# CONFINEMENT AS A CONSEQUENCE OF MONOPOLE CONDENSATION

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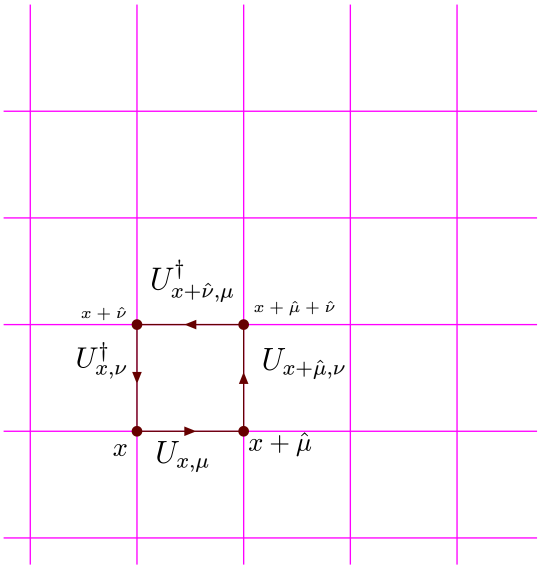
The role of monopoles in compact Euclidean QED is discussed. Some ideas how to reduce non-Abelian gauge theories to QED with monopoles are reviewed.

## 1 Lattice Action and Confinement Criterion




The lattice (Wilson) action of the $SU(N)$-invariant gauge theory can be represented in the form

$$S = \frac{2N}{g^2}\sum_{P=x,\mu,\nu}\left(1-\frac{1}{N}\,\mathrm{Re}\;Tr\;U_P\right)\ , \tag{1}$$

where

$$U_P = U_{x,\mu}U_{x+\hat\mu,\nu}U^\dagger_{x+\hat\nu,\mu}U^\dagger_{x,\nu} \tag{2}$$

and $U_{x,\mu}$ are the link variables that can be expressed in terms of the vector potential $A_\mu(x) = A^a_\mu(x)T^a$:

$$U_{x,\mu} = \exp\left(\frac{igA_\mu}{2}\right), \tag{3}$$

$T^a$ being the generators of the $SU(N)$ group.

This action is invariant under gauge transformations:

$$\Lambda:\quad U_{x,\mu}\to\Lambda^\dagger_x U_{x,\mu}\Lambda_{x+\hat\mu}\ , \tag{4}$$

$U,\Lambda\in SU(N)$.

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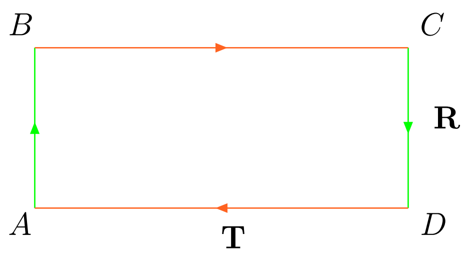
The gauge-invariant operator for creation of the quark–antiquark pair has the form

$$\bar{q}(x)W(x,x';C)q(x'), \tag{5}$$

where

$$W(x,x';C) = Pexp(ig\int_x^{x'} A_\mu^a(x)T^a dx^\mu). \tag{6}$$




The function

$$\Omega(T,R) = \langle 0| \ \ (\bar{q}(A)W((A,B)q(B))^\dagger \qquad \bar{q}(C)W(C,D)q(D) \ |0\rangle \tag{7}$$

can be represented in the form

$$\Omega(T,R) = \sum_n |\langle 0|\,(\bar{q}(A)W((A,B)q(B))^\dagger\,|n\rangle|^2 \exp(-E_nT), \tag{8}$$

where $E_n = E(R)$ is the rest energy of the quark-antiquark pair associated with the potential of quark–antiquark interaction. In the approximation of infinitely heavy static quarks, the quark propagator in the external fields takes the form

$$\langle 0|q(D)\bar{q}(A)|0\rangle \simeq e^{-2m_qT}\ W(D,A), \tag{9}$$

where the factor $W(D,A) = \int_D^A A_0(\vec{x},t)dt$ accounts for the energy of the quark in the chromoelectric field. Thus the function $\Omega(T,R)$ is approximated by

$$\Omega(T,R) \simeq W(x,x;C)e^{-2mT}\ . \tag{10}$$

The behavior of the Wilson loop

$$W(C) = Pexp(ig\int_C A_\mu^a(x)T^a dx^\mu) \sim e^{-T(E(R)-2m)} \tag{11}$$

at $T \to \infty$ and $R \to \infty$ is the most widely used confinement criterion.

