

# CAUSALITY, CONFINEMENT AND THE RENORMALIZATION GROUP IN QCD: RECENT RESULTS ON THE EFFECTIVE COUPLING CONSTANT

A.I. Alekseev<sup>\*†</sup>

*Institute for High Energy Physics, 142281 Protvino, Moscow Region, Russia*

Based on a study of the analytic running coupling obtained from the perturbation theory results up to four-loop order, the QCD “synthetic” running coupling  $\alpha_{\text{syn}}$  is built. In so doing the perturbative time-like discontinuity is preserved and nonperturbative contributions not only remove the nonphysical singularities of the perturbation theory in the infrared region but also decrease very rapidly in the ultraviolet region. In the framework of the approach, on the one hand, the running coupling is enhanced at zero and, on the other hand, the dynamical gluon mass  $m_g$  arises. Fixing the parameter  $\sigma$  corresponding to the string tension parameter of the string models and normalization, say, at  $M_\tau$  completely define the synthetic running coupling. In this case the dynamical gluon mass appears to be fixed and the higher loop stabilization property of  $m_g$  is observed. For  $\sigma = (0.42 \text{ GeV})^2$  and  $\alpha_{\text{syn}}(M_\tau^2) = 0.33 \pm 0.01$  it is obtained that  $m_g = (530 \pm 80) \text{ MeV}$ .

## 1 Introduction

We describe the running coupling constant (the invariant charge) of QCD in which on the basis of the perturbative study up to four-loop order the attempt has been undertaken to include the nonperturbative contributions in such a way that nonphysical singularities of the perturbation theory in the infrared region be cancelled and the essentials of the QCD dynamics for all energy scales accumulated in its framework.

Doing that it seems notably attractive, on the theoretical grounds of the analytic running coupling  $\alpha_{\text{an}}(Q^2)$  of QCD, to build the running coupling including additional nonperturbative terms. We will consider a variation of such couplings, the “synthetic” running coupling  $\alpha_{\text{syn}}(Q^2)$ , which on the practical grounds of the perturbation theory successfully used for description of the region of large  $Q^2$ , contains the nonperturbative terms, determining the main features of QCD in low  $Q^2$  region without the dramatic changes of this qualitatively different regimes.

The analytic approach in QFT was formulated late in the 50s in Refs. [1] and [2] for QED and other theories. For QCD the analytic approach was applied in [3–7]. The analyticity requirement, which follows from the general principles of QFT, enables one to resolve the difficulties connected with nonphysical singularities of the perturbation theory in the infrared region. In the “analytically improved” running coupling this singularities are canceled out by the nonperturbative contributions. In the ultraviolet region the nonperturbative contributions rapidly decrease and the perturbative contributions are decisive. Nonetheless, the behavior of the nonperturbative contributions in the ultraviolet region originating from the procedure is of a considerable interest. Their behavior appears to be important in the construction of the synthetic running coupling.

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<sup>†</sup>Electronic address: alekseev@th1.ihep.su

Whereas for the one-loop case a separation of the analytic running coupling into the perturbative and the nonperturbative components and behavior of the nonperturbative component are obvious, for the multi-loop analytic running coupling this is not the case. For the two-loop case such separation in an explicit form and study of the nonperturbative component of the analytic running coupling was made in Ref. [8], for the three-loop case it was made in [9,10], and for the four-loop case it was made in [11,12]. The nonperturbative contributions in the analytic running coupling were extracted explicitly and their expansion in powers of  $\Lambda^2/Q^2$  was obtained. The effective method of the precise calculation of the analytic running coupling was developed on a basis of this expansion.

It is known [3] that the analytic running coupling is finite at zero. In Refs. [11] and [12] it was shown that the finiteness of the analytic coupling at zero was a consequence of the asymptotic freedom property of the initial perturbation theory,  $\alpha_{\text{an}}^{\text{n-loop}}(0) = 4\pi/b_0 \simeq 1.396$ . The running coupling is finite at zero in case of “freezing” of the interaction [13], which is appealing with respect to the phenomenology. However, with such infrared behavior of the running coupling the description of the confinement and the dynamical breaking of the chiral symmetry is not immediate. The behavior of the running coupling  $\alpha_V \sim 1/Q^2$  at  $Q^2 \rightarrow 0$  in the so-called V-scheme of the renormalization corresponds to the linear confining quark-antiquark static potential with the universal string tension parameter  $\sigma$ . In QCD the static potential is defined in an explicit gauge invariant form through the vacuum expectation value of the Wilson loop. The synthetic running coupling under consideration also belongs to the singular type couplings,  $\alpha_{\text{syn}} \sim 1/Q^2$  at  $Q^2 \rightarrow 0$ , and yet it has an additional motivation related to the study of the nonperturbative contributions at  $Q^2 \rightarrow \infty$ .

The main methods of nonperturbative study of the Green’s functions in QCD and the running coupling which can be built out of these functions are solving the Dyson–Schwinger (DS) equations and lattice calculations of the functional integrals. In Ref. [14] a summary of the results of such recent studies is given, which can be supplemented with papers [15–17] where analytic methods were used and [18] with the lattice stimulation results. The variety of the results for the behavior of  $\alpha_s(Q^2)$  in the infrared region is connected with an intrinsic for the gauge theories *vertex dependence* [14] of the definitions of the running coupling when the mass dependent subtraction schemes in the momentum space used, and different truncation methods applied to close the DS equations. Besides, solving the closed integral equations (or systems of equations) requires, as a rule, the simplifying assumptions quite often breaking the gauge symmetry and the pure technical approximations. It is not surprising, than, that the results of the infrared behavior study of  $\alpha_s$  differ greatly from one another and should not be compared literally. Let us note the review papers [19–21] of the investigations on the IR behavior of the Green’s functions, the running coupling in QCD, and their applications in the hadron physics.

The possibility of the singular behavior  $\alpha_s \sim 1/Q^2$  at  $Q^2 \rightarrow 0$  which we consider has been studied in a number of papers. In particular, in Ref. [22] the infrared behavior of the gluon Green’s functions was studied by the analytical calculations of the corresponding Feynman integrals of the DS equation for the gluon propagator in the ghost-free axial gauge where the running coupling was defined by the full gluon propagator. It was shown that the singular behavior of the gluon propagator of the form  $D(Q) \sim 1/(Q^2)^2$  at  $Q^2 \rightarrow 0$  was possible but it is essential to give up the commonly used approximation of the three-gluon vertex function by its longitudinal part and to take into account the transverse part of the three-gluon vertex function of a definite form.

