# QUANTUM GRAVITY THROUGH NON-PERTURBATIVE RENORMALIZATION GROUP AND IMPROVED BLACK HOLE

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Usually, General Relativity (GR) is known to be unrenormalizable perturbatively from the viewpoint of quantum field theory. But in the modern sense of renormalizability, there still remains the possibility to investigate whether GR is "nonpertubatively" renormalizable or not. Here I review the basics and results in this topic based on the "Effective Average Action" approach which was proposed by M.Reuter, and discuss its application to the balck hole geometry.

## Introduction and Motivation

General Relativity(GR) has been investigated for a long time and widely accepeted as classical theory of gravity, but the quantum structure of the gravity is still not completely understood. At present, there are many candidates of quantum gravity. Among them, in this paper, I introduce the recent progress of one approach by "non-perturvative" renormalization group, which is the search for UV fixed point that S.Weinberg called as "asymptotic safety" [1].

If we proceed along the course of traditional quantum field theory, we meet the difficulty of nonrenormalizability because its method depends on the perturbative series expansion of the Newton coupling as is the case of other field theory. On the other hand, there exists the modern viewpoint of renormalization theory started by K.G.Wilson [2]. In the gravity, "asymptotic safety" which was pointed out by Weinberg is essentialy the same idea with Wilsonian approach. In his article, he practiced this idea in $(2+\epsilon)$ dimension by $\epsilon-$expansion. However this method does not work in higher dimensions. In this direction, the first analytical application of the modern "nonperturbative renormalization group method" to General Relativity was started by M.Reuter [6].

This methodology has been developed and showed its efficiency in other field theories recently [7]. The problem is reduced to the flow equation of effective (average) action. We review the basics of this formulation in this aritcle, and discuss about the application to try to see the quantum correction for the black hole.

In the end of this section, I would like to mention related works on this UV fixed point analysis in the discretized numerical simulation, Dynamical Triangulation and so on [4], [10]. In these cases also, the searches for a fixed point or a phase transition structure have been reported by many authors. Their connection with continuum theory which I will concern below was also studied in [3].

## 1 Asymptotic safety

"Criteria of quantum fundamental theory"

(A) perturbative renormalizability in ordinary sense of quantum field theory. (Gaussian fixed point)

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(B) perturbative nonrenormalizable but well-defined in nonperturbative sense (non-Gaussian fixed point).

$*$ Renormalization theory of K.G.Wilson (1974) et al.

$*$ Gravity version , "asymptotic safety" (S.Weinberg, 1979) [1].

Define demensionless couplings $g_i(\mu) = \mu^{d_i}\hat{g}_i(\mu)$

$d_i$: mass dimension of original couplings

$\mu$ : momentum scale of the renormalization point.

Then, the partial or total reaction rate $R$ is

$$R = \mu^D f(\frac{E}{\mu}, X, \hat{g}(\mu)). \tag{1}$$

Here $D$ is ordinary mass dimension of $R$, and $E$ is some energy characterizing the process, $X$ are all other dimensionless physical variables.

We set $\mu = E$,

$$R = E^D f(E, X, g(E)). \tag{2}$$

Depends on the behavior of the coupling $g(\mu)$ as $\mu \to \infty$.

Then, we concentrate on the flow equation of dimensionless coupling of $g(\mu)$.

# 2 Formulation of nonperturbative RG approach to quantum gravity

## 2.1 Strategy

The formulation of this problem is the following [6]

(1) We construct the "effective average action" of gravity in terms of background field method by introducing suitable infrared cutoff term (denoted its momentum scale as "$k$"). We suppress the contribution of infrared momentum which is lower than the cutoff scale in the integral of partition function.

(2) We integrate the mode from the UV scale down to the IR scale gradually, and read out the "flow equation" of effective action.

(3) In the original sence, the theory is defined in the infinite dimensional theory space, infinite number of operators,

$$\{\Lambda, R, , R^2, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, R^3, .....\}. \tag{3}$$

To solve the problem, we "truncate" the theory space into the lower mass dimensional operators.

$$\{\Lambda, R : \text{Einstein-Hilbert truncation}\}. \tag{4}$$

Most simple sketch of the evolution equation of the effective average acion $\Gamma_k$ is

$$k\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}[(\Gamma_k^{(2)} + \mathcal{R})^{-1}k\partial_k\mathcal{R}_k]. \tag{5}$$

Ansatz of "Einstein-Hilbert truncation" is

$$\Gamma_k[g, \bar{g}] = \frac{1}{16\pi G_k}\int d^dx\sqrt{g}\{-R(g) + 2\bar{\lambda}_k\} + classical\ \ gauge\ \ fixing. \tag{6}$$

### 2.2 Quantization based on the background field method

We consider the fluctuations $h_{\mu\nu}$ around the "classical" background $\bar{g}_{\mu\nu}$ in d-dimensional Euclidean sense.

Classical action $S[\gamma] = S[\bar{g} + h]$ is invariant under the gauge symmetry

$$\delta\gamma_{\mu\mu} = \mathcal{L}_v\gamma_{\mu\nu} = \delta h_{\mu\nu} = v^\rho\partial_\rho\gamma_{\mu\nu} + \partial_\nu v^\rho\gamma_{\rho\nu} + \partial_\mu v^\rho\gamma_{\rho\nu}, \tag{7}$$

$$(\gamma_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}); \tag{8}$$

$\mathcal{L}_v$ is the Lie derivative with respect to the infinitesimal coordinate transformation, $v^\mu$ is the gauge parameter.