In the strong-coupling limit the Wilson loop has the form

$$\begin{aligned} W(C) &= \frac{1}{Z}\int \prod_l DU_l W(x,x;C)\exp(\frac{-1}{2g^2}\sum_p \mathrm{Tr}\,U_P) \\ &\simeq \left(\frac{1}{g^2}\right)^{N_p} = \exp\left(-\frac{TR\ln(Ng^2)}{a^2}\right). \end{aligned} \tag{12}$$

In the weak-coupling limit we obtain the perimeter law

$$W(C) \simeq \exp(-\gamma(R+T)).$$

# 2 Magnetic Monopoles

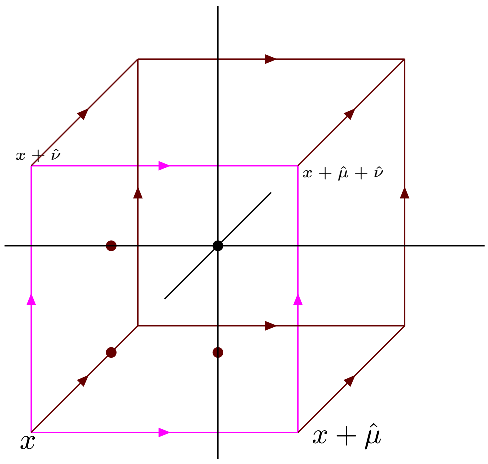
## 2.1 Monopoles on a lattice

Let us consider QED in three-dimensional Euclidean space using lattice regularization. We assume that the action be given by the sum over plaquettes

$$S = \sum_{x,\mu,\nu} \frac{-1}{2}(1 - cos(\phi_p)), \tag{13}$$

where the plaquette variable $\phi_p = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ coincides with the field strength tensor ($\partial_\mu A_\nu$ here is nothing but $A_\nu(x+\hat{\mu}) - A_\nu(x)$). In this case, the vector potential represents a compact variable because the change of $A_\nu(x+\hat{\mu}) - A_\nu(x)$ by $2\pi N, N \in Z$ leaves the action invariant. The partition function for such a theory is dominated by the solutions of the Euclidean "equations of motion". Dirac monopoles in the **compact** theory can exist because Dirac threads needed for flux conservation do not contribute to the action.

Thus the partition function of the $QED_3$ is dominated by monopoles, and the Wilson loop, as it was evaluated in [1], gives evidence for confinement. Therewith, the monopoles cannot provide confinement in compact $QED_4$ (in four dimensions, "monopoles" are loops rather than point-like objects). Since these are the only classical solutions with finite action, there is no confinement in the $QED_4$ at small couplings.




The plaquette variable has the form (for notation see the Appendix)

$$\phi_{\mu\nu} = \theta_\mu(x) + \theta_\nu(x+\hat{\mu}) - \theta_\mu(x+\hat{\nu}) - \theta_\nu(x). \tag{14}$$

$$-\pi < \theta_\lambda(y) < \pi. \tag{15}$$

The abelian magnetic flux through the surface of the cube $C$ is given by

$$m = \frac{1}{2\pi} \sum_{P \in \partial C} \bar{\phi}, \tag{16}$$

where $\bar{\phi}_{P=(x;\mu\nu)}$ is defined by the relation

$$\phi_{\mu\nu}(x) = \bar{\phi}_{\mu\nu}(x) + 2\pi n_{\mu\nu}(x) \tag{17}$$

with

$$-\pi < \bar{\phi}_{\mu\nu} < \pi.$$

It should be emphasized that the monopole on a lattice furnishes a fictitious object, “quasiparticle”, associated with a definite field configuration. For example, the action in 3D Euclidean QED approaches its minimum at the configurations that can be interpreted as the monopole fields. In other words, solutions of classical Euclidean “equations of motions” are parameterized in terms of monopoles.