This paper is organized as follows. In Section 2 the one-loop synthetic running coupling model of QCD for all  $Q^2$  is considered. In Section 3 we study the analytic running coupling for the standard perturbation theory approximations up to four-loop order and its separation into the perturbative and the nonperturbative components. In Section 4 the synthetic running coupling with the nonperturbative contributions suppressed at large  $Q^2$  is build on the basis of the analytic running coupling. Setting the parameters of the synthetic running coupling is made. In the concluding section the main results are summarized and some remarks made.

## 2 One-loop synthetic running coupling of QCD

Let us consider the following additive modification of the one-loop running coupling of QCD by means of the nonperturbative pole type terms <sup>1</sup>

$$\alpha_{\text{syn}}^{(1)}(Q^2) = \frac{4\pi}{b_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{c\Lambda^2}{Q^2} + \frac{(1-c)\Lambda^2}{Q^2 + m_g^2} \right], \quad (1)$$

where

$$m_g^2 = \frac{\Lambda^2}{c-1}. \quad (2)$$

Here  $Q^2$  is the Euclidean momentum squared, constant  $b_0 = 11 - 2n_f/3$  ( $n_f$  is the number of active quark flavors),  $\Lambda$  is the dimensional parameter of the one-loop model (1),  $c$  is the dimensionless parameter of this model (it is convenient to introduce the dimensional parameter  $\Lambda_1 = \sqrt{c}\Lambda$ ,  $c \in (1, +\infty)$ ). The parameter  $\Lambda$  can be fixed, for example, by the normalization condition at large  $Q^2$ , whereas the parameter  $c$  of the model, as can be seen further, describes the relation between the parameter  $\Lambda_{\text{QCD}}$  and the string tension parameter  $\sigma$  of the string models <sup>2</sup>. It stands to reason that for the realistic definition of the parameters  $\Lambda_{\text{QCD}}$ ,  $\sigma$  and their connection it is necessary to go out of the one-loop approximation.

The first term of Eq. (1) is the solution of the renormalization group equation for the QCD running coupling  $\alpha_s(Q^2)$

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s) \quad (3)$$

in the one-loop approximation,  $\beta(\alpha_s) \simeq -\beta_0 \alpha_s^2$ ,  $\beta_0 = b_0/4\pi$ . Introducing the renormalization invariant parameter  $\Lambda$  (the integration constant of the differential equation) we obtain

$$\alpha_s^{(1)}(Q^2) = \frac{4\pi}{b_0} \frac{1}{\ln(Q^2/\Lambda^2)}, \quad (4)$$

where a nonphysical singularity (the Landau pole) at  $Q^2 = \Lambda^2$  is presented. The vanishing of expression (4) for  $Q^2 \rightarrow \infty$  corresponds to the remarkable property of asymptotic freedom [30] of non-Abelian gauge theories, while the growth of  $\alpha_s$  (to some critical value or to infinity) with decreasing  $Q^2$  can be connected with the confinement problem.

The pole terms in Eq. (1) are nonperturbative,  $\Lambda^2 \simeq \mu^2 \exp\{-4\pi/b_0 \alpha_s(\mu^2)\}$ . The sum of the first two terms in Eq. (1) is an analytic function in the complex  $Q^2$ -plane with a cut from 0 to  $-\infty$ ,

$$\alpha_{\text{an}}^{(1)}(Q^2) = \frac{4\pi}{b_0} \left[ \frac{1}{\ln(Q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - Q^2} \right]. \quad (5)$$

This function can be presented by the dispersion relation

$$\alpha_{\text{an}}^{(1)}(Q^2) = \int_0^\infty \frac{d\sigma \tilde{\rho}^{(1)}(\sigma)}{\sigma + Q^2}, \quad (6)$$

where the function  $\tilde{\rho}^{(1)}(\sigma)$  is called the one-loop spectral density

$$\tilde{\rho}^{(1)}(\sigma) = \frac{4\pi}{b_0} \frac{1}{\ln^2(\sigma/\Lambda^2) + \pi^2}. \quad (7)$$

<sup>1</sup>This one-loop model was considered in Refs. [23–25] and finally formulated in Ref. [26] with determination of the parameters from the minimality condition of the nonperturbative vacuum energy density.

<sup>2</sup>The approximations of QCD which take no account of the effects connected with masses of the heavy quarks contain only one dimensional parameter. For example, it is  $\Lambda_{\text{QCD}}$  for large  $Q^2$  or the string tension parameter  $\sigma$  which is adequate for low  $Q^2$ . In the potential models the connection of this parameters is studied in description of the bound states of the heavy quarks [27–29].

The last equation can be obtained by the analytic continuation of  $\alpha_s$  into the Minkowski space  $Q^2 \rightarrow -\sigma - i0$  and calculation of the imaginary part  $\tilde{\rho}^{(1)}(\sigma) = \frac{1}{\pi} \Im \alpha_s^{(1)}(-\sigma - i0)$ . For real  $Q^2 > 0$  function (5) is positive monotone decreasing function with maximum at zero  $\alpha_{\text{an}}^{(1)}(0) = 4\pi/b_0$ . The second nonperturbative term in Eq. (5) does not contribute to the imaginary part of  $\alpha_{\text{an}}^{(1)}(Q^2)$  in going to the Minkowski space, so that  $\tilde{\rho}^{(1)}(\sigma) = \frac{1}{\pi} \Im \alpha_{\text{an}}^{(1)}(-\sigma - i0)$ . The synthetic running coupling (1) can also be presented in the form of the dispersion relation

$$\alpha_{\text{syn}}^{(1)}(Q^2) = \int_{-0}^{\infty} \frac{d\sigma \tilde{\rho}_{\text{syn}}^{(1)}(\sigma)}{\sigma + Q^2}. \quad (8)$$

The function  $\tilde{\rho}_{\text{syn}}^{(1)}(\sigma)$  will be called the one-loop spectral density for the synthetic running coupling. It contains additional terms in the form of the delta functions,

$$\tilde{\rho}_{\text{syn}}^{(1)}(\sigma) = \tilde{\rho}^{(1)}(\sigma) + \frac{4\pi}{b_0} [c\Lambda^2\delta(\sigma) + (1-c)\Lambda^2\delta(\sigma - m_g^2)]. \quad (9)$$

Introducing two pole terms at  $Q^2 = 0$  and  $Q^2 = -m_g^2 < 0$  does not change the analyticity domain of the analytic running coupling (5). In Eq. (9) for the spectral density the additional terms in the form of two  $\delta$ -functions localized at  $\sigma = 0$  and  $\sigma = m_g^2 > 0$  emerged (in comparison with expression (7) of the perturbative origin).