The procedure is the same as usual BRS gauge fixing,$v^\mu \to C^\mu$, here $C^\mu$ is the Faddeev-Popov ghost, and we introduce an auxiliary $B$-field, the gauge fixing function $F[\bar{g}, h]$ (this imposes the gauge condition $F_\mu[\bar{g}, h] = 0$). Then the BRS transformations are

$$\delta_B h_{\mu\nu} = \kappa^{-2}\mathcal{L}_v\gamma_{\mu\nu}, \tag{9}$$

$$\delta_B C^\mu = \kappa^{-2}C^\nu\partial_\nu C^\mu, \tag{10}$$

$$\delta_B \bar{C}_\mu = B_\mu, \tag{11}$$

$$\delta_B B_\mu = 0. \tag{12}$$

Here $\kappa \equiv (32\pi G)^{-1}$.

The lagrangians of the gauge fixing part and the ghost are

$$\mathcal{L}_{GF} = \kappa B_\mu(F^\mu + \frac{\alpha\kappa}{2}B^\mu), \tag{13}$$

$$\mathcal{L}_{gh} = -\kappa\bar{C}_\mu(\delta_B F^\mu). \tag{14}$$

Here, we choose

$$F_\mu = \sqrt{2}\kappa(\delta^\rho_\mu\bar{D}^\sigma - \frac{1}{2}\bar{g}^{\rho\sigma}\bar{D}_\mu)h_{\rho\sigma}. \tag{15}$$

We integrate $B$-field at first. Next, we introduce external sources as $\{t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu\}$ which couple to $\{h_{\mu\nu}, C^\mu, \bar{C}_\mu\}$, and $\{\beta^{\mu\nu}, \tau_\mu\}$ couple to the BRS variations of $\{h_{\mu\nu}, C^\mu\}$ respectively. These lead to the following action for the source part

$$\begin{aligned} S_{source} = -\textstyle\int d^dx\sqrt{\bar{g}}\{&t^{\mu\nu}h_{\mu\nu} + \bar{\sigma}_\mu C^{mu} + \sigma^\mu\bar{C}_\mu \\ &+ \beta^{\mu\nu}\mathcal{L}_C(\bar{g}_{\mu\nu} + h_{\mu\nu}) + \tau_\mu C^\nu\partial_{nu}C^\mu\}. \end{aligned} \tag{16}$$

Then we get the $k$-dependent connected Green's functions

$$\begin{aligned} &\exp\{W_k[t^{\mu\nu}, \sigma^\mu, \bar{\sigma}_\mu; \beta^{\mu\nu}, \tau_\mu; \bar{g}_{\mu\nu}]\} \\ &= \int \mathcal{D}h_{\mu\nu}\mathcal{D}C^\mu\mathcal{D}\bar{C}_\mu \exp\Big\{ - S[\bar{g} + h] - S_{gf}[h; \bar{g}] - S_{gh}[h, C, \bar{C}; \bar{g}] \\ &\qquad - S_{source} - \Delta_k S[h, C, \bar{C}; \bar{g}]\Big\}. \end{aligned} \tag{17}$$

Here, the cutoff term $\Delta_k S[h, C, \bar{C}; \bar{g}]$ was introduced, and the gauge fixing part and the ghost part are

$$S_{gf} = \frac{1}{2\alpha}\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}F_\mu F_\nu, \tag{18}$$

$$S_{gh} = -\sqrt{2}\int d^dx\sqrt{\bar{g}}\bar{C}_\mu\mathcal{M}[g, \bar{g}]^\mu_\nu C^\nu, \tag{19}$$

$$\mathcal{M}[g, \bar{g}]^\mu_\nu \equiv \delta^\mu_\nu\bar{D}^2 + \bar{R}^\mu_\nu. \tag{20}$$

### 2.3 The cutoff term and the effective average action

The infrared cutoff term is

$$\Delta_k S[h, C, \bar{C} : \bar{g}] = \tfrac{1}{2}\kappa^2 \int d^dx \sqrt{\bar{g}} h_{\mu\nu} R_k^{grav}[\bar{g}]^{\mu\nu\rho\sigma} + \\ + \sqrt{2} \int d^dx \sqrt{\bar{g}} \bar{C}_\mu R_k^{gh}[\bar{g}] C^\mu. \tag{21}$$

Here, the cutoff function $R_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\bar{D}^2/k^2)$ is the interpolating smooth founction which satisfies

$$\lim_{x\to\infty} R^{(0)}(x) = 0, \tag{22}$$

$$R^{(0)}(x) = \begin{cases} 0 & (x \to \infty) \\ 1 & (x \to 0) \end{cases}. \tag{23}$$

This means that eigenvalues, $p^2$, of operator $-\bar{D}^2$ are integrated out for $p^2 \gg k^2$ and suppressed for $p^2 \ll k^2$ by the $k$-dependent mass term. For example, we can take

$$R^{(0)}(x; s) = \frac{sx}{\exp{(sx)} - 1)}. \tag{24}$$

Here $s$ is an arbitrary constant.

We introduce the $k$-dependent effective average as follows.

