KOLMOGOROV's PROBABILITY THEORY IN QUANTUM PHYSICS

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A connection between the probability space measurability requirement and the complementarity principle in quantum mechanics is established. It is shown that measurability of the probability space implies that the results of the quantum measurement depend not only on properties of the quantum object under consideration but also on classical characteristics of the measuring instruments used. It is also shown that if the measurability requirement is taken into account, then the hypothesis that the objective reality exists does not lead to the Bell inequality.

Recently we commemorated the centenary of quantum mechanics. This year we celebrate the Einstein year. Einstein was one of founders of quantum mechanics. Simultaneously he was one of the most known critics of quantum mechanics. First of all he criticized quantum mechanics because in it there is no a mathematical counter-part of a physical reality which becomes apparent in individual experiment. Seventy years ago he wrote [1]: "a) every element of the physical reality must have a counter-part in the complete physical theory; b) if, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there is an element of physical reality corresponding to this physical quantity".

In the report I want to talk about this mathematical counter-part, probability, and quantum measurement.

At the same time, when the quoted statement of Einstein was written, Kolmogorov created modern probability theory [2]. Thirteen years ago in review [3] dedicated to interpretations of quantum mechanics, Home and Whitaker stated: "The fundamental difficulty in interpreting quantum theory is that it provides in general only probabilities of obtaining given results. Thus much of any discussion of quantum theory must depend on what one means by probability — how one defines or interprets the term". In this review, much space was allocated to describing different interpretations of the probability concept, but Kolmogorov's approach to the probability problem was mentioned only briefly.

The Kolmogorov probability theory is nowadays the most developed mathematically. It is commonly assumed that a special quantum probability theory is required for quantum systems. In this work, I defend the opinion that the classical Kolmogorov probability theory is also quite sufficient for the quantum case if we take the peculiarity of quantum measurements into account.

The Kolmogorov probability theory is based on the notion of the so-called probability space $(\Omega, \mathcal{F}, P)$ (see, e.g., [2, 4]). The first component Ω is the set (space) of elementary events. Along with elementary event, the notion of a "random event" or simply "event" is introduced. Each event F is identified with some subset of the set Ω. The event F is assumed to be realized if one of the elementary events belonging to this set ($\varphi \in F$) is realized. It is assumed that we can find out whether an event is realized or not in each trial. For elementary events, this requirement is not imposed.

Sets of subsets of the set Ω (including the set Ω itself and the empty set $\emptyset$) are endowed with the structure of Boolean algebras. The algebraic operations are the intersection of subsets, the union of subsets, and the complement of a subset up to Ω. A Boolean algebra that is closed with respect to countable unions and intersections is called a σ-algebra. The second component of a probability space is some σ-algebra $\mathcal{F}$. The set Ω, where a fixed σ-algebra $\mathcal{F}$ is chosen, is called a measurable space.

Finally, the third component of a probability space is a probability measure P. This is a map from the algebra $\mathcal{F}$ to the set of real numbers satisfying the conditions (a)$0 \leq P(F) \leq 1$ for all $F \in \mathcal{F}$, $P(\Omega) = 1$ and (b)$P(\sum_j F_j) = \sum_j P(F_j)$ for every countable family of disjoint subsets $F_j \in \mathcal{F}$. We note that the probability measure is defined only for the events belonging to the algebra $\mathcal{F}$. The probability is generally not defined for elementary events.

We now consider the application of the basic principles of probability theory to the problem of quantum measurements (see [5, 6]). In the probability theory the physical meaning of elementary events is not explicitly specified, but it is assumed that they are mutually exclusive and that one and only one elementary event is realized in each trial.

We begin by considering a classical physical system. For such a system, "observable" is a basic notion.The main property of observables is that they can be multiplied by real numbers, added, and multiplied by each other. In other words, they form a real algebra $\mathfrak{A}_{cl}$. In the classical case, the algebra turns out to be commutative. Fixing an observable A still tells nothing about the value $\varphi(A)$ that will be obtained as a result of a measurement in a concrete situation. Fixing the values of observables is realized by fixing the state of a physical system. In mathematical terms, this corresponds to fixing a functional $\varphi(A)$ on the algebra $\mathfrak{A}_{cl}$.