Let us bring equation (1) for  $\alpha_{\text{syn}}^{(1)}(Q^2)$  to the explicit renormalization invariant form. It can be done without solving the differential renormalization group equations. Writing the normalization condition for  $\alpha_{\text{syn}}^{(1)}(Q^2)$  we obtain an equation for the required dependence of the parameter  $\Lambda^2$  on the values  $\alpha_{\text{syn}}^{(1)}(\mu^2)$  and  $\mu^2$  of the form

$$\alpha_{\text{syn}}^{(1)}(\mu^2) = \frac{4\pi}{b_0} \left[ \frac{1}{\ln(\mu^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - \mu^2} + \frac{c\Lambda^2}{\mu^2} + \frac{(1-c)\Lambda^2}{\mu^2 + m_g^2} \right]. \quad (10)$$

From dimensional considerations

$$\Lambda^2 = \mu^2 \exp\{-\varphi(a(\mu^2))\}, \quad (11)$$

where  $a(\mu^2) = (b_0/4\pi)\alpha_{\text{syn}}^{(1)}(\mu^2)$ . Then for  $\varphi(a)$  we have a transcendental equation

$$a = \frac{1}{\varphi(a)} + \frac{1}{1 - e^{\varphi(a)}} + ce^{-\varphi(a)} - \frac{(c-1)^2}{1 + (c-1)e^{\varphi(a)}}. \quad (12)$$

The function  $\varphi(a)$  has the behavior  $\varphi(a) \simeq 1/a \rightarrow +\infty$  as  $a \rightarrow +0$  for all values of  $c$ . This behavior corresponds to the perturbative region. The behavior of this solution at  $a \rightarrow +\infty$  is  $\varphi(a) \simeq -\ln(a/c) \rightarrow -\infty$ . The beta function  $\beta_{\text{syn}}(\alpha_{\text{syn}})$  for the synthetic running coupling can be found by the equation which is analogous to the equation (3),

$$Q^2 \frac{\partial \alpha_{\text{syn}}(Q^2)}{\partial Q^2} = \beta_{\text{syn}}(\alpha_{\text{syn}}). \quad (13)$$

Differentiating the running coupling (1) with the use of equations (11) and (12), we obtain

$$\beta_{\text{syn}}(\alpha_{\text{syn}}) = \frac{4\pi}{b_0} \left\{ -a + \frac{1}{\varphi(a)} - \frac{1}{\varphi^2(a)} + \frac{1}{(1 - e^{\varphi(a)})^2} - \frac{(c-1)^2}{(1 + (c-1)e^{\varphi(a)})^2} \right\} \Bigg|_{a=b_0\alpha_{\text{syn}}/4\pi}. \quad (14)$$

Therefore, using the behavior of function  $\varphi(a)$  as  $a \rightarrow 0$  and  $\infty$ , we find the asymptotic behavior

$$\beta_{\text{syn}}(\alpha_{\text{syn}}) \simeq -\frac{b_0}{4\pi}\alpha_{\text{syn}}^2 + o(\alpha_{\text{syn}}^2), \quad \alpha_{\text{syn}} \rightarrow 0, \quad (15)$$

$$\beta_{\text{syn}}(\alpha_{\text{syn}}) \simeq -\alpha_{\text{syn}} - \frac{4\pi}{b_0}c(c-2) + o(1), \quad \alpha_{\text{syn}} \rightarrow \infty. \quad (16)$$

Doing the corresponding expansions we make sure that the singularity of the  $\beta$ -function (14) at  $\varphi \rightarrow 0$  is apparent. We also make sure that for all  $\alpha_{\text{syn}} > 0$  the function  $\beta_{\text{syn}}(\alpha_{\text{syn}})$  is negatively defined.

Let us write the last three terms of the synthetic running coupling (1), taking into account (2), in the form

$$\begin{aligned} \alpha_{\text{syn}}^{\text{npt}(1)}(Q^2) &= \frac{4\pi}{b_0} \left[ \frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{c\Lambda^2}{Q^2} + \frac{(1-c)\Lambda^2}{Q^2 + m_g^2} \right] \\ &= \frac{4\pi}{b_0} \frac{c\Lambda^6}{Q^2(\Lambda^2 - Q^2)(\Lambda^2 + (c-1)Q^2)}. \end{aligned} \quad (17)$$

For  $a \rightarrow +0$ , according to equation (12) the function  $\phi(a) \simeq 1/a$ , thus for  $\Lambda^2$  from equation (11) we obtain  $\Lambda^2 \simeq \mu^2 \exp(-1/a)$  at  $\alpha_{\text{syn}}(\mu^2) = (4\pi/b_0)a \rightarrow +0$  so expression (17) must be considered as a nonperturbative component of the synthetic running coupling. The behavior of the nonperturbative “tail” at large  $Q^2$  is the following:

$$\alpha_{\text{syn}}^{\text{npt}(1)}(Q^2) = -\frac{4\pi}{b_0} \left[ \frac{c}{c-1} \frac{\Lambda^6}{Q^6} \right] + O\left(\frac{\Lambda^8}{Q^8}\right). \quad (18)$$

As seen from Eq. (18), the nonperturbative contributions of the synthetic running coupling decreases at large  $Q^2$  substantively faster than that of the analytic running coupling (5).