The $k$-dependent expectation values are

$$\bar{h}_{\mu\nu} = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta t^{\mu\nu}}, \qquad \xi = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta \bar{\sigma}_\mu}, \qquad \bar{\xi}_\mu = \frac{1}{\sqrt{\bar{g}}} \frac{\delta W_k}{\delta t^{\mu\nu}}. \tag{25}$$

Then, the $k$-dependent effective action is

$$\tilde{\Gamma}_k[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}] = \int d^dx \sqrt{\bar{g}} \{ t^{\mu\nu} \bar{h}_{\mu\nu} + \bar{\sigma}_\mu \xi^\mu + \sigma^\mu \bar{\xi}_{mu} \} - W_k[t, \sigma, \bar{\sigma}; \beta, \tau; \bar{g}]. \tag{26}$$

We define the “effective average action” as

$$\Gamma_k[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}] = \tilde{\Gamma}_k[\bar{h}, \xi, \bar{\xi}; \beta, \tau; \bar{g}] - \Delta_k S[h, C, \bar{C} : \bar{g}]. \tag{27}$$

If we write the expected value of the quantum metric $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$ , and rewrite $\Gamma_k$ as

$$\Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}; \xi^\mu, \bar{\xi}_\mu; \beta, \tau] \equiv \Gamma_k[g_{\mu\nu} - \bar{g}_{\mu\nu}; \xi^\mu, \bar{\xi}_\mu; \beta, \tau; \bar{g}_{\mu\nu}]. \tag{28}$$

In this language, the effective average action $\Gamma_k$ reduces to the conventional effective action if we set $\beta, \tau = 0$ and $k \to 0$ ,

$$\Gamma[g_{\mu\nu}] = \lim_{k\to 0} \Gamma_k[g_{\mu\nu}, g_{\mu\nu}; 0, 0; 0, 0]. \tag{29}$$

This is invariant under the original symmetry $\delta g_{\mu\nu} = \mathcal{L}_v g_{\mu\nu}$. In addition, the $k$-dependent version of this is defined as

$$\bar{\Gamma}_k[g_{\mu\nu}] \equiv \Gamma_k[g_{\mu\nu}, g_{\mu\nu}; 0, 0; 0, 0]. \tag{30}$$

From the original definition of connected Green’s function, $(t \equiv \ln k)$

$$\partial_t W_k = -\frac{1}{2} \mathrm{Tr} \left[ < h \otimes h > (\partial_t \hat{R}_k)_{\bar{h}\bar{h}} \right] - \mathrm{Tr} \left[ < \bar{C} \otimes C > (\partial_t \hat{R}_k)_{\bar{\xi}\xi} \right] . \tag{31}$$

Here

$$(\hat{R}_k)_{\bar{h}\bar{h}}^{\mu\nu\rho\sigma} = \kappa^2 (R_k^{grav}[\ barg])^{\mu\nu\rho\sigma}, \quad (\hat{R}_k)_{\bar{\xi}\xi} = \sqrt{2} R_k^{gh}[\bar{g}]. \tag{32}$$

After some manipulations in terms of the connected 2-point function $G_{ij}(x,y)$ and its inverse $\tilde{\Gamma}_k^{ij}(x,y)$,

$$\partial_t\Gamma_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}] = \frac{1}{2}\mathrm{Tr}\left[(\Gamma_k^{(2)}+\hat{R}_k)^{-1}_{\bar{h}\bar{h}}(\partial_t\hat{R}_k)_{\bar{h}\bar{h}}\right] - \\ -\frac{1}{2}\mathrm{Tr}\left[\left\{(\Gamma_k^{(2)}+\hat{R}_k)^{-1}_{\bar{\xi}\xi} - (\Gamma_k^{(2)}+\hat{R}_k)^{-1}_{\xi\bar{\xi}}\right\}(\partial_t\hat{R}_k)_{\bar{\xi}\xi}\right]. \tag{33}$$

On the other hand, $\Gamma_k$ satisfies the integrodifferential equation

$$\exp\Big\{-\Gamma_k[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}]\Big\} = \\ = \int\mathcal{D}h\mathcal{D}C\mathcal{D}\bar{C} + \exp\Big[-\bar{S}[h,C,\bar{C};\beta,\tau;\bar{g}] + \\ + \int d^dx\big\{(h_{\mu\nu}-\bar{h}_{\mu\nu})\frac{\delta\Gamma_k}{\delta\bar{h}_{\mu\nu}} + (C^\mu-\xi^\mu)\frac{\delta\Gamma_k}{\delta\xi^\mu} + (\bar{C}_\mu-\bar{\xi}_\mu)\frac{\delta\Gamma_k}{\delta\bar{\xi}_\mu}\big\}\Big]\times \\ \times\exp\{-\Delta_k S[h-\bar{h},C-\xi,\bar{C}-\bar{\xi};\bar{g}]\}. \tag{34}$$

Here

$$\tilde{S} \equiv S + S_{gf} + S_{gh} - \int d^dx\sqrt{\bar{g}}\{\beta^{\mu\nu}\mathcal{L}_C(\bar{g}_{\mu\nu}+h_{\mu\nu}) + \tau_\mu C^\nu\partial_\nu C^\mu\}. \tag{35}$$

The main contribution of this functional integral as $k\to\infty$ comes from

$$(h,C,\bar{C}) \sim (\bar{h},\ \xi,\ \bar{\xi}). \tag{36}$$

Then, at the UV cutoff scale, the effective average action as initial condition

$$\Gamma_\Lambda[\bar{h},\xi,\bar{\xi};\beta,\tau;\bar{g}] = S[\bar{g}+\bar{h}] + S_{gf}[\bar{h};\bar{g}] + S_{gh}[\bar{h},\xi,\bar{\xi};\bar{g}] \\ -\int d^dx\sqrt{\bar{g}}\{\beta^{\mu\nu}\mathcal{L}_\xi(\bar{g}_{\mu\nu}+\bar{h}_{\mu\nu}) + \tau_\mu\xi^\nu\partial_\nu\xi^\mu\}. \tag{37}$$

