1, f_2\}_M = \{f_{1M}, f_{2M}\}.$$

**c)** An orbit is invariant under the diffeomorphisms generated by an *arbitrary* strictly Hamiltonian vector field on $g^*$. Therefore, if the initial state of a dynamical system is represented by a point on an orbit $\mathcal{O}$, then during the time evolution with *any* Hamiltonian $H \in C^\infty(g^*, \mathbb{R})$ the states of this system are constrained to the same orbit $\mathcal{O}$.

This is a quite general result valid for any connected Lie group (see, for example, [Ar, Ol]). Below we give a simple proof of this statement for the (semisimple) orbits of $GL(n,\mathbb{R})$. For this purpose, we should analize relations which define an orbit. This is a subject of the next subsections.

### 2.4. Generic semisimple orbits

We shall restrict ourselves to the class of *semisimple* orbits. The orbit is called semisimple if it contains a matrix, which can be diagonalized by a similarity transformation (a semisimple matrix by definition). In accordance with the property **a)**, all the points of a semisimple orbit are semisimple matrices.

Fix a diagonal matrix $X = \mathrm{diag}(\mu_1, \ldots, \mu_n) \in g^*$ and consider its orbit $\mathcal{O}_X$ under the coadjoint action (13) of the group $G$. Let us first suppose that all eigenvalues $\mu_i$ are pairwise distinct numbers.

Such an orbit will be called *generic*. As is known from linear algebra, any $n \times n$ matrix $X$ satisfies the polynomial Cayley-Hamilton identity of the form

$$\sum_{k=0}^{n} (-X)^{n-k} \sigma_k(X) \equiv 0, \quad X^0 = I, \tag{15}$$

$I$ being the $n \times n$ unit matrix. The coefficient $\sigma_k(X)$ is the sum of all principal minors of $X$ of the $k$-th order. In terms of the spectral values $\mu_i$ of $X$ the coefficient $\sigma_k(X)$ is the $k$-th order elementary symmetric function in $\mu_i$

$$\sigma_k(X) = \sum_{1 \le i_1 < \ldots < i_k \le n} \mu_{i_1} \ldots \mu_{i_k}, \quad 1 \le k \le n.$$

The functions $p_k(X)$ defined in (12) (recall, that we identify $\boldsymbol{x}$ with the corresponding matrix $X$) are the power sums of the eigenvalues of $X$

$$p_k(X) = \sum_{i=1}^{n} \mu_i^k.$$

The functions $\sigma_k$ and $p_k$ are connected by the well known Newton relations

$$k\sigma_k - p_1\sigma_{k-1} + p_2\sigma_{k-2} - \ldots + (-1)^k p_k \equiv 0, \quad 1 \le k \le n.$$

With these relations one can easily prove that $\{x_{ij}, \sigma_k(X)\} \equiv 0$ as soon as we have $\{x_{ij}, p_k(X)\} \equiv 0$.

The elements $p_k(X)$, $1 \le k \le n$ are algebraically independent, as well as $\sigma_k(X)$, $1 \le k \le n$. The higher power sums $p_r(X)$ at $r > n$ are polynomials in the first $n$ functions $p_k$. This follows from (15) and Newton relations.

Let us now find a set of relations, which define the orbit $\mathcal{O}_X$ as a submanifold in $g^*$. For this purpose, observe, that the coadjoint action (13) preserve the spectrum of $X$. Due to this fact, all the points of $\mathcal{O}_X$ have the same eigenvalues $\mu_i$ and, therefore, the same values of the central functions $p_k(X)$, $1 \le k \le n$. So, the necessary condition for a matrix $Y \in g^*$ to belong to the orbit $\mathcal{O}_X$ is

$$Y \in \mathcal{O}_X \;\Rightarrow\; p_k(Y) = p_k(X), \quad k = 1, 2, \ldots, n$$

Since we consider a semisimple generic orbit (all eigenvalues are pairwise distinct) this condition is also sufficient.