Hence the one-loop synthetic running coupling of QCD has the following interesting properties:

- (i) By the construction, as a function of  $Q^2$ , it has an analytic structure corresponding to the causality; that is, it is a holomorphic function in the complex  $Q^2$ -plane with a cut along the negative real semiaxis.
- (ii) As a function of its value  $\alpha_{\text{syn}}(\mu^2)$  at the normalization point  $\mu^2$ , it has an essential singularity at the origin; the asymptotic expansion of its nonperturbative part in  $\alpha_{\text{syn}}(\mu^2)$  for  $\alpha_{\text{syn}}(\mu^2) \rightarrow +0$  is equal to zero, which ensures conformity to the initial perturbation theory.
- (iii) In the ultraviolet region, it coincides with the usual result of perturbation theory (with renormalization invariance taken into account) apart from fast decreasing power terms. The nonperturbative component behaves as  $\sim 1/(Q^2)^3$  when  $Q^2 \rightarrow \infty$ .
- (iv) In the infrared region the synthetic running coupling does not have nonphysical singularities of the perturbation theory. There is the mass term and the singular at the origin term responsible for the confinement of quarks.

As it will be evident from the subsequent considerations, all these properties are valid for the two-, three- and four-loop synthetic running coupling. It is significant that for the one-loop case we have not only representation (8), but for one thing, the nonperturbative contributions are extracted from the synthetic running coupling in the explicit form

$$\alpha_{\text{syn}}^{(1)}(Q^2) = \alpha^{\text{pt}(1)}(Q^2) + \alpha_{\text{syn}}^{\text{npt}(1)}(Q^2) \quad (19)$$

and for another, for the nonperturbative contributions, a simple formula is on hand. The case  $c = 2$  is the particular one from the symmetry considerations,

$$\alpha_{\text{syn}}^{\text{npt}(1)}(Q^2) = \frac{4\pi}{b_0} \left[ \frac{\Lambda^2}{\Lambda^2 - Q^2} + \frac{2\Lambda^2}{Q^2} - \frac{\Lambda^2}{\Lambda^2 + Q^2} \right] = \frac{4\pi}{b_0} \frac{2\Lambda^2}{Q^2} \frac{\Lambda^4}{\Lambda^4 - Q^4}, \quad (20)$$

for which the nonperturbative component is the odd function of  $Q^2$ ,

$$\alpha_{\text{syn}}^{\text{npt}(1)}(-Q^2) = -\alpha_{\text{syn}}^{\text{npt}(1)}(Q^2). \quad (21)$$

Here, in the ultraviolet expansion as well as in the infrared expansion there are no terms of the even powers of  $Q^2$ . In particular, the infrared expansion does not have the Coulomb's mode.

Indicate for completeness two boundary cases for the values  $c \in (1, +\infty)$  considered. The first case is  $c = 1$ , for which

$$\alpha_{\text{syn}}^{\text{npt}(1)}(Q^2) = \frac{4\pi}{b_0} \frac{\Lambda^4}{Q^2(\Lambda^2 - Q^2)},$$

and the nonperturbative component decreases at infinity not so fast as in expression (18). The second one is  $c = \infty$  for which

$$\alpha_{\text{syn}}^{\text{npt}(1)}(Q^2) = \frac{4\pi}{b_0} \frac{\Lambda^6}{Q^4(\Lambda^2 - Q^2)},$$

and the singularity in the infrared region is stronger,  $\alpha_{\text{syn}}^{(1)}(Q^2) \sim 1/(Q^2)^2$ ,  $Q^2 \rightarrow 0$ .

### 3 Multi-loop analytic running coupling of QCD

For the multi-loop case the renormalization group equation (3) for the QCD running coupling  $\alpha_s(Q^2)$  is of the form

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s) = -\beta_0 \alpha_s^2 - \beta_1 \alpha_s^3 - \beta_2 \alpha_s^4 - \beta_3 \alpha_s^5 + O(\alpha_s^6). \quad (22)$$

The coefficients  $\beta_0, \beta_1$  do not depend on the renormalization scheme choice, whereas the next coefficients do depend on this choice. For the numerical calculations we use its values within the  $\overline{MS}$ -scheme.

Let us write down the solution of equation (22) for  $\alpha_s(Q^2)$  at  $L = \ln(Q^2/\Lambda^2) \rightarrow \infty$  in the form of the standard expansion in inverse powers of the logarithms

$$\begin{aligned} \alpha_s(Q^2) = \frac{1}{\beta_0 L} \left\{ 1 - \frac{\beta_1}{\beta_0^2 L} \ln L + \frac{\beta_1^2}{\beta_0^4 L^2} \left[ \ln^2 L - \ln L - 1 + \frac{\beta_0 \beta_2}{\beta_1^2} \right] \right. \\ \left. - \frac{\beta_1^3}{\beta_0^6 L^3} \left[ \ln^3 L - \frac{5}{2} \ln^2 L - \left( 2 - \frac{3\beta_0 \beta_2}{\beta_1^2} \right) \ln L + \frac{1}{2} - \frac{\beta_0^2 \beta_3}{2\beta_1^3} \right] + O\left(\frac{1}{L^4}\right) \right\}. \end{aligned} \quad (23)$$

The sum of the terms of Eq. (23) up to  $1/L^n$  order ( $n = 1, 2, 3, 4$ ) will be referred to further on by the  $n$ -loop perturbative component of the running coupling and denoted as  $\alpha^{\text{pt}}(Q^2)$ . It can be written in the form

$$\alpha^{\text{pt}}(Q^2) = \frac{4\pi}{b_0} a(x), \quad (24)$$

$$\begin{aligned} a(x) = \frac{1}{\ln x} - b \frac{\ln(\ln x)}{\ln^2 x} + b^2 \left[ \frac{\ln^2(\ln x)}{\ln^3 x} - \frac{\ln(\ln x)}{\ln^3 x} + \frac{\kappa}{\ln^3 x} \right] \\ - b^3 \left[ \frac{\ln^3(\ln x)}{\ln^4 x} - \frac{5}{2} \frac{\ln^2(\ln x)}{\ln^4 x} + (3\kappa + 1) \frac{\ln(\ln x)}{\ln^4 x} + \frac{\bar{\kappa}}{\ln^4 x} \right]. \end{aligned} \quad (25)$$

Here  $x = Q^2/\Lambda^2$  and the coefficients

$$b = \frac{\beta_1}{\beta_0^2}, \quad \kappa = -1 + \frac{\beta_0 \beta_2}{\beta_1^2}, \quad \bar{\kappa} = \frac{1}{2} - \frac{\beta_0^2 \beta_3}{2\beta_1^3}. \quad (26)$$

The values of  $b_0$ ,  $b$ ,  $\kappa$ ,  $\bar{\kappa}$  depend on  $n_f$ . Within the standard picture of matching the solutions at heavy quark thresholds the parameter  $\Lambda$  becomes dependent on  $n_f$  <sup>3</sup>.