## 2.4 Modified Ward identities and truncations

The BRS variations come from the cutoff and the source terms. Assuming the invariance of the measure

$$<\delta_\epsilon S_{source} + \delta_\epsilon\Delta_k S> = 0. \tag{38}$$

leads to

$$\partial_t\Gamma_k[g,\bar{g}] = \frac{1}{2}\mathrm{Tr}\left[\kappa^{-2}(\Gamma_k^{(2)}[g,\bar{g}] + R_k^{grav}[\bar{g}])^{-1}(\partial_t R_k^{grav}[\bar{g}])\right] - \\ -\mathrm{Tr}\left[(\mathcal{M}[g,\bar{g}] + R_k^{gh}[\bar{g}])^{-1}\partial_t R_k^{gh}[\bar{g}]\right]. \tag{39}$$

Here, we take as truncation

$$\begin{aligned}\bar{\Gamma}_k[g_{\mu\nu}] &= \Gamma_k[g_{\mu\nu},g_{\mu\nu};0,0;0,0]. \\ &= \bar{\Gamma}_k[g] + S_{gf}[g-\bar{g};\bar{g}] + \hat{\Gamma}_k[g,\bar{g}]\end{aligned} \tag{40}$$

In the first approximation, $\hat{\Gamma}_k[g,\bar{g}]$ will be neglected.

## 2.5 Einstein-Hilbert truncation

At the UV scale, we set the initial condition as

$$S = \frac{1}{16\pi\bar{G}}\int d^dx\sqrt{-g}\{-R(g)+2\bar{\lambda}\}, \tag{41}$$

$$G_k \equiv \bar{G}Z_{N_k}. \tag{42}$$

Here, $G_k \equiv \bar{G}/Z_{N_k}$ is the dimensionful renormalized Newton constant. And

$$\begin{aligned} \Gamma_k[g,\bar{g}] = 2\kappa^2 Z_{N_k} \textstyle\int d^dx\sqrt{g}\{-R(g)+2\bar{\lambda}_k\}+ \\ +\kappa^2 Z_{N_k} \textstyle\int d^dx\sqrt{\bar{g}}\bar{g}^{\mu\nu}(\mathcal{F}_\mu^{\alpha\beta}g_{\alpha\beta})(\mathcal{F}_\nu^{\rho\sigma}g_{\rho\sigma}). \end{aligned} \tag{43}$$

here we take the gauge parameter as $\alpha = \frac{1}{Z_{N_k}}$.

After evaluating the "Trace" of (39)(R.H.S of the evolution equation), we will calculate the LHS of (39)

$$\partial_t\Gamma_k[g,g] = 2\kappa^2 \int d^dx\sqrt{g}\left[-R(g)\partial_t Z_{N_k} + 2\partial_t(Z_{N_k}\bar{\lambda}_k)\right]. \tag{44}$$

that is, we project the solution onto the theory space $\{\sqrt{g},\sqrt{g}R\}$.

To evaluate the functional integral, we contrive in the following manner. The quadratic part of the graviton action is

$$\begin{aligned} \Gamma_k^{quad}[\bar{h};\bar{g}] = Z_{N_k}\kappa^2 \textstyle\int d^dx\sqrt{\bar{g}}\{\frac{1}{2}\hat{h}_{\mu\nu}[-\bar{D}^2-2\bar{\lambda}_k+\bar{R}]\hat{h}^{\mu\nu} - \\ -\left(\frac{d-2}{4d}\right)\phi\left[-\bar{D}^2-2\bar{\lambda}_k+\frac{d-4}{d}\bar{R}\right]\phi + \\ -\bar{R}_{\mu\nu}\bar{h}^{\nu\rho}\bar{h}^\mu_{rho} + \bar{R}_{\alpha\beta\nu\mu}\bar{h}^{\beta\nu}\bar{h}^{\alpha\mu} + \frac{d-4}{d}\phi\bar{R}_{\mu\nu}\bar{h}^{\mu\nu}\}. \end{aligned} \tag{45}$$

Then, we divide the graviton field into trace $\phi \equiv \bar{g}^{\mu\nu}h_{\mu\nu}$ and traceless $\hat{h}_{\mu\nu}$ part

$$h_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{d}\bar{g}_{\mu\nu}\phi, \qquad \bar{g}_{\mu\nu}\hat{h}_{\mu\nu} = 0. \tag{46}$$

For technical reasons, we make use of the concrete form of the background spacetime, maximally symmetric spacetime

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{d(d-1)}(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho})\bar{R}, \qquad \bar{R}_{\mu\nu} = \frac{1}{d}\bar{g}_{\mu\nu}\bar{R}. \tag{47}$$

Then, we get

$$\begin{aligned} \Gamma_k^{quad}[\bar{h};\bar{g}] = \tfrac{1}{2}Z_{N_k}\kappa^2 \textstyle\int d^dx\sqrt{\bar{g}}\{\hat{h}_{\mu\nu}[-\bar{D}^2-2\bar{\lambda}_k+C_T\bar{R}]\hat{h}^{\mu\nu} - \\ -\left(\frac{d-2}{2d}\right)\phi\left[-\bar{D}^2-2\bar{\lambda}_k+C_S\bar{R}\right]\}. \end{aligned} \tag{48}$$

From the view point of the tensor structure, we take the renormalization factor as

$$\begin{aligned} (\mathcal{Z}_k^{grav})^{\mu\nu\rho\sigma} = \left[(I-P_\phi)^{\mu\nu\rho\sigma} - \tfrac{d-2}{2d}P_\phi^{\mu\nu\rho\sigma}\right]Z_{N_k}, \\ \text{where } \ (P_\phi)_{\mu\nu}{}^{\rho\sigma} = \tfrac{1}{d}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}. \end{aligned} \tag{49}$$