Therefore, a semisimple generic orbit in $gl(n, \mathbb{R})^*$ is defined by the set of $n$ polynomial relations

$$p_k(X) = \mathrm{Tr}(X^k) = c_k, \quad c_k = \sum_{i=1}^{n} \mu_i^k, \quad 1 \le k \le n, \tag{16}$$

where the numbers $\mu_i$, $1 \le i \le n$, are pairwise distinct.

At last, since $p_k$ are central functions with respect to the Poisson-Lie bracket (11), we get

$$\{p_k(X), H(X)\} \equiv 0, \quad \forall H \in C^\infty(g^*, \mathbb{R}).$$

This means that a Hamiltonian vector field $\xi_H = \{\ , H\}$, generated by an arbitrary Hamiltonian $H$, is tangent to any generic semisimple orbit, defined by (16). This proves the property **c)** for this type of orbits.

### 2.5. Generic orbits in $gl(2,\mathbb{R})^*$

Let us consider a simple example of generic orbits in $gl(2,\mathbb{R})^*$. This is a space of $2\times 2$ real valued matrices which is isomorphic (as a vector space) to $\mathbb{R}^4$. The Cayley-Hamilton identity reads

$$X^2 - X\mathrm{Tr}(X) + I\det X \equiv 0, \quad X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

Fix two different number $\mu_1$ and $\mu_2$. In order they could represent the spectrum of a matrix $X \in gl(2,\mathbb{R})^*$, the numbers $\mu_1$ and $\mu_2$ must either be both real or be complex conjugate to each other: $\bar{\mu}_1 = \mu_2$.

Then a generic orbit is a submanifold in $\mathbb{R}^4$ defined by a couple of relations

$$x_{11} + x_{22} = \mu_1 + \mu_2, \quad x_{11}x_{22} - x_{12}x_{21} = \mu_1\mu_2. \tag{17}$$

Parametrizing $x_{ij}$ by the new coordinates $x$, $y$ and $t$

$$x_{11} = \frac{\mu_1+\mu_2}{2} + x, \quad x_{22} = \frac{\mu_1+\mu_2}{2} - x, \quad x_{12} = y - t, \quad x_{21} = y + t$$

we reduce system (17) to the following equation

$$t^2 - x^2 - y^2 = -\frac{1}{4}(\mu_1 - \mu_2)^2. \tag{18}$$

If $\mu_1$ and $\mu_2$ are real, this equation defines a one sheet hyperboloid around the $t$-axis.

If the eigenvalues are complex conjugate then the above equation transforms to

$$t^2 - x^2 - y^2 = (\mathrm{Im}\mu_1)^2.$$

This is an equation of a two sheet hyperboloid around the $t$-axis.

Note, that a cone, corresponding to the zero right hand side in (18), is not a generic orbit (since $\mu_1 = \mu_2$).

### 2.6. The non-generic orbits

Consider now the case, when the representative matrix $X$ of the orbit $\mathcal{O}_X$ possesses coincident eigenvalues. Suppose that the spectrum of $X$ consists of $r < n$ pairwise distinct numbers $\mu_i$, each of them entering the spectrum with a multiplicity $m_i \geq 1$

$$\mathrm{Spec}(X) = \{(\mu_i, m_i)_{\,1\leq i\leq r} \,|\, m_1 + m_2 + \ldots + m_r = n\}.$$

The functions $p_k$ take the values

$$p_k(X) - c_k = 0, \quad c_k = \sum_{i=1}^{r} m_i\mu_i^k. \tag{19}$$

But now their fixation does not define a semisimple orbit. Indeed, if we choose the values of $p_k(X)$ as in (19), the Cayley-Hamilton identity (15) can be written in the factorized form

$$(X - \mu_1 I)^{m_1}(X - \mu_2 I)^{m_2}\ldots(X - \mu_r I)^{m_r} \equiv 0. \tag{20}$$

If all multiplicities $m_i = 1$ (the generic case) then identity (20) guarantees the matrix $X$ to be semisimple. Therefore, on fixing $p_k$ as in (16), one uniquely defines (up to a permutation of eigenvalues) a semisimple matrix $X$ with a spectrum $\mu_i$.