The analytic running coupling is defined through the dispersion relation

$$a_{\text{an}}(x) = \frac{1}{\pi} \int_0^{\infty} \frac{d\sigma}{x + \sigma} \rho(\sigma), \quad (27)$$

where the spectral density  $\rho(\sigma) = \Im a_{\text{an}}(-\sigma - i0)$ . The analytic approach suggests <sup>4</sup> that  $\Im a_{\text{an}}(-\sigma - i0) = \Im a(-\sigma - i0)$ . The corresponding spectral density  $\rho(\sigma)$  is shown in Fig. 1.

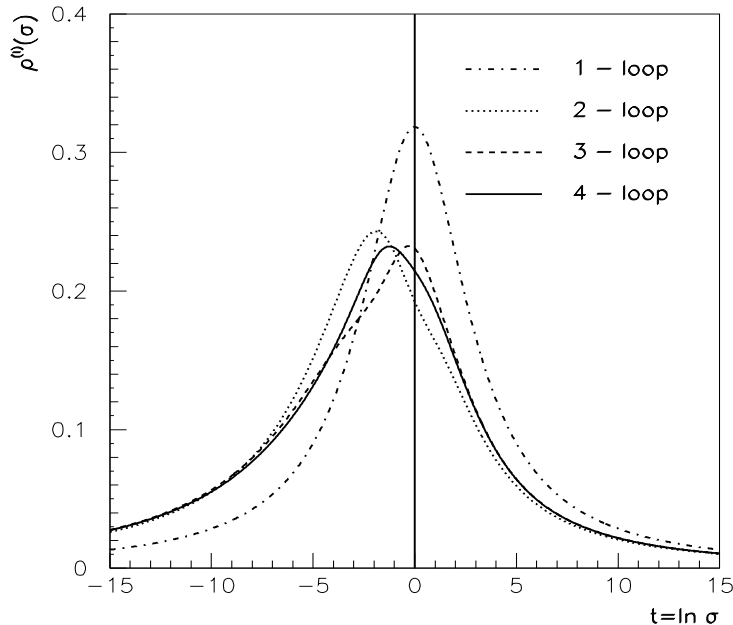


Fig. 1. The spectral density of the analytic running coupling up to four loop order.

As a result from function  $a(x)$  of the form (25) with nonphysical singularities on the positive real semiaxis of the complex plane  $x = Q^2/\Lambda^2$  we come to the function  $a_{\text{an}}(x)$  of the form (27), which is a single-valued analytic function in the complex plane  $x$  with a cut from 0 to  $-\infty$  (with a standard definition of the cuts of the logarithmic function). In Refs. [11,12] up to four-loop order the separation of the analytic running coupling into the perturbative and nonperturbative components was obtained,

$$\alpha_{\text{an}}(Q^2) = \alpha^{\text{pt}}(Q^2) + \alpha_{\text{an}}^{\text{npt}}(Q^2). \quad (28)$$

In Eq. (28) we take  $\alpha^{\text{pt}}(Q^2)$  as the initial (in our case the standard) solution of the renormalization group equation up to four-loop order and  $\alpha_{\text{an}}^{\text{npt}}(Q^2)$  appears to arise additionally as a result of the procedure. The following expansion was obtained in the power series

$$\alpha_{\text{an}}^{\text{npt}}(Q^2) = \frac{4\pi}{b_0} \sum_{n=1}^{\infty} c_n \left( \frac{\Lambda^2}{Q^2} \right)^n, \quad (29)$$

where the coefficients  $c_n$  were defined by the beta function coefficients. Note the important properties of expansion (29) such as the higher loop stability of the coefficient of the leading term and the slow increase of the coefficients  $c_n$  with number  $n$ .

<sup>3</sup>In perturbation theory the parameter  $\Lambda$  depends also on the renormalization scheme choice.

<sup>4</sup>This is precisely the variant of the “analytic improvement” procedure which we consider. The running coupling obtained in this way we call the analytic running coupling.

## 4 Multi-loop synthetic running coupling of QCD

The one-loop synthetic running coupling can be naturally extended to the multi-loop cases. Thus modify the analytic running coupling introducing two additional nonperturbative terms, the singular at zero term of the form  $\sim 1/Q^2$  and the mass term of the form  $\sim 1/(Q^2 + m_g^2)$ . As a result, we come to the expression

$$\alpha_{\text{syn}}(Q^2) = \alpha_{\text{an}}(Q^2) + \frac{4\pi}{b_0} \left[ \frac{c\Lambda^2}{Q^2} - \frac{d\Lambda^2}{Q^2 + m_g^2} \right], \quad (30)$$

containing, besides  $\Lambda$ , three subsidiary parameters:  $c$ ,  $d$  and  $m_g$  ( $m_g \equiv m_\Lambda \Lambda$ ), which are assumed to be nonzero. We will define this parameters in the following way. At large  $Q^2$  ( $Q^2 > \Lambda^2$ ), using expansion (29), we can find

$$\alpha_{\text{syn}}(Q^2) = \alpha^{\text{pt}}(Q^2) + \frac{4\pi}{b_0} \left[ (c_1 + c - d) \frac{\Lambda^2}{Q^2} + (c_2 + dm_\Lambda^2) \left( \frac{\Lambda^2}{Q^2} \right)^2 + (c_3 - dm_\Lambda^4) \left( \frac{\Lambda^2}{Q^2} \right)^3 \right] + O\left( \left( \frac{\Lambda^2}{Q^2} \right)^4 \right). \quad (31)$$