Eventually, the quadratic part of the graviton action is

$$\begin{aligned} \left(\kappa^{-2}(\Gamma_k^{(2)}[g,g]+R_k^{grav}[\bar{g}]\right)_{\hat{h}\hat{h}} = \\ = Z_{N_k}\left[-\bar{D}^2 + k^2R^{(0)}(-D^2/k^2) - 2\bar{\lambda}_k + C_T R\right], \end{aligned} \tag{50}$$

$$\begin{aligned} \left(\kappa^{-2}(\Gamma_k^{(2)}[g,g]+R_k^{grav}[\bar{g}]\right)_{\phi\phi} = \\ = -\tfrac{d-2}{2d}Z_{N_k}\left[-\bar{D}^2 + k^2R^{(0)}(-D^2/k^2) - 2\bar{\lambda}_k + C_S R\right]. \end{aligned} \tag{51}$$

In our truncation here, we don't take account of the evolution for ghost part ($Z_k^{gh}=1$). Therefore

$$\begin{aligned} -\mathcal{M} + R_k^{gh} = -D^2 + k^2R^{(0)}(-D^2/k^2) + C_V R \\ \text{with} \qquad C_V \equiv -\tfrac{1}{d}. \end{aligned} \tag{52}$$

Then, the RHS of renormalization group equation (denoted $\mathcal{S}_k(R)$) with $\bar{g} = g$

$$\mathcal{S}_k(R) = \mathrm{Tr}_T\left[\mathcal{N}(\mathcal{A}+C_T R)^{-1}\right] + \mathrm{Tr}_S\left[\mathcal{N}(\mathcal{A}+C_S R)^{-1}\right] - \\ -2\mathrm{Tr}_V\left[\mathcal{N}_0(\mathcal{A}_0+C_V R)^{-1}\right]. \tag{53}$$

with

$$\mathcal{A} \equiv -D^2 + k^2 R^{(0)}(-D^2/k^2) - 2\bar{\lambda}_k, \tag{54}$$

$$\mathcal{N} \equiv \tfrac{1}{2Z_{N_k}}\partial_t\left[Z_{N_k}k^2R^{(0)}(-D^2/k^2)\right] = \\ = \left[1-\tfrac{1}{2}\eta_N(k)\right]k^2R^{(0)}(-D^2/k^2) + D^2R^{(0)'}(-D^2/k^2). \tag{55}$$

Here $\eta_N(k)$ is an anomalous dimension of $\sqrt{g}R$.

$$\eta_N(k) \equiv -\partial_t \ln Z_{N_k}. \tag{56}$$

We expand (53) in terms of the curvature $R$ up to the lowest order.

$$\mathcal{S}_k(R) = \mathrm{Tr}_T\left[\mathcal{N}\mathcal{A}^{-1}\right] + \mathrm{Tr}_S\left[\mathcal{N}\mathcal{A}^{-1}\right] - 2\mathrm{Tr}_V\left[\mathcal{N}_0{\mathcal{A}_0}^{-1}\right] - \\ -R(C_T\mathrm{Tr}_T\left[\mathcal{N}\mathcal{A}^{-2}\right] + C_S\mathrm{Tr}_S\left[\mathcal{N}\mathcal{A}^{-2}\right] - 2C_V\mathrm{Tr}_V\left[\mathcal{N}_0{\mathcal{A}_0}^{-2}\right]). \tag{57}$$

We can take advantage of the heat kernel formula on the general background

$$\mathrm{Tr}\left[e^{-isD^2}\right] = \left(\frac{i}{4\pi s}\right)^{d/2} tr(I)\int d^dx\sqrt{g}\left\{1-\frac{1}{6}isR+\mathcal{O}(R^2)\right\}. \tag{58}$$

to evaluation the general functional of operator $W(-D^2)$. Here $tr_S(I)=1$, $tr_V(I)=d$, $tr_T(I)=(d-1)(d+2)/2$. Using the Fourier transform $\tilde{W}(s)$

$$\mathrm{Tr}\left[W(-D^2)\right] = \int_{-\infty}^{\infty} ds\tilde{W}(s)\mathrm{Tr}\left[e^{-isD^2}\right]. \tag{59}$$

and Mellin transform. Then the functional trace is

$$\mathrm{Tr}\left[W(-D^2)\right] = \left(\tfrac{1}{4\pi}\right)^{\frac{d}{2}} tr(I)\Big\{Q_{d/2}[W]\int d^dx\sqrt{g} + \\ +\tfrac{1}{6}Q_{d/2-1}[W]\int d^dx\sqrt{g}R + \mathcal{O}(R^2)\Big\}. \tag{60}$$

Here

$$Q_0[W] = W(0), \\ Q_n[W] = \tfrac{1}{\Gamma(n)}\int_0^\infty dz z^{n-1}W(z). \tag{61}$$

# 3 The running Newton and cosmological constant

We get two equations of the coupling flow, by comparing the LHS and the RHS from each coefficient of $\sqrt{g}$ and $\sqrt{g}R$

$$\partial_t(Z_{N_k}\bar{\lambda}_k) = \tfrac{1}{4\kappa^2}\left(\tfrac{1}{4\pi}\right)^{\frac{d}{2}}\Big\{tr_T(I)Q_{d/2}[\mathcal{N}/\mathcal{A}] + \\ +tr_S(I)Q_{d/2}[\mathcal{N}/\mathcal{A}] - 2tr_V(I)Q_{d/2}[\mathcal{N}_0/\mathcal{A}_0]\Big\}, \tag{62}$$

$$\partial_t(Z_{N_k}) = -\tfrac{1}{12\kappa^2}\left(\tfrac{1}{4\pi}\right)^{\frac{d}{2}}\Big\{tr_T(I)Q_{d/2}[\mathcal{N}/\mathcal{A}] + \\ +tr_S(I)Q_{d/2}[\mathcal{N}/\mathcal{A}] - 2tr_V(I)Q_{d/2}[\mathcal{N}_0/\mathcal{A}_0]\Big\}. \tag{63}$$