If there are the multiplicities $m_i > 1$ then (20) has several solutions with the same spectrum, but only one of them is a semisimple matrix.

Consider an example for the case $n = 3$. Let the spectrum consists of two eigenvalues $(\mu_1, m_1 = 1)$ and $(\mu_2, m_2 = 2)$. The Cayley-Hamilton identity (20) looks as follows

$$(X - \mu_1 I)(X - \mu_2 I)^2 \equiv 0 \tag{21}$$

and it has two non-equivalent solutions

$$X_1 = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 1 \\ 0 & 0 & \mu_2 \end{pmatrix}.$$

Evidently, only the matrix $X_1$ defines a semisimple orbit, but $p_k(X_1) = p_k(X_2)$ for all $k$.

So, relations (19) where some $m_i > 1$ define a *union* of orbits and only one orbit of them is semisimple. That is why we should find some additional relations in order to uniquely extract a semisimple orbit.

This problem is easy to solve. Let us pay attention to the fact that the semisimple matrix $X_1$ in the above example actually obeys a polynomial identity of a *lower* order than that of identity (21). Indeed

$$(X_1 - \mu_1 I)(X_1 - \mu_2 I) \equiv 0. \tag{22}$$

This is the so called *minimal polynomial identity* for the matrix $X_1$. As is known from matrix analysis, a matrix $X$ is semisimple if and only if its minimal polynomial is a product of linear factors with pairwise distinct spectral values and with all exponents to be equal to unity:

$$X \text{ is semisimple} \quad \Leftrightarrow \quad (X - \mu_1 I) \dots (X - \mu_r I) \equiv 0. \tag{23}$$

So, in order to extract a semisimple non-generic orbit we should complete (19) with $n^2$ new relations

$$\Phi_{ij}(X) = 0, \quad 1 \le i, j \le n, \tag{24}$$

where $\Phi_{ij}$ are the matrix elements of (23)

$$\Phi_{ij}(X) = (X^r)_{ij} - \sum_{a=1}^{r} \mu_a \, (X^{r-1})_{ij} + \dots + (-1)^r \, \delta_{ij} \, \mu_1 \dots \mu_r.$$

Since the minimal polynomial identity (23) is of the $r$-th order, then in (19) only first $r$ functions $p_k$ should be fixed (others are polynomials in these $p_k$-s).

It is worth stressing, that in general we cannot disregard relations[1] (19). Having fixed the identity (23) alone, we again define a union of orbits, each of them being semisimple. The representative matrices of these orbits have the same spectral values but different multiplicities. For example, the identity (22) is satisfied by the two non-equivalent semisimple matrices

$$X_1 = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \quad \text{and} \quad X_1' = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}.$$

The role of relations (19) consists in fixing multiplicities $m_i$ of spectral values $\mu_i$.

[1] It is only possible in the most degenerate case when the spectrum consists of the single eigenvalue $\mu$. Then the multiplicity is uniquely defined by the size of the matrix and the only semisimple matrix is proportional to the unit matrix $X = \mu I$.

The question on the invariance of the non-generic orbit under the Hamiltonian evolution is slightly more involved in comparing with the generic case. The matter is that the matrix elements $\Phi_{ij}$ of the identity (23) do not Poisson-commute with the coordinates $x_{ij}$ (in contrast with the power sums $p_k$). Nevertheless, the *full* set of relations (24) defines a manifold which is invariant under the diffeomorphisms of the Hamiltonian vector fields. To prove this we first calculate the bracket $\{x_{ks}, (X^m)_{ij}\}$ on the base of (11)

$$\{x_{ks}, (X^m)_{ij}\} = \delta_{si}(X^m)_{kj} - \delta_{kj}(X^m)_{is}, \quad X = \|x_{ij}\|.$$

Then, taking into account the structure of $\Phi_{ij}$, we come to the analogous result for the bracket of $x_{ks}$ and $\Phi_{ij}$

$$\{x_{ks}, \Phi_{ij}\} = \delta_{si}\Phi_{kj} - \delta_{kj}\Phi_{is}. \tag{25}$$