## 2.2 A sketch of computations in the Abelian case

The partition function of the compact $U(1)$ gauge theory

$$\mathcal{Z} = \int_{-\pi}^{\pi} \prod_{links} \mathcal{D}\theta_l \exp(-S[d\theta]) \tag{18}$$

(for notation and useful relations see the Appendix). The Wilson action in terms of the plaquette variable $\phi_P = d\theta$

$$S_W(\phi) = \frac{1}{g^2}(1 - cos(\phi)) \tag{19}$$

gives the generating functional (Fourier transform of $\exp(-S_W)$)

$$\int_{-\pi}^{\pi} \prod_P d\phi_P \exp(-S_W[\phi_P] + in\phi_P) = \int_{-\pi}^{\pi} dz \cos(zn_P) \exp(\beta \cos(z)), \tag{20}$$

where $\beta = \dfrac{1}{g^2}$. For this reason, one should begin with the Villain action

$$S_V(\phi_P) = -\ln \sum_{l=-\infty}^{\infty} \exp\left(\frac{-1}{g^2}(\phi_P + 2\pi l)^2)\right). \tag{21}$$

The Fourier transform of $\exp(-S_V)$ with respect to $\phi_P$ is rather simple:

$$\exp(-S_V(\phi_P)) = \frac{g}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \exp(in\phi_P)\ \exp\left(-\frac{1}{2}g^2 n^2\right). \tag{22}$$

The expectation value of the Wilson loop is readily expressed in terms of the introduced chains. Let $j_l = 1$ if the link $l$ belongs to the Wilson loop and 0 otherwise. Then

$$\langle W_C \rangle = \int_{-\pi}^{\pi} \prod_{links} d\theta_l \sum_{l_p=-\infty}^{\infty} \exp\left(\frac{\langle d\theta - 2\pi l_P, d\theta - 2\pi l_P \rangle}{g^2}\right) e^{i\langle j_l \theta_l \rangle}. \tag{23}$$

This expression can be rewritten in terms of the dual lattice:

$$\langle W_C \rangle = \int_{-\pi}^{\pi} \prod_{cubes} d\theta_c \sum_{l_p=-\infty}^{\infty} \exp\left(\frac{\langle \delta\theta - 2\pi l_P, \delta\theta - 2\pi l_P \rangle}{g^2}\right) e^{i\langle, \tilde{j}_c \theta_c \rangle}, \tag{24}$$

where $\tilde{j}_c = j_l^*$ and $\theta_c = \theta_l^*$.

After the Banks–Myerson–Kogut transformation[4], the expectation value of the Wilson loop is expressed in terms of the magnetic currents

$$\langle W_C \rangle = A_0[C] E[C], \tag{25}$$

where

$$A_0[C] = \exp\left(\frac{-g^2}{2}\langle \tilde{j}_c \Box^{-1} \tilde{j}_c, \rangle\right) \tag{26}$$

and

$$E[C] = \sum m_l : \delta m = 0 \exp\left(\frac{-2\pi^2}{g^2}\langle m, \Box^{-1} m\rangle + 2\pi i \langle \tilde{j}_c, \Box^{-1}\nabla G\{m\}\rangle\right), \tag{27}$$

where $G\{m\}$ is a particular choice of $\delta^{-1}\{m\}$. Here, the link variables $m$ describe the magnetic current density. The functional $E[C]$ was estimated in [3], where it was shown that the confinement–deconfinement phase transition in compact $\mathrm{QED}_4$ occurs at $g^2 = 0.168$. It was also observed [6] in simulations that monopoles are numerous in the confinement phase and rare in the deconfinement phase. For this reason, it is tempting to search for the

# 3 Monopoles in Non-Abelian Theories

## 3.1 Abelian Projection

The principle of the Abelian projection is to fix a gauge as locally as possible (in the case of the covariant gauge,

$$\partial_\mu A^a_\mu = 0 \quad \longrightarrow \qquad \text{FP operator} \quad \partial_\mu D^{ab}_\mu \Lambda^a \tag{28}$$

the inverse of the Faddeev–Popov operator is extremely unlocal).

There exists a possibility to fix a gauge in two stages.

First we pick a field $X$ in the adjoint representation of the gauge group. Say, $X = G^i_{12} j$ ($A^a_\mu$ is ruled out owing to the inhomogeneous term). It transforms as follows:

$$X \to X^\Lambda = \Lambda^\dagger X \Lambda. \tag{29}$$

Second, a gauge transformation at each point is performed so that $X$ be diagonal:

$$X^\Lambda = \begin{pmatrix} \lambda_1 & ... & 0 \\ ..... & ... & ..... \\ 0 & ... & \lambda_n \end{pmatrix}, \quad \text{where} \quad \lambda_1 > \lambda_2 > ... > \lambda_n \ .$$

There remains $U(1)^{N-1}$ gauge invariance.