From here, we express the function $Q_n[z]$ in terms of some functions $\Phi^p_n(w)$ by the definition, for $n > 0$

$$\Phi^p_n(w) = \tfrac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \tfrac{R^{(0)}(z) - zR^{(0)\prime}(z)}{[z+R^{(0)}(z)+w]},$$
$$\tilde{\Phi}^p_n = \tfrac{1}{\Gamma(n)} \int_0^\infty dz z^{n-1} \tfrac{R^{(0)}(z)}{[z+R^{(0)}(z)+w]}, \tag{64}$$

and for $n = 0$

$$\Phi^p_0(w) = \tilde{\Phi}^p_0(w) = (1+w)^{-p}. \tag{65}$$

In addition, we use the "dimensionless" renormalized coupling constant

$$g_k \equiv k^{d-2} G_k \equiv Z^{-1}_{N_k} \bar{G},$$
$$\lambda_k = k^{-2} \bar{\lambda}_k. \tag{66}$$

Eventually, the evolutions of the couplings, $g_k$ and $\lambda_k$ are

$$\partial_t g_k = [d - 2 + \eta_N(k)] g_k, \tag{67}$$

$$\partial_t \lambda_k = -(2 - \eta_N)\lambda_k + \tfrac{1}{2} g_k \left(\tfrac{1}{4\pi}\right)^{1-\frac{d}{2}} \times$$
$$\times [2d(d+1)\Phi^1_{d/2}(-2\lambda_k) - 8d\Phi^1_{d/2}(0) - d(d+1)\tilde{\Phi}^1_{d/2}(-2\lambda_k)]. \tag{68}$$

Here the anomalous dimension $\eta_N(k)$ is

$$\eta_N(k) = g_k B_1(\lambda_k) + \eta_N(k) g_k B_2(\lambda_k), \tag{69}$$

and

$$B_1(\lambda_k) \equiv \tfrac{1}{3}\left(\tfrac{1}{4\pi}\right)^{1-\frac{d}{2}} \Big[d(d+1)\Phi^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\Phi^2_{d/2}(-2\lambda_k) -$$
$$-4d\Phi^1_{d/2-1}(0) - 24\Phi^2_{d/2}(0)\Big],$$
$$B_2(\lambda_k) \equiv -\tfrac{1}{6}\left(\tfrac{1}{4\pi}\right)^{1-\frac{d}{2}} \Big[d(d+1)\tilde{\Phi}^1_{d/2-1}(-2\lambda_k) - 6d(d-1)\tilde{\Phi}^2_{d/2}(-2\lambda_k)\Big]. \tag{70}$$

The anomalous dimension is also expressed in terms of $g_k$ and $\lambda_k$

$$\eta_N = \frac{g_k B_1(\lambda_k)}{1 - g_k B_2(\lambda_k)}. \tag{71}$$

# 4 Summary of recent results

We summarize the main results of the numerical computation [11], [13]. This was reported in [12] concisely. The extended theory space which includes the invariant $R^2$ with a coupling $\beta_k$, that is $\{\sqrt{g}, \sqrt{g}R, \sqrt{g}R^2\}$, was also investigated (we call it "$R^2$-truncation" here). In each case, the existence of the nontrivial UV (non-gaussian) fixed point was studied for various ways of the cutoff function. The following points are their main features in 4-dimension.

(1) Universal existence: Non-Gaussian fixed point exists for various type of cutoff operators.

(2) Positive Newton constant for any cutoff operators.

(3) Stability: Non-Gaussian fixed point is UV attractive for any cutoff.

(4) Scheme and gauge dependence: The values of the critical exponents or each quantity $g_k$, $\lambda_k$ at UV fixed point are dependent on the cutoff scheme and the gauge fixing condition. But the value of the product $g_k\lambda_k$ at fixed point seemed to be strongly universal.

In addition, the same investigation was performed for $R^2$-truncation [12]. The UV fixed point exists for all admissible cutoff in this case also. This situation is the same for Einstein-Hilbert truncation. At the fixed point, the values of $\beta_k$ are always significantly smaller than $\lambda_k$ or $g_k$. This means the validity of the approximation of Einstein-Hilbert truncation for the approximation scheme employed ("Einstein-Hilbert dominance").

In higher dimension of spacetime, the existence of UV fixed point becomes more dependent on the approximation scheme than that in 4 dimensions.

As is always the case with the effective action approach, the trunacation with nonlocal invariants was also studied and interesting possibility was also pointed out [14].

# 5 Quantum Schwarzschild spacetime and curvature singularity

## 5.1 Approximate solution for the running coupling in 4 dimensions

To investigate the quantum metric of black hole by the renormalization group flow in 4 dimension, we study the solution of the flow equations (67), (68) at first [15]. We consider the case without cosmological constant ($\lambda_k = 0$).

$$\partial_t g(t) = 2\frac{1-\omega' g}{1-B_2}g, \tag{72}$$

here

$$\omega' \equiv \omega + B_2, \tag{73}$$

and

$$\omega = \frac{4}{\pi}\left(1 - \frac{\pi^2}{144}\right), \quad B_2 = \frac{2}{3\pi}. \tag{74}$$

The fixed point is

$$\begin{cases} g_*^{\mathrm{IR}} = 0 & : \text{Gaussian} \\ g_*^{\mathrm{UV}} = \frac{1}{\omega'} \sim 0.71 & : \text{Non-Gaussian.} \end{cases}$$

The analytical integration is possible

$$\frac{g}{(1-\omega' g)^{\omega/\omega'}} = \frac{g(k_0)}{(1-\omega' g(k_0))^{\omega/\omega'}}\left(\frac{k}{k_0}\right)^2. \tag{75}$$

We use the approximation

$$\frac{\omega'}{\omega} \sim 1.18 \to 1, \tag{76}$$

which corresponds to the first order approximation in the anomalous dimension.