This result is quite obvious except for the question about the constant term proportional to the product $\mu_1 \dots \mu_r$. Being constant, the term has vanishing Poisson bracket with $x_{ks}$ and at the first glance it is not clear, how it can appear in the right hand side of (25). The mater is that the constant term is only contained in the diagonal matrix elements $\Phi_{ii}$. A simple analysis shows that the right hand side of (25) depends on the *difference* of diagonal matrix elements. The only bracket with such a dependence is of the form

$$\{x_{ji}, \Phi_{ij}\} = \Phi_{jj} - \Phi_{ii}.$$

The constant terms are cancelled and the right hand side of (25) is actually independent of them.

Now it is a straightforward consequence of (25) that the manifold defined by the system of relations (24) is tangent to a Hamiltonian vector field generated by an arbitrary Hamiltonian $H \in C^\infty(g^*, \mathbb{R})$.

### 2.7. Physical systems on orbits of Lie groups

There are known a plenty of examples of mechanical systems the phase space of which can be presented as an orbit of some Lie group. A great number of them can be found, for example, in [Pe, FT] with the detailed consideration.

Here we turn to a few of such examples just for the illustration of the above consideration.

**Example 1.** Take the Lie group $G = SO(3)$ — the group of rotations of the three dimensional Euclidean space $\mathbb{R}^3$. The Poisson-Lie structure on the space $g^* = so(3)^*$ is

$$\{x_i, x_j\} = \epsilon_{ijk}x_k,$$

where $\epsilon_{ijk}$ are the components of the full antisymmetric tensor.

The orbits of the coadjoint $SO(3)$-action are the two dimensional spheres $\mathcal{O}_r$

$$\mathcal{O}_r : \quad x_1^2 + x_2^2 + x_3^2 = r^2.$$

Given a Hamiltonian $H(x)$, the dynamical equations read

$$\frac{d\vec{x}}{dt} = \vec{x} \times \vec{\nabla} H.$$

The quadratic Hamiltonian $H = \frac{1}{2}\sum_i J_i x_i^2$ leads to the Euler equation describing the rotation of a rigid body with the principle moments of inertia $J_i$ around a fixed point. The vector $\vec{x}$ represents the angular velocity of the body.

**Example 2 [Pe].** The Lie group $G = E(3)$ is a group of motions of the three dimensional Euclidean space $\mathbb{R}^3$. This group is a semidirect product of the group of rotations and the group of translations.

The dual space to its Lie algebra $g^* = \mathrm{Lie}^*(E(3))$ is a six dimensional vector space with coordinates $(x_i, y_i)$ obeying to the following bracket relations

$$\{x_i, x_j\} = \epsilon_{ijk} x_k, \quad \{x_i, y_j\} = \epsilon_{ijk} y_k, \quad \{y_i, y_j\} = 0.$$

The orbits are four dimensional manifolds parametrized by the two numbers $a \geq 0$ and $b$

$$\mathcal{O}_{a,b} = \{\vec{x}, \vec{y} : \quad \vec{x} \cdot \vec{y} = ab,\ |\vec{y}|^2 = a^2,\ a \geq 0\}.$$

A quadratic Hamiltonian leads to the Kirchhoff equations for the motion of a rigid body in an ideal liquid.

## 3. Quantization of algebra of functions on an orbit

In this section we extend the field of real numbers to the complex field $\mathbb{C}$. This means that we shall work with complex manifolds and shall consider complex valued functions on the manifolds. Besides, we constrain ourselves to the *regular* functions on a manifold. By definition, a function is called regular on a manifold $M$ if it coincides with a polynomial in each coordinate chart of the manifold. The set of all regular functions on a complex manifold $M$ is called a *coordinate ring* of $M$ and is denoted as $\mathbb{C}[M]$. Of course, it is also an algebra over $\mathbb{C}$.

Let us consider a problem of quantization of the algebra of functions over an orbit $\mathcal{O}_X$ of the coadjoint action of the Lie group $G = GL(n)$ on the space $g^* = gl(n, \mathbb{C})^*$. The construction of the orbit presented in the previous section can be extended to the complex case in a straightforward way.