However, there are singularities such that $\lambda_i = \lambda_j$. To treat them, let us restrict our attention to the case of $SU(2)$ symmetry. Let the above-mentioned field $X$ be represented in the form

$$X = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix}.$$

The eigenvalues of this matrix are $\lambda_{1,2} = X_0 \pm |\vec{\mathbf{X}}|$, they coincide provided that $X_1 = X_2 = X_3 = 0$ **three conditions are met.**

$\Longrightarrow$ the singularities occur at isolated points.

Let us consider a vicinity of such a point, where the field $X$ is a diagonal term plus a small 3-vector $\vec{\varepsilon}$:

$$X = \begin{pmatrix} X_0 + \varepsilon_3 & \varepsilon_1 - i\varepsilon_2 \\ \varepsilon_1 + i\varepsilon_2 & X_0 - \varepsilon_3 \end{pmatrix}.$$

The gauge transformation $\Lambda$ at this point transforms the gauge potential of the remaining $U(1)$ symmetry to the potential of the magnetic monopole located at the point $\vec{x}$ where $X$ has the diagonal form. Dirac monopoles here arise from.

### The 't Hooft–Polyakov monopoles

The equations of motion determined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \frac{1}{2}D_\mu\phi^a D_\mu\phi^a - \frac{1}{4}\lambda(\phi^a\phi^a - F^2)^2 \tag{30}$$

allow a **soliton** solution of the form

$$\begin{aligned} \phi^a &= \delta_{ia}\frac{x_i}{r}F(r), \\ A^a_i &= \epsilon^{aij}\frac{x_j}{r}W(r), \\ A^a_0 &= 0, \qquad \text{where} \\ F(r) &\to F(r\to\infty), \\ W(r) &\sim \frac{1}{gr}(r\to\infty), \end{aligned} \tag{31}$$

which can be interpreted as a monopole. Let us introduce a gauge-invariant quantity

$$F_{\mu\nu} = \hat{\phi}^a G^a_{\mu\nu} - \frac{1}{g}\epsilon^{abc}\hat{\phi}^a D_\mu\phi^b D_\nu\phi^c, \tag{32}$$

For the above soliton solution it behaves as

$$F_{ij} = \epsilon_{ijk}B_k = \epsilon_{ijk}\frac{r_k}{gr^3}\ , \tag{33}$$

as $x$ tends to the infinity. Then, in the unitary gauge $\phi^1 = \phi^2 = 0,\ \phi^3 > 0$, $F_{\mu\nu}$ has the form

$$F_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu\ , \tag{34}$$

$B_k$ being associated with the magnetic field. The above soliton solution can be transformed to the unitary gauge by a singular gauge transformation, the resulting vector potential is characteristic of the Dirac monopole.

## 3.2 Maximum Abelian Gauge (MAG)

However, the most simple "diagonal" gauge consistent with the renormalizability requirements is only approximately non-propagating.

In the case of $SU(2)$ group, the gauge-fixing functional has the form

$$\Phi = \sum_{x,\mu}\mathrm{Tr}\left(U_{x,\mu}\sigma_3 U^\dagger_{x,\mu}\sigma_3\right). \tag{35}$$

where

$$U_{x,\mu} = \begin{pmatrix} u_0 + iu_3 & u_2 + iu_1 \\ -u_2 + iu_1 & u_0 - iu_3 \end{pmatrix}, \qquad U^\dagger_{x,\mu} = \begin{pmatrix} u_0 - iu_3 & -u_2 - iu_1 \\ u_2 - iu_1 & u_0 + iu_3 \end{pmatrix}.$$

The stationarity condition of this functional makes the gauge condition:

$$\begin{aligned} a\nabla^B_\mu(u_2u_3 - u_0u_1) - 2u_2u_3 &= 0, \\ a\nabla^B_\mu(u_1u_3 + u_0u_2) - 2u_1u_3 &= 0, \end{aligned} \tag{36}$$

$u_i$ denotes $u_i(x,\mu)$.

In the naive continuum limit, these gauge conditions have the form

$$(\partial_\mu + igA^3_\mu)(A^1_\mu + iA^2_\mu) = 0, \qquad (37)$$
$$(\partial_\mu - igA^3_\mu)(A^1_\mu - iA^2_\mu) = 0.$$

The maximum Abelian gauge can be used to perform the so called Abelian projection - construct an effective model in which nondiagonal components of the vector potential are neglected and the Wilson loop is dominated by the contribution of the $U(1)$ fields and the monopoles, which are responsible for confinement.