Within this approximation, we obtain

$$g(k) = \frac{g(k_0)k^2}{\omega g(k_0)k^2 + [1-\omega g(k_0)]k_0^2}. \tag{77}$$

In terms of $G(k) \equiv g(k)/k^2$ and $G_0 \equiv G(k_0)$ one gets: ($k_0 \sim 0$).

$$G(k) = \frac{G_0}{1+\omega G_0 k^2}. \tag{78}$$

This implies the asymptotic freedom. The Newton coupling $G(k)$ vanishes as $k^2 \to \infty$.

### 5.2 Cutoff function in the Schwarzschild spacetime

The flow equations (67), (68) and its solution (78) are expressed in terms of momentum $k$. On the other hand, we want to know the information about the solution in the coordinate space, since the information of curvature or geometrical quantity should be written in the coordinate space. To get such information, we start from the approach suggested in [15]. This "cutoff identification" program has been studied in many papers. It is a generalization of the relation between the quantum Coulomb potential (Uehling potential) and the solution for the renormalization group equation of electric charge. That is the substitution

$$k \to \frac{1}{r} \tag{79}$$

in the solution for the running coupling solution

$$e^2(k) = \frac{e^2}{1 - \frac{e^2}{6\pi^2}\ln\frac{k}{k_0}} \ . \tag{80}$$

This manipulation surely restores the well-known behaviour of the effective potential up to constant in the leading order at small enough distances [15]. We consider a generalization of (79) for the curved spacetime below.

There is no universal meaning in a coordinate itself in curved spacetime. One way for the identification in the curved background was proposed in [15]

$$k(P) = \frac{\xi}{d(P)} \ , \tag{81}$$

$$d(P) \equiv \int_{\mathcal{C}} \sqrt{|ds^2|} \ , \tag{82}$$

here $d(P)$ is the proper distance from the center of black hole to the reference point $P$ which corresponds to the infrared cutoff scale in momentum space. $\xi$ is some constant which will be determined later. We take up the Schwarzschild spacetime, whose metric in the Hilbert gauge is

$$ds^2 = -\left(1 - \frac{2G_0M}{r}\right)dt^2 + \left(1 - \frac{2G_0M}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \tag{83}$$

For $r < 2G_0M$, the proper distance (82) is

$$d(r) = 2G_0M \arctan\sqrt{\frac{r}{2G_0M - r}} - \sqrt{r(2G_0M - r)}, \tag{84}$$

for $r > 2G_0M$

$$d(r) = \pi G_0M + 2G_0M \ln\left(\sqrt{\frac{r}{2G_0M}} + \sqrt{\frac{r}{2G_0M} - 1}\right) + \sqrt{r(2G_0M - r)}, \tag{85}$$

and asymptotically

$$\begin{aligned} d(r) &= r + \mathcal{O}(\ln r) \quad \text{for} \quad r \to \infty, \\ d(r) &= \tfrac{2}{3}\tfrac{1}{\sqrt{2G_0M}}r^{3/2} + \mathcal{O}(r^{5/2}) \quad \text{for} \quad r \to 0. \end{aligned} \tag{86}$$

Therefore we adopt the following function which interpolates the behaviours at $r = 0$ and $r = \infty$

$$d(r) = \left(\frac{r^3}{r + \gamma G_0M}\right)^{1/2} . \tag{87}$$

Here $\gamma = 9/2$.

### 5.3 RG-modified Schwarzschild spacetime

We move to the coordinate space by making use of solution (78) through the identification $k = \xi/d(r)$ and the proper distance function adopted in (87). We call the Schwarzschild spacetime so modified "the RG-modified Schwarzschild spacetime".

The relation of the Newton constant between some scale $G(k(r))$ and the experimentally observed (infrared) scale $G_0$ is

$$G(r) = \frac{G_0 d(r)^2}{d(r)^2 + \omega\xi^2 G_0} \,. \tag{88}$$

At large distances,

$$G(r) = G_0 - \omega\xi^2 \frac{G_0^2}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) . \tag{89}$$

At this stage, we can adjust the parameter to the results from the effective theory calculations in various papers. For example, in [16]

$$\omega\xi^2 = \frac{118}{15\pi} \,. \tag{90}$$

Then, the lapse function of RG-corrected spacetime at some distance scale $r$ is

$$f(r) = 1 - \frac{2G(r)M}{r} = 1 - \frac{2G_0 M r^2}{r^3 + \omega\xi^2 G_0 [r + \gamma G_0 M]} \,. \tag{91}$$

The character of this spacetime was investigated in detail in [15].

$*$ For large masses $M$, quantum effects are negligible;

and its property depends on the mass of the object. For some critical mass $M_{cr}$;

$*$ If $M > M_{cr}$, there is 2 horizons (similar to Reissner-Nordström);

$*$ If $M < M_{cr}$, there is no horizon;

$*$ When $M = M_{cr}$, it corresponds to the "Extremal Black Hole" ( Hawking evaporation stops). This seems to be the final state of black hole;

Then, we study its behaviour near the center of the RG-modified black hole.