We know from experience that the sum and product of observables correspond to the sum and product of the measurement results:

$$\begin{aligned} \alpha A_1 + \beta A_2 &\to \alpha\varphi(A_1) + \beta\varphi(A_2), \\ A_1 A_2 &\to \varphi(A_1)\varphi(A_2), \end{aligned} \tag{1}$$

where α and β are real numbers. If the functional $\varphi(A)$ satisfies to the conditions (1) then it refers to as character of real commutative algebra. The state of classical physical system, i.e. character of algebra of observables $\mathfrak{A}_{cl}$, can play a role of the elementary event in probability theory.

Let us pass to quantum system. Although the observables also have algebraic properties in the quantum case, it is impossible to construct a closed algebra from them that would be real, commutative, and associative. In the quantum case the observables form a set $\mathfrak{A}_+$ of Hermitian elements of an involutive, associative, and (generally) noncommutative algebra $\mathfrak{A}$. The elements of the algebra $\mathfrak{A}$ are called dynamical quantities.

Let $\mathfrak{Q}$ $\quad(\mathfrak{Q} \subset \mathfrak{A}_+)$ denote a maximal real commutative subalgebra of the algebra $\mathfrak{A}$. This is the subalgebra of compatible (simultaneously measurable) observables. If the algebra $\mathfrak{A}$ is commutative (an algebra of classical dynamical quantities), then such a subalgebra is unique. If the algebra $\mathfrak{A}$ is noncommutative (an algebra of quantum dynamical quantities), then there are many different subalgebras $\mathfrak{Q}_\xi$ $(\xi \in \Xi)$.

In quantum measurements in each individual experiment, we deal only with observables belonging to one of the subalgebras $\mathfrak{Q}_\xi$. The result of such a measurement is determined by the functional φ_ξ which is character of a subalgebra $\mathfrak{Q}_\xi$. Let us name the elementary state of quantum system a totality $\varphi = \{\varphi_\xi\}$ of functionals φ_ξ where φ_ξ are characters of all real commutative subalgebras $\mathfrak{Q}_\xi$ $(\xi \in \Xi)$ of algebra $\mathfrak{A}$.

Fixing an elementary state φ, we fix all such functionals. Thus, the result of each individual measurement of observables of the physical system is determined by the elementary state of this system. Therefore, the elementary state of quantum system can play a role of the elementary event in probability theory.

In quantum measurement, the elementary state cannot be fixed unambiguously. Indeed, in one experiment, we can measure observables belonging to the same maximal commutative subalgebra $\mathfrak{Q}_\xi$ because instruments measuring incompatible observables are incompatible. As a result, we find only the values of the functional φ_ξ. The rest of the elementary state φ remains undetermined. We can perform repeated measurement using an instrument of other type which allows to measure observables of other subalgebra $\mathfrak{Q}_{\xi'}$. It will give the new information but will uncontrollably perturb the elementary state that arose after the first measurement. The information obtained in the first measurement will therefore become useless. In this connection, it is convenient to adopt the following definition.

Elementary states φ are said to be φ_ξ-equivalent if they have the same restriction φ_ξ to the subalgebra $\mathfrak{Q}_\xi$.

In quantum measurement, we can thus find only the equivalence class to which the studied elementary state belongs. In a measurement of observables belonging to the subalgebra $\mathfrak{Q}_\xi$, we easily recognize the state preparation procedure of the standard quantum mechanics. Thus, the class of equivalent elementary states is a quantum state. Therefore, from the probability theory standpoint, the quantum state is random event but not the elementary event.

The main purpose of a quantum experiment is to find the probability distributions for some observable quantities. Using a definite measuring instrument, we can obtain such a distribution for a set of compatible observables. From the probability theory standpoint, choosing a certain measuring instrument corresponds to fixing the σ-algebra $\mathcal{F}$.

We suppose that we conduct a typical quantum experiment. We have an ensemble of quantum systems in a definite *quantum state*. For example, we consider particles with the spin 1/2 and the projection of the spin on the x axis equal to 1/2. We suppose that we want to investigate the distribution of the projections of the spin on the directions having the angles θ_1 and θ_2 to the x axis. The corresponding observations are incompatible, and we cannot measure both observables in one experiment. We should therefore conduct two groups of experiments using different measuring instruments. In our concrete case, the magnets in the Stern–Gerlach instrument should have different spatial orientations.

These two groups of experiments can be described by the respective probability spaces $(\Omega, \mathcal{F}_1, P_1)$ and $(\Omega, \mathcal{F}_2, P_2)$. Although the space of elementary events Ω is the same in both cases, the probability spaces are different. To endow these spaces with the measurability property, they are given different σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$.