Consider the main features of the quantization from the physical point of view. Let we are given a system $S$ and let $\mathcal{A}$ be the algebra of its dynamical variables (observables) endowed with a Poisson structure. The algebra $\mathcal{A}$ is an associative commutative algebra with respect to pointwise multiplication and addition of functions and is a (infinite dimensional) Lie algebra with respect to the Poisson bracket.

To quantize a system $S$ means the following:

1. One should pass from the commutative associative algebra $\mathcal{A}$ to some non-commutative associative algebra $\mathcal{A}_\hbar$ which is called a quantized algebra of dynamical variables (observables). In practice one usually works not with the algebra $\mathcal{A}_\hbar$ itself but with some its representation in a Hilbert space, the observables being represented by hermitian operators.

2. Being a Lie algebra with respect to the commutator, the quantum algebra should be isomorphic to the Lie algebra obtained from $\mathcal{A}$ with the help of the Poisson structure. That is, if $f \to \hat{f}$ and $g \to \hat{g}$ then
$$\{f, g\} \to \frac{1}{i\hbar} [\hat{f}, \hat{g}].$$

3. A quasiclassical limit
$$\lim_{\hbar \to 0} \mathcal{A}_\hbar = \mathcal{A}$$
should be defined (in some sense).

4. At the quantization the number of degrees of freedom must not changed and the symmetries of the classical system should be maximally retained.

The second point is central in the above scheme and causes the main difficulties. The matter is that even if the brackets of generators have the simplest form (1) then there is an ambiguity in the construction of the correspondence $f \to \hat{f}$ – the well known problem of ordering. In general case we can represent a quantum "function" $\hat{f}$ as a (formal) series in the Planck constant $\hat{f} = \sum \hbar^s \hat{f}_s$ and try to find the coefficients $\hat{f}_s$ in such a way that the correspondence described in the point 2 could be satisfied. This means, that generally the associative multiplication in the quantum algebra $\mathcal{A}_\hbar$ can be restored only up to a finite order in $\hbar$.

The mathematical definition of the quantization procedure accepted in the non-commutative geometry is very close to the physical one.

Given a coordinate ring $\mathcal{A} = \mathbb{C}[M]$ of some affine algebraic variety[2] $M$ we should pass to the non-commutative algebra $\mathcal{A}_\hbar$ parametrized by a formal parameter $\hbar$ in such a way that the following requirements should be satisfied:

- The initial algebra $\mathcal{A}$ should be isomorphic to the following quotient

  $$\mathcal{A}_\hbar / \hbar \mathcal{A}_\hbar \cong \mathcal{A}$$

  and the first order term of the quantum product must be defined by the Poisson bracket in $\mathcal{A}$

  $$\hat{a} \star \hat{b} = ab + \frac{\hbar}{2}\{a, b\} + o(\hbar^2).$$

  This is analogous to the points 2 and 3 of the physical scheme.

- The quantization should be a *flat* deformation. This means that the "supply" of elements of $\mathcal{A}_\hbar$ is as large as that of $\mathcal{A}$. This condition is equivalent to the physical requirement that the quantization should not alter the number of degrees of freedom of a system.

- If the algebra $\mathcal{A}$ is a module over some another algebra (or a group) $\mathcal{B}$ then the quantum algebra $\mathcal{A}_\hbar$ should also be a module over $\mathcal{B}$ (or, possibly, over some its deformation[3]). This is equivalent to the physical requirement that the symmetries of the classical system (represented by $\mathcal{B}$) should be extended to the quantum case.

Let us shortly describe the result of quantization of the algebra of regular functions on a semisimple orbit $\mathcal{O}_X$ of the coadjoint action of the Lie group $G = GL(n)$. The importance of the result consists, in particular, in the fact that the quantum algebra can be described explicitly as a quotient of the universal enveloping algebra $U(gl(n))$ over some ideal generated by a polynomial relations (not as a series in the quantization parameter $\hbar$).