Demand the nonperturbative contributions to be minimal at large  $Q^2$ , i.e. the terms of the form  $\sim 1/Q^2$ ,  $\sim 1/(Q^2)^2$  to be absent in Eq. (31)<sup>5</sup>. Then two of three parameters are fixed by the following equations

$$d = c + c_1, \quad m_\Lambda^2 = -c_2/(c + c_1). \quad (32)$$

The parameter  $\Lambda_1 = \sqrt{c}\Lambda$  will be considered as fixed. The coefficients  $c_n < 0$ , therefore the absence of tachion condition is  $\Lambda < \Lambda_1/\sqrt{-c_1}$ . With a given number of loops the free parameter of the synthetic running coupling is only one parameter  $\Lambda$ . Then taking into account Eq. (32) we have

$$\alpha_{\text{syn}}(Q^2) = \alpha^{\text{pt}}(Q^2) + \frac{4\pi}{b_0} \left[ c_3 - \frac{c_2^2 \Lambda^2}{\Lambda_1^2 + c_1 \Lambda^2} \right] \left( \frac{\Lambda^2}{Q^2} \right)^3 + O\left( (Q^2)^{-4} \right). \quad (33)$$

As seen from Eq. (33), the leading power nonperturbative term decreases rapidly at  $Q^2 \rightarrow \infty$  and in the absence of the tachion is negative<sup>6</sup>. For the dynamical gluon mass from Eqs. (32) we obtain

$$m_g = \Lambda \sqrt{\frac{-c_2 \Lambda^2}{\Lambda_1^2 + c_1 \Lambda^2}}. \quad (34)$$

Let us turn to the interpretation of the parameter  $\Lambda_1$  which describes the value of the singular term. As it has already been noted, the behavior of the running coupling of the form  $\alpha_V \sim 1/Q^2$  at  $Q^2 \rightarrow 0$  in the V-scheme corresponds to the linear confining quark-antiquark static potential which is defined by the vacuum expectation value of the Wilson loop. We set up a correspondence of the potential and the running coupling using the equation (to compare with Refs. [31] and [27])

$$V(r) = -\frac{4}{3} \int \frac{d^n q}{(2\pi)^n} \exp(i\mathbf{q}\mathbf{r}) \frac{4\pi\alpha_V(q^2)}{q^2} \Big|_{n=3}, \quad (35)$$

where  $\alpha_V(q^2)$  is defined as an effective charge, which is the renormalization scheme independent and gauge invariant quantity. The color factor corresponds to the  $SU(N_c)$  group,  $N_c = 3$ . Let us assume that in the infrared region

$$\alpha_V(q^2) \simeq \frac{3}{2} \frac{\sigma}{q^2}, \quad q^2 \rightarrow 0. \quad (36)$$

<sup>5</sup>This condition corresponds to the principle of minimality of the nonperturbative contributions in the perturbative ultraviolet region [24, 25].

<sup>6</sup>The nonperturbative ‘‘tail’’ of the running coupling  $\alpha_{\text{syn}}(Q^2)$  as a whole turns out to be negative at large  $Q^2$ .



Then the integral over three dimensional momentum space in Eq. (35) formally diverges at the origin. We define this integral introducing the dimensional regularization. After integration over the  $n$ -dimensional Euclidean momentum space we put  $n = 3$ . Then as the divergences do not occur, the transition to the  $n$ -dimensional integration provides not only the regularization but the definition of the divergent integral for the case  $n = 3$ . As a result, for the infrared behavior of the effective charge (36) the behavior of the potential at large distances is as follows:

$$V(r) \simeq \sigma r, \quad r \rightarrow \infty, \quad (37)$$

where  $\sigma \equiv a^2$  is the string tension parameter. Let us define the parameter  $\Lambda_1$  of the synthetic running coupling  $\alpha_{\text{syn}}$  from the correspondence of the singular at zero term in Eq. (30) to the infrared behavior (36) of the running coupling  $\alpha_V$ . Then

$$\frac{3}{2}\sigma = \frac{4\pi}{b_0}\Lambda_1^2, \quad \Lambda_1^2 = c\Lambda^2. \quad (38)$$

Therefore, if the string tension parameter is given, the parameter  $\Lambda_1$  can be specified by Eq. (38). Then with  $a \simeq 0.42$  GeV,  $b_0 = 9$  we obtain <sup>7</sup>  $\Lambda_1 \simeq 435$  MeV. The parameter  $\Lambda$  (as well as the parameter  $c$ ) can be fixed by the normalization condition, and the synthetic running coupling will be fixed completely.

Consider the dependence of the dynamical gluon mass on  $\Lambda$  for different number of loops of the initial perturbation theory approximation.

In Fig. 2 the dynamical gluon mass  $m_g(\Lambda)$  is shown for one – four cases. Up to 400 MeV the curves do not diverge too much and at  $\Lambda = 375$  MeV  $m_g \simeq 0.6$  GeV.

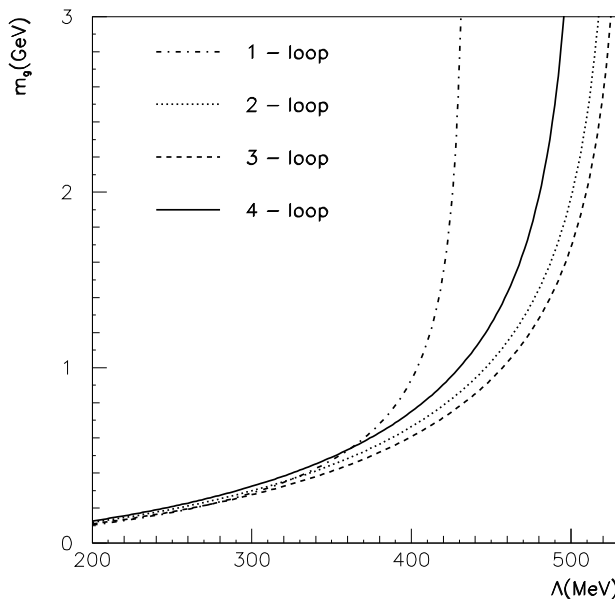


Fig. 2. The dynamical gluon mass  $m_g$  as a function of  $\Lambda$  for different number of loops of the initial perturbation theory approximation.

We normalize the running couplings  $\alpha_{\text{syn}}(Q^2)$ ,  $\alpha_{\text{an}}(Q^2)$  and  $\alpha^{\text{pt}}(Q^2)$  at the  $\tau$ -lepton mass by [33, 34]  $\alpha(M_\tau^2) = 0.33$ ,  $M_\tau = 1.777$  GeV. For this normalization condition the values of the parameters  $\Lambda$ , the dynamical gluon mass  $m_g$  and the parameters  $c$ ,  $d$  are given in Table 1. Point to two things. The parameters  $\Lambda_{\text{syn}}$  and  $\Lambda_{\text{pt}}$  are close in value whereas the parameters  $\Lambda_{\text{an}}$  are considerably larger.