Formally and purely mathematically, we can construct a σ-algebra $\mathcal{F}_{12}$ including both the algebras $\mathcal{F}_1$ and $\mathcal{F}_2$. Such an algebra is called the algebra generated by $\mathcal{F}_1$ and $\mathcal{F}_2$. In addition to the subsets $F_i^{(1)} \in \mathcal{F}_1$ and $F_j^{(2)} \in \mathcal{F}_2$ of the set Ω, it also contains all intersections and unions of these subsets. But such a σ-algebra is unacceptable from the physical standpoint.

Indeed, the event $F_{ij} = F_i^{(1)} \cap F_j^{(2)}$ means that the values of two incompatible observables lie in a strictly fixed domain for one quantum object. For a quantum system, it is impossible in principle to conduct an experiment that could distinguish such an event. For such an event, the notion "probability" therefore does not exist at all, i.e., the subset F_{ij} does not correspond to any probability measure, and the σ-algebra $\mathcal{F}_{12}$ cannot be used to construct the probability space.

Here, an important peculiar feature of the application of probability theory to quantum systems is revealed: not every mathematically possible σ-algebra is physically allowable.

An element of the measurable space $(\Omega, \mathcal{F})$ thus corresponds in the experiment to a pair consisting of a quantum object (for example, in a definite quantum state) and certain type of measuring instrument that allows fixing an event of a certain form. Each such instrument can separate events corresponding to some set of compatible observable quantities, i.e., belonging to the same subalgebra $\mathfrak{Q}_\xi$. If we assume that each measuring instrument has some type ξ, then the σ-algebra $\mathcal{F}$ depends on the parameter ξ: $\mathcal{F} = \mathcal{F}_\xi$.

In view of the peculiarity of quantum experiments, we should take care in defining one of the basic notions of probability theory – the real random variable. A real random variable is usually defined as a map from the space Ω of elementary events to the extended real axis $\bar{R} = [-\infty, +\infty]$. But such a definition does not take peculiarities of quantum experiments, where the result may depend on the type of measuring instrument, into account. We therefore adopt the following definition.

A real random variable is a map from the measurable space $(\Omega, \mathcal{F}_\xi)$ of elementary events to the extended real axis.

For an observable A, this means that

$$\varphi \xrightarrow{A} A_\xi(\varphi) \equiv \varphi_\xi(A) \in \bar{R}.$$

We assume that an event $\tilde{A}$ is realized in the experiment if the registered value of the observable A does not exceed $\tilde{A}$. Let $P_\xi(\tilde{A}) = P(\varphi : \varphi_\xi(A) \leq \tilde{A})$ denote the probability of this event and $P_\xi(d\,\varphi) \equiv [P(\varphi : \varphi_\xi(A) \leq (\tilde{A} + d\,\tilde{A})) - P(\varphi : \varphi_\xi(A) \leq \tilde{A})]$.

Then the average of the observable A is determined by the equation

$$\langle A \rangle = \int P_{\xi(A)}(d\varphi)\, \varphi(A). \tag{2}$$

Here $\xi(A)$ must be such that $A \in \mathfrak{Q}_\xi$.

We can investigate how the measurability condition on the probability space is manifested in the important case of the derivation of the Bell inequality [7]. Forty years ago Bell has derived the inequality assuming that there is a physical reality which can be is marked with some parameter. There are many forms of this inequality. Hereinafter, we refer to the version proposed in [8]. This variant is usually designated by the abbreviation CHSH.

Let particle with the spin 0 decay into two particles $\mathbf{A}$ and $\mathbf{B}$ with spin 1/2. These particles move apart, and the distance between them becomes large. The projections of their spins are measured by two independent devices D_a and D_b. Let the device D_a measures the spin projection of the particle $\mathbf{A}$ on a direction, and the device D_b measures the spin projection of the particle $\mathbf{B}$ on the b direction. We let A and B denote the corresponding observables and let A_a and B_b denote the measurement results.