Let us fix a matrix $X \in gl(n)^*$ with the spectrum

$$\mathrm{Spec}(X) = (\mu_i, m_i), \quad 1 \le i \le r. \tag{26}$$

Recall, that the eigenvalues $\mu_i$ are all distinct and $m_i \ge 1$ are the corresponding multiplicities. Consider then the orbit $\mathcal{O}_X$ of the coadjoint action of $G$. The number $r \le n$ is called the rank of the orbit. As was explained in Section 2, the orbit is defined by the set of relations (19) and (24).

Turn now to the coordinate ring $\mathbb{C}[g^*]$ of $g^*$ which is a $\mathbb{C}$-algebra of all polynomials in the coordinates $x_{ij}$. Let us construct a two-sided ideal $\mathcal{J}_X$ in $\mathbb{C}[g^*]$ generated by the elements $\Phi_{ij}$ and $\pi_k = p_k - c_k$. This ideal is a set of all elements of $\mathbb{C}[g^*]$ which can be presented in the form

$$\mathcal{J}_X = \{a\Phi_{ij}b,\ g\pi_k h \mid a, b, g, h \in \mathbb{C}[g^*],\ 1 \le i, j \le n,\ 1 \le k \le r\}.$$

[2]This simply means that a variety is defined by a system of polynomial equations in some affine space.

[3]The quantum groups is one of the most known examples of such a deformation.

Then as is well known (see, for example, a textbook [Sh]), the algebra of regular functions on the orbit $\mathcal{O}_X$ is given by the following quotient

$$\mathbb{C}[\mathcal{O}_X] = \mathbb{C}[g^*]/\mathcal{J}_X.$$

After the quantization of the Poisson brackets (11) the algebra $\mathbb{C}[g^*]$ turns into the universal enveloping algebra $U(gl(n)_\hbar)$. The coordinates $x_{ij}$ map into the $U(gl(n)_\hbar)$ generators $e_{ij}$ with the commutation relations

$$[e_{ij}, e_{ks}] = \hbar(\delta_{jk}e_{is} - \delta_{si}e_{kj}).$$

Consider the matrix $L = \|e_{ij}\|$ composed of the non-commutative generators $e_{ij}$. It can be shown, that the matrix $L$ satisfies the Cayley-Hamilton identity analogous to (15) but with modified coefficients [GS1]. The quantities $p_k = \mathrm{Tr}(L^k)$ are central elements of $U(gl(n)_\hbar)$.

Let us first suppose that all the multiplicities are equal to unity (the generic orbit). Define a *quantum spectrum* as $n$ numbers $\nu_i = \mu_i$ (that is in the generic case the quantum spectrum coincides with the classical one). Then consider the two-sided ideal $\mathcal{J}(\nu)$ in $U(gl(n))_\hbar$ which is generated by the set of $n$ elements

$$\pi_k = \mathrm{Tr}(L^k) - c_k, \quad c_k = \sum_{i=1}^{n} \nu_i^k d_i,$$

where the *quantum multiplicities* are not equal to unity even in the generic case

$$d_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(\nu_i - \nu_j - \hbar)}{\nu_i - \nu_j}.$$

Then the quantized algebra $\mathcal{A}_\hbar$ of functions on the generic orbit $\mathcal{O}_X$ turns out to be the following quotient of the $U(gl(n)_\hbar)$ (see [GS2])

$$\mathcal{A}_\hbar = U(gl(n)_\hbar)/\mathcal{J}(\nu).$$

Consider now the non-generic case when there exist multiplicities greater than 1. With each pair $(\mu_i, m_i)$ we associate a *string* of *quantum eigenvalues*

$$(\mu_i, m_i) \ \to \ (\mu_i,\ \mu_i + \hbar,\ \mu_i + 2\hbar,\ \ldots,\ \mu_i + (m_i - 1)\hbar).$$

Note that all the elements of the string are distinct.