<sup>7</sup>The string tension parameter  $\sigma = (0.42 \text{ GeV})^2$  in particular, in the relativistic string model with massive quarks at the ends of the string [32]. Then the slope of Regge trajectories  $\alpha' = 1/(2\pi\sigma) \simeq 0.90 \text{ GeV}^{-2}$ .

This is a consequence of conditions (32) which give the fast decrease of the nonperturbative terms of  $\alpha_{\text{syn}}(Q^2)$  at large  $Q^2$ . For all quantities considered the one-loop case turns out to be exceptional, and then the stabilization is observed with the number of loops of the initial perturbation theory approximation.

**Table 1.** The parameters  $\Lambda_{\text{pt}}$  (MeV),  $\Lambda_{\text{an}}$  (MeV),  $\Lambda_{\text{syn}}$  (MeV), the dynamical gluon mass  $m_g$  (MeV) and the parameters  $c$ ,  $d$  on the number of loops. The number of active quark flavors  $n_f = 3$ ,  $\Lambda_1 = 435$  MeV. The normalization condition is  $\alpha(M_\tau^2) = 0.33$ ,  $M_\tau = 1.777$  GeV.

	1-loop	2-loop	3-loop	4-loop
$\Lambda_{\text{pt}}$	214.25	395.10	364.19	357.32
$\Lambda_{\text{an}}$	254.51	636.02	523.86	535.54
$\Lambda_{\text{syn}}$	214.26	397.10	365.26	358.70
$m_g$	121.14	648.09	461.66	526.57
$c$	4.1282	1.2018	1.4204	1.4728
$d$	3.1282	0.5358	0.7706	0.7447

## 5 Conclusions

In the construction of the QCD analytic running coupling (27) the nonphysical singularities of the perturbation theory in the infrared region disappear and in the ultraviolet region the nonperturbative power corrections arise decreasing rapidly at large  $Q^2$ , in comparison with the main perturbative component. However, when considering the nonperturbative quantities it may happen that the decrease of the nonperturbative contributions is not fast enough for the consistent definition of these quantities. In the synthetic running coupling it is proposed to provide the highest possible suppression of the nonperturbative contributions at large  $Q^2$  by means of a minimal number of the pole type terms. The parameters characterizing the additional nonperturbative terms have clear physical meaning and take the reasonable values. The running coupling (1) built on the basis of the analytic running coupling (5) is called the one-loop synthetic running coupling because it contains the parameters related to the ultraviolet region as well as to the infrared region. We introduce the singular at zero term of the form  $\sim 1/Q^2$  which corresponds to the linear quark confinement and the mass term of the form  $\sim 1/(Q^2 + m_g^2)$  with the parameter  $m_g^2$  corresponding to the non-vanishing dynamical gluon mass. We impose the condition of the fastest decrease of the nonperturbative component at large  $Q^2$  and receive the running coupling model of form (1). The model  $\alpha_{\text{syn}}^{(1)}(Q^2)$  has two independent parameters, the dimensional parameter  $\Lambda$  and dimensionless parameter  $c$  which defines the value of the singular term.

In Section 2 the one-loop model of the synthetic running coupling and its nonperturbative component properties for  $c \in (1, +\infty)$  are considered. A study of the multi-loop analytic running coupling and its nonperturbative component provide a possibility of natural generalization of the synthetic running coupling model to the multi-loop cases. The multi-loop synthetic running coupling (30) as the one-loop model is made by introducing two additional nonperturbative terms of the form  $\sim 1/Q^2$  and  $\sim 1/(Q^2 + m_g^2)$ . The minimality principle of the nonperturbative contributions in the perturbative region leads to two equations (32) for the introduced nonperturbative parameters.

As a result, the synthetic running coupling has two independent parameters. First, the parameter  $\Lambda$  which owing to highly fast decrease of the nonperturbative contributions at large  $Q^2$  practically coincides with the parameter  $\Lambda_{\text{QCD}}$  in the region of application of the perturbative solutions. Second, the dimensionless parameter  $c$  (or the dimensional parameter  $\Lambda_1 = \sqrt{c}\Lambda$ ) determining the value of the singular term. Correlating this parameter responsible for the infrared enhancement by Eqs. (38) with the string tension parameter  $\sigma$  of the string models, we arrive at the dynamical gluon mass as a function of the parameter  $\Lambda$  defined by Eq. (34).

The corresponding dependencies for 1–4-loop cases are given in Fig. 1 with  $\Lambda_1 = 435$  MeV (that is  $\sigma^{1/2} = 0.42$  GeV). The normalization completely defines the synthetic running coupling and for  $\alpha(M_\tau^2) = 0.33$  the values of the parameters are shown in Table 1. The parameter  $m_g$  in Eq. (34) for  $\Lambda < \Lambda_1/\sqrt{-c_1}$  is real, and the synthetic running coupling  $\alpha_{\text{syn}}(Q^2)$  in Eq. (30) is a holomorphic function in the complex plane  $Q^2$  with a cut along the real negative semiaxis. Thus the synthetic running coupling of QCD  $\alpha_{\text{syn}}(Q^2)$  has the properties outlined in Section 2 for the one-loop synthetic running coupling model.

As seen in Table 1 the parameters of the synthetic running coupling fixed in such a way show the higher loop stabilization with the one-loop case excluded. In particular, the dynamical gluon mass  $m_g$  can be estimated as 400–600 MeV. For  $\sigma = (0.42 \text{ GeV})^2$  and  $\alpha_{\text{syn}}(M_\tau^2) = 0.32, 0.33, 0.34$  it is obtained that  $m_g = 453, 527, 613$  MeV (for 4-loop case). Hence the string parameter identification of the parameter of the synthetic running coupling defining the value of the singular term results in the consistent values for the other parameters considered.