Around the UV fixed point momentum or coordinate dependence of the Newton coupling is

$$G(k) \approx \frac{1}{\omega\ k^2} \qquad \Leftrightarrow \qquad G(r) \approx \frac{d(r)^2}{\omega\xi^2}, \tag{92}$$

for the Schwarzschild background, the distance function $d(r)$ is

$$d_{Sch} \propto r^{3/2} \qquad \to \qquad G_{Sch} \propto r^3. \tag{93}$$

As $r \to 0$, the metric behaves as a "de-Sitter"-like one.

$$f(r) = 1 - cr^2, \qquad\qquad c \text{ is some constant.} \tag{94}$$

On the other hand, if the spacetime becomes de-Sitter background approximately near the origin, its distance function is

$$d_{dS} \propto r \qquad \to \qquad G_{dS} \propto r^2 \tag{95}$$

in this case, as $r \to 0$, the lapse behaves

$$f(r) = 1 - cr + \mathcal{O}(r^2), \qquad\qquad c \text{ is some constant.} \tag{96}$$

By the way, for the lapse function $f(r) = 1 - cr^\nu$ $(\nu > 0)$, the curvature invariants are

$$R = c(\nu+1)(\nu+2)r^{\nu-2}, \tag{97}$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = c^2(\nu^4 - 2\nu^3 + 5\nu^2 + 4)r^{2\nu-4}. \tag{98}$$

There is a curvature singularity at $r = 0$ for $\nu < 2$. Then the (96) is singular, while the spacetime (94) is regular. This shows that the above prescription for the cutoff identification is not consistent.

## 5.4 Improvement of the cutoff identification

The prescription for the cutoff identification in the previous subsection consists of the following steps.

(1) We assume the fixed background $g_{\mu\nu}$, then calculate the distance function $d(r)$.(2) Identification with momentum space by $k(P) = \frac{\xi}{d(P)}$.(3)The metric modification $\hat{g}_{\mu\nu}$ from the solution of the coupling flow equation.

To make the above prescription consistent, we propose the next procedure. This improvement is essentially the same as in the recent works [17], [18]. More detailed explanation will be published [21].

(1) We start from the "unfixed" form of the metric $g_{\mu\nu}(r)$,

$$f(r) = 1 - \frac{2G(r)M}{r} .$$

(2) Then, the distance function is

$$d(r) = \int \sqrt{|ds^2|} = \int_0^r \sqrt{\left|\frac{r}{r - 2G(r)M}\right|} . \tag{99}$$

(3) On the other hand, we obtained the solution (88) of the flow equation already in coordinate space.

$$G(r) = \frac{G_0 d(r)^2}{d(r)^2 + \omega\xi^2 G_0} .$$

Then, we must solve the above two equations simultaneously. Differentiating both equations in $r$, and equating them we have

$$\frac{1}{2}\sqrt{\frac{G_0 - G(r)}{\tilde{\omega} G_0 G(r)}} \frac{\tilde{\omega} G_0^2 G'(r)}{(G_0 - G(r))^2} = \sqrt{\left|\frac{r}{r - 2G(r)M}\right|} . \tag{100}$$

This equation cannot be solved exactly. We have to rely on a numerical solution by computer under suitable boundary condition. Here we only study the behaviour near the origin since whether the central singularity of the classical black hole is regularized or not by quantum gravity is of general interest. The spacetime singularity is generated in quite general situations in classical General Relativity [20].

When we are near the center $r \ll 2G(r)M$ then

$$\frac{1}{4}\tilde{\omega} G_0^3 \left(\frac{dG}{dr}\right)^2 = -\frac{r}{r - 2G(r)M} G(r)(G_0 - G(r))^3. \tag{101}$$

We search for a consistent solution assuming its asymptotic form as a series expansion around $r = 0$:

$$G(r) = \sum_{n=1}^{\infty} a_n r^n = a_1 r + a_2 r^2 + a_1 r^3 + \cdots .$$

This assumption implies the asymptotic freedom as $r \to 0$.

We get up to the third order

$$a_1 = 0 , \quad a_2 = \frac{1}{\tilde{\omega}} , \quad a_3 = \frac{M}{2\tilde{\omega}^2} , \quad \cdots . \tag{102}$$

As a result, the lapse function in this case starts with the linear term. Therefore this RG-corrected spacetime has curvature singularity still at its origin. The consistent scheme of the cutoff identification does not save the situation in terms of the resolution of singularity.

## 6 Summary and Discussion

In this talk, we review the basic concept and the formulation of the nonperturbative renormalization group approach to quantum gravity. The recent results in numerical survey to this direction are getting to increase the possibility of "asymptotic safety" in 4 dimension in spite of its nonrenormalizability in the sense of the perturbative theory.

If this "asymptotic safety" takes place, the Newton constant shows asymptotic freedom in the high energy region, which will correspond to the weak coupling between the gravity and the other matter field probably. Then this position-dependent Newton constant will modify our concepts of spacetime in the classical theory. There are many works which studied the effects of the position-dependent Newton constant in the cosmological problems. The crucial point to discuss the phenomenological impacts is the way in which we interpret the information on the coupling flow solution for momentum space in the coordinate space. We discussed the quantum modified Schwarzschild spacetime through the renormalization group by the cutoff identification method which was introduced in [15] and tried improving it. This improvement is essentially the same concept as the "consistent cutoff identification" which was proposed already in [17], [18]. Here we concentrated on the possibility of curvature singularity at the origin in this method. Whether the central singularity of black hole is regularized or not is of general interest in this region. As a result, the curvature singularity still exists at the center of this RG-corrected spacetime also by the improving identification method.

Although this identification method shows the interesting effect in the cosomological phenomenology (see for example [17],[18]), still we could not find the regularization mechanism for the black hole singularity or some prescriptions to describe the quantum spacetime by regular geometry. More close argument will be reported eslewhere [21].

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