For the $SU(2)$ group, the monopole currents under consideration are defined by [7]

$$j^* = \frac{1}{2\pi} * d(d\theta \mathrm{mod} 2\pi), \qquad (38)$$

where $\theta$ is the phase of the diagonal element of a link matrix: $U_{ii} = |U_{ii}| \exp(i\theta_i)$. It was shown (for a review, see [8]) that the expectation value of the Wilson loop, after transformations like in the case of the compact QED$_4$, indicates confinement in the case of non-Abelian theories and the concentration of monopole currents is strongly correlated with the confinement–deconfinement phases, as in the Abelian case. However, it should be emphasized, that monopoles in the non-Abelian case appear as a result of gauge fixing or, figuratively speaking, they account for the contribution of non-Abelian degrees of freedom.

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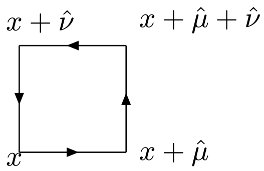
# Appendix

## Differential forms on a lattice

Let $C_k$ be a $k$-dimensional primitive cell.

- A site $C_0$ is defined by a dot $x$.
- A link $\{x, \mu\}$ is defined by a dot with an index $x_\mu$, $\mu = 1,\ 2,\ 3,\ 4$. $x$ indicates the beginning of a link, $\mu$—its direction.
- A plaquette $\{x, \mu, \nu\}$ $(\mu < \nu)$:




- A cube ($C_3$ chain) is designated by $\{x, \mu, \nu, \lambda\}$.
- and so on

The oriented chains form an Abelian group.

Then we consider "forms", that is, functions on $k$-chains: $\phi = \phi(C_k)$.
Scalar field is a function on sites $x$,
Vector field $U_{x,\mu}$ is a function on links,
Tensor fields (as the curvature tensor) is a function on plaquettes $F_{(x,\mu,\nu)}$,
$J_{\mu\nu\lambda}$ is a function on cubes.
The incidence function of two cells is defined by

$$I(C_k, C_{k+1}) = I(\{x, \mu_1, ..., \mu_k\}\{y, \nu_1, ..., \nu_{k+1}\}) = \tag{39}$$

$$= \sum_{i=1}^{k} (\delta_{x,y} - \delta_{x,y+\hat{\nu}_i})\delta^{\nu_1}_{\mu_1} ......\delta^{\nu_{i-1}}_{\mu_{i-1}} \delta^{\nu_{i+1}}_{\mu_i} ......\delta^{\nu_{k+1}}_{\mu_k} \ . \tag{40}$$

For any function $f(C_k)$ we define the exterior derivative

$$df(C_{k+1}) = \sum_{C_k} I(C_k, C_{k+1}) f(C_k) \tag{41}$$

and the coderivative

$$\delta f(C_{k+1}) = \sum_{C_k} I(C_{k-1}, C_k) f(C_k). \tag{42}$$

Examples of external derivatives are provided by the derivative of a scalar field

$$df(\{x, \mu\}) = f(x + \mu) - f(x)$$

and the derivative of a vector field

$$dF(\{x, \mu, \nu\}) = F(x, \mu) + F(x + \hat{\mu}, \nu) - F(x + \hat{\nu}, \mu) - F(x, \nu).$$

$$d^2 = 0, \qquad \delta^2 = 0, \qquad \Delta = d\delta + \delta d \ . \tag{43}$$

A scalar product $\langle f, g \rangle = \sum_p f(p)g(p)$ is defined on chains; operations $d$ and $\delta$ are dual to each other:

$$\langle f, dg \rangle = \langle \delta f, g \rangle.$$

**A Dual lattice**

Let $\Lambda$ be a 3-dimensional lattice. The dual lattice $\Lambda^*$ is defined so that its sites are the centers of the cubes of $\Lambda$: $x^* \sim \{x, 1, 2, 3\}$. Then, say

$$\{x, \mu\}^* = \frac{1}{2}\varepsilon_{\mu\nu\lambda}\{x^* - \hat{\nu} - \hat{\lambda}, \nu, \lambda\}.$$

The duality transformation on chains satisfies the relations

$$** = -1, \quad \text{and} \quad * d * = \delta. \tag{44}$$