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## References

- [1] P.J. Redmond, Phys. Rev. **112**, 1404 (1958).
- [2] N.N. Bogoliubov, A.A. Logunov, and D.V. Shirkov, Zh. Éksp. Teor. Fiz. **37**, 805 (1959) [Sov. Phys. JETP **37**, 574 (1960)].
- [3] D.V. Shirkov and I.L. Solovtsov, JINR Rapid Comm. **2[76]-96**, 5 (1996); D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. **79**, 1209 (1997).
- [4] Yu.L. Dokshitzer, G. Marchesini, and B.R. Webber, Nucl. Phys. **B469**, 93 (1996).
- [5] I.L. Solovtsov and D.V. Shirkov, Teor. Mat. Fiz. **120**, 482 (1999) [Theor. Math. Phys. **120**, 1210 (1999)].
- [6] D.V. Shirkov, Teor. Mat. Fiz. **136**, 3 (2003) [Theor. Math. Phys. **136**, 893 (2003)].
- [7] B.A. Magradze, Int. J. Mod. Phys. A **15**, 2715 (2000); B.A. Magradze, Comm. of JINR, E2-2000-222, Dubna, 2000; D.S. Kourashev and B.A. Magradze, Teor. Mat. Fiz. **135**, 95 (2003) [Theor. Math. Phys. **135**, 531 (2003)].
- [8] A.I. Alekseev, Phys. Rev. D **61**, 114005 (2000).
- [9] A.I. Alekseev, in *Proceedings of the XVth Workshop on High Energy Physics and Quantum Field Theory*, Tver, Russia, September 7-13, 2000, edited by M.N. Dubinin and V.I. Savrin (INP Moscow State University, Moscow, 2001), p. 290, hep-ph/0011178.
- [10] A.I. Alekseev, Yad. Fiz. **65**, 1722 (2002) [Phys. Atom. Nucl. **65**, 1678 (2002)].
- [11] A.I. Alekseev, J. Phys. G **27**, L117 (2001).
- [12] A.I. Alekseev, Few-Body Syst. **32**, 193 (2003).
- [13] Yu.A. Simonov, Yad. Fiz. **65**, 140 (2002); Yad. Fiz. **66**, 796 (2003) [ Phys. Atom. Nucl. **66**, 764 (2003)].

- [14] D.V. Shirkov, *Teor. Mat. Fiz.* **132**, 484 (2002) [*Theor. Math. Phys.* **132**, 1309 (2002)].
- [15] A. V. Nesterenko, *Int. J. Mod. Phys. A* **18**, 5475 (2003).
- [16] V. Gogohia, *Phys. Lett. B* **584**, 225 (2004).
- [17] A.I. Alekseev and B.A. Arbuzov, *Mod. Phys. Lett. A* **20**, 103 (2005).
- [18] G. Burgio, F. Di Renzo, G. Marchesini, and E. Onofri, *Phys. Lett. B* **422**, 219 (1998), *Nucl. Phys. B (Proc. Suppl.)* **63**, 805 (1998); G. Burgio, F. Di Renzo, C. Parrinello, and C. Pittori, *ibid.* **73**, 623 (1999), *ibid.* **74**, 388 (1999); Ph. Boucaud *et al.*, *JHEP* **0004**, 006 (2000).
- [19] L. von Smekal, A. Hauck, and R. Alkofer, *Ann. Phys.* **267**, 1 (1998).
- [20] C.D. Roberts and S.M. Schmidt, *Prog. Part. Nucl. Phys.* **45**, S1 (2000).
- [21] R. Alkofer and L. von Smekal, *Phys. Rept.* **353**, 281 (2001).
- [22] A.I. Alekseev, *Yad. Fiz.* **33**, 516 (1981) [*Sov. J. Nucl. Phys.* **33**, 269 (1981)]; A.I. Alekseev, B.A. Arbuzov, and V.A. Baikov, *Teor. Mat. Fiz.* **52**, 187 (1982) [*Theor. Math. Phys.* **52**, 739 (1982)]; A.I. Alekseev and V.F. Edneral, *Yad. Fiz.* **45**, 1105 (1987) [*Sov. J. Nucl. Phys.* **45**, 685 (1987)].
- [23] A.I. Alekseev, in *Proc. of the XII International Workshop on High Energy Physics and Quantum Field Theory*, Samara, Russia, September 4-10, 1997, edited by B.B. Levchenko (MSU, Moscow, 1999), p. 334; IHEP Preprint 97-90, Protvino, 1997, hep-ph/9802372.
- [24] A.I. Alekseev and B.A. Arbuzov, *Yad. Fiz.* **61**, 314 (1998) [*Phys. Atom. Nucl.* **61**, 264 (1998)].
- [25] A.I. Alekseev and B.A. Arbuzov, *Mod. Phys. Lett. A* **13**, 1447 (1998).
- [26] A.I. Alekseev, in *Proc. of the Workshop on Nonperturbative Methods in Quantum Field Theory*, Adelaide, Australia, February 2-13, 1998, edited by A.V. Schreiber, A.G. Williams, and A.W. Thomas (World Scientific, Singapore, 1998), p. 104; IHEP Preprint 98-41, Protvino, 1998, hep-ph/9808206.
- [27] W. Buchmüller, G. Grunberg, and S.-H.H. Tye, *Phys. Rev. Lett.* **45**, 103 (1980); W. Buchmüller and S.-H.H. Tye, *Phys. Rev. D* **24**, 132 (1981).
- [28] V.V. Kiselev, A.E. Kovalsky, and A.I. Onishchenko, *Phys. Rev. D* **64**, 054009 (2001).
- [29] V.V. Kiselev and A.K. Likhoded, *Usp. Fiz. Nauk* **172**, 497 (2002) [*Phys. Usp.* **45** (5), (2002)].
- [30] D.J. Gross and F. Wilczek, *Phys. Rev. Lett.* **30**, 1343 (1973); H.D. Politzer, *Phys. Rev. Lett.* **30**, 1346 (1973).
- [31] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1980).
- [32] L.D. Soloviev, *Phys. Rev. D* **58**, 035005 (1998); *ibid.* **61**, 015009 (2000).
- [33] *The Review of Particle Physics*, S. Eidelman *et al.*, *Phys. Lett. B* **592**, 1 (2004).
- [34] A.A. Pivovarov, Institute for Nuclear Research, Preprint No. INR-TH-03-3, Moscow, 2003, hep-ph/0301074.