Then we define the quantum spectrum as the set of all elements of all strings

$$\nu_{i,s} = \mu_i + s\hbar, \quad 1 \leq i \leq r,\ 0 \leq s \leq m_i - 1.$$

That is the degenerate classical eigenvalues split into the set of non-degenerate quantum spectral values.

The quantized algebra $\mathcal{A}_\hbar$ of functions on a semisimple orbit $\mathcal{O}_X$ passing through the matrix (26) is the quotient of $U(gl(n)_\hbar)$ over the two sided ideal (see [DM, GS2]) generated by the matrix elements of the polynomial identity

$$(L - \mu_1 I)\ldots(L - \mu_r I) = 0$$

and by the elements

$$\pi_k = \mathrm{Tr}(L^k) - c_k, \quad c_k = \sum_{i,s} \nu_{i,s}^k d_{i,s}, \quad 1 \leq k \leq r.$$

The quantum multiplicities

$$d_{i,s} = \prod_{(j,r)\neq(i,s)} \frac{(\nu_{i,s} - \nu_{j,r} - \hbar)}{\nu_{i,s} - \nu_{j,r}}$$

are correctly defined since the quantum spectrum is not degenerated.

It is worth noting, that only the multiplicities $d_{i,0}$ are non-zero. All the numbers $d_{i,s}$ with $s \geq 1$ vanish due to the structure of the quantum spectral values $\nu_{i,s}$. This means that only the first elements $\mu_i$ of each string contribute to the value of the trace

$$\mathrm{Tr}(L^k) = \sum_{i=1}^{r} \mu_i^k d_{i,0} \ .$$

## Conclusion

In my talk I considered the simplest linear brackets $\{\,,\}_{PL}$ defined by (11). But on any semisimple orbit in $gl(n)^*$ there exists another Poisson structure $\{\,,\}_r$ connected with the classical R-matrix $r \in g \wedge g$ [DGK]. This Poisson structure is compatible with (11) and allows us to construct the so called Poisson pencil on the orbit, that is a family of brackets

$$\{\,,\}_{a,b} = a\{\,,\}_{PL} + b\{\,,\}_r$$

with some constants $a$ and $b$. The quantization of such a Poisson pencil leads to an orbit in the reflection equation algebra connected with the corresponding quantum R-matrix $R$.

The reflection equation algebra plays a significant role in the theory of integrable systems with boundaries and in the non-commutative geometry. The detailed consideration of the orbits in this algebra is given in [GS2].

## References


[Ar] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, "Nauka" publ., Moscow, 1974 (in Russian).

[Di] P.A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer graduate school of sciense, Yeshiva Univ., New York, 1964.

[DGK] J. Donin, D. Gurevich, S. Khoroshkin, *Double quantization on $CP^n$ type orbits by generalized Verma modules*, J. Geom. Phys., **28** (1998) 384–486.

[DM] J. Donin, A. Mudrov, *Explicit equivariant quantization on (co)adjoint orbits of $GL(n, C)$*, `math.QA/0206049`.

[FT] A.T. Fomenko, V.V. Trofimov, *Integrable Systems on Lie Algebras and Symmetric Spaces*, Gordon and Breach Sci. Publ., 1988.

[GS1] D.I. Gurevich, P.A. Saponov *Quantum line bundles via Cayley-Hamilton identity*, J. Phys. A: Math. Gen., **3**4 (2001) 4553 – 4569.

[GS2] D.I. Gurevich, P.A. Saponov, *Geometry of non-commutative orbits related to Hecke symmetries*, `math.QA/0411579`, to appear in Contemporary Mathematics.

[Ki] A.A. Kirillov, *Elements of Representation Theory*, 2-nd ed., "Nauka" publ., Moscow, 1978 (in Russian).

[Ol] P. Olver, *Applications of Lie groups to differential equations*, Springer Verlag, 1986.

[Pe] A.M. Perelomov, *Integrable Systems of Classical Mechanics and Lie Algebras*, Birkhäuser Verlag, Berlin, 1990.

[Sh] I.R. Shafarevich, *Foundations of Algebraic Geometry*, 2-nd ed., "Nauka" publ., Moscow, 1988 (in Russian).