Let us assume that the initial particle has some physical reality that can be marked by the parameter λ. We use the same parameter to describe the physical realities for the decay products. Accordingly, it is possible to consider measurement results of the observables $\hat{A}$, $\hat{B}$ as the function $A_a(\lambda)$, $B_b(\lambda)$ of the parameter λ. Let the distribution of the events with respect to the parameter λ be characterized by the probabilistic measure $P(\lambda)$:

$$\int P(d\lambda) = 1, \qquad 0 \leq P(\lambda) \leq 1.$$

We introduce the correlation function $E(a, b)$:

$$E(a, b) = \int P(d\lambda)\, A_a(\lambda)\, B_b(\lambda) \tag{3}$$

and consider the combination

$$\begin{aligned} I &= |E(a, b) - E(a, b')| + |E(a', b) + E(a', b')| \\ &= \left| \int P(d\lambda)\, A_a(\lambda)\, [B_b(\lambda) - B_{b'}(\lambda)] \right| \\ &+ \left| \int P(d\lambda)\, A_{a'}(\lambda)\, [B_b(\lambda) + B_{b'}(\lambda)] \right|. \end{aligned} \tag{4}$$

The equalities

$$A_a(\lambda) = \pm 1/2, \quad B_b(\lambda) = \pm 1/2 \tag{5}$$

are satisfied for any directions a and b. Therefore,

$$\begin{aligned} I &\leq \int P(d\lambda)\,[|A_a(\lambda)|\,|B_b(\lambda) - B_{b'}(\lambda)| \\ &+ |A_{a'}(\lambda)|\,|B_b(\lambda) + B_{b'}(\lambda)|] \\ &= 1/2 \int P(d\lambda)\,[|B_b(\lambda) - B_{b'}(\lambda)| + |B_b(\lambda) + B_{b'}(\lambda)|]. \end{aligned} \tag{6}$$

Due to the equality (5) for each λ one of the expressions

$$|B_b(\lambda) - B_{b'}(\lambda)|, \qquad |B_b(\lambda) + B_{b'}(\lambda)| \tag{7}$$

is equal to zero and the other is equal to unity. Here it is crucial that the same value of the parameter λ appears in both expressions. Hence, the Bell inequality then follows:

$$I \leq 1/2 \int P(d\lambda) = 1/2. \tag{8}$$

The correlation function can be easily calculated within standard quantum mechanics. We obtain

$$E(a,b) = -1/4 \cos\theta_{ab},$$

where θ_{ab} is the angle between the directions a and b. For the directions $a = 0$, $b = \pi/8$, $a' = \pi/4$, $b' = 3\pi/8$ we have

$$I = 1/\sqrt{2}\,.$$

It contradicts the inequality (8).

Experiments that have been performed corresponded to quantum-mechanical calculations and did not confirm the Bell inequality. These results have been interpreted as decisive argument against the hypothesis of the existence of local objective reality in quantum physics. It is easy to see that if my variant of the probability theory is properly applied to quantum system, then the above derivation of the Bell inequality is invalid.

Because the σ-algebra and accordingly probability measure depend on the measuring device used in a quantum case, it is necessary to make replacement $P(d\lambda) \to P_{\xi(AB)}(d\varphi)$ in the equation (3). If we are interested in correlation function $E(a',b')$, it is necessary to make replacement $P(d\lambda) \to P_{\xi(A'B')}(d\varphi)$ in the equation (3). Although we used the same symbols $d\varphi$ in both cases for notation of the elementary volume in the space of the physical states, it is necessary to remember that sets of the physical states corresponding $d\varphi$, are different. The matter is that these sets should be elements of the σ-algebras. If observables A, B are incompatible with observables A', B', then σ-algebras are different. Moreover, there are no physically allowable σ-algebra which has these algebras as subalgebras. Finally, the equation (3) should be replaced on

$$E(a,b) = \int P_{\xi(AB)}(d\varphi)\varphi(AB).$$

Accordingly, the equation (4) now has the form

$$\begin{aligned} I &= \left| \int P_{\xi(AB)}(d\varphi)\,\varphi(AB) - \int P_{\xi(AB')}(d\varphi)\,\varphi(AB') \right| + \\ &+ \left| \int P_{\xi(A'B)}(d\varphi)\,\varphi(A'B) + \int P_{\xi(A'B')}(d\varphi)\,\varphi(A'B') \right|. \end{aligned}$$

If the directions a and a' (b and b') are not parallel to each other, then the observables AB, AB', $A'B$, $A'B'$ are mutually incompatible. Therefore, there is no physically acceptable universal σ-algebra that corresponds to the measurement all these observables. It follows that there is no probability measure common for these observables. As a result, the reasoning which have led to to an inequality (8), appears unfair for the elementary states.

On the other hand, we can consider the elementary state as mathematical counter-part for local objective reality. Therefore, the hypothesis that local objective reality does exist in the quantum case does not lead to the Bell inequalities. Thus, the numerous experimental verifications of the Bell inequalities that have been undertaken in the past and at present largely lose theoretical grounds.

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