I. GENERAL ASPECTS OF HIGH ENERGY PHYSICS

TO WHAT DEGREE CAN WE TRUST IN EXPERIMENTAL DATA ?

Vladimir Anikeev

IHEP, Protvino,Russian[1]

In this report we discuss popular methods of estimation of true distributions from the observed/measured ones. We also try to give statistical properties of solution obtained where it is possible. For details and illustrations we send reader to the references [1, 2, 3] cited in chronological order.

1. Model of measurements

We consider normalized differential cross section $p(x) = d\sigma/dx/\sigma_{tot}$ in some kinematical variable x (one dimension, continuous).

$$\tilde{p}(x') = c\int dx\tilde{E}(x')K(x'|x)E(x)p(x), \tag{1}$$

where $0 \leq E(x) \leq 1$ is the probability of detection by the detector of event with given x (detector acceptance or efficiency), $K(x'|x) \geq 0$ is resolution of experiment with respect to the variable x, i.e. conditional probability density to obtain instead of true value x some other value x'. $K(x'|x)$ is calculated by the Monte Carlo simulation of detector operation and is not as a rule normal distribution. $0 \leq \tilde{E}(x')$ is the efficiency of reconstruction program and probability of inclusion of an event with given x' after cuttoffs with respect to the kinematical variables have been made in the analysis of events. c is a normalization constant, integral is over all allowed values of x at the given energy of the beam particle.

The integration of $p(x)$ with kernel $K(x'|x) \geq 0$ possesses a filtering property and leads to smoothing of the distribution $p(x)$ and loss of its fine structure. The presence of $E(x), \tilde{E}(x')$ leads to additional distortions. Therefore the difference between the true distribution $p(x)$ and experimental one $\tilde{p}(x')$ (systematic error, bias) may be significant and exceed the statistical error, and this necessitates a "correction" of the experimental data.

[1]e-mail: anikeev@mx.ihep.su

These data are the numbers of events in the channels of the histogram with the respect to the value x', which we denote Y_n.

$$\bar{Y}_n = L \int_{x'_n}^{x'_{n+1}} dx' \tilde{E}(x') \int dx K(x'|x) E(x) \frac{d\sigma}{dx},$$

where L – is the integrated luminosity of the experiment. Introducing the notation

$$S(x) = L d\sigma/dx, \quad A_n(x) = \int_{x_n}^{x_{n+1}} dx' \tilde{E}(x') K(x'|x) E(x), \tag{2}$$

we can express the mathematical model of the experiment in the form

$$Y_n = \int dx A_n(x) S(x) + \varepsilon_n, \quad n = 1, \dots N, \tag{3}$$

where ε_n is the fluctuation of the number of events Y_n in n-th bin. We assume $\bar{\varepsilon}_n = 0$, $\overline{\varepsilon_n \varepsilon_m} = K_{nm} \approx \delta_{nm}/Y_n$ for big N. The functions $A_n(x)$ have a transparent meaning: $A_n(x)$ is the probability for event with given x to enter n-th bin, so called apparatus functions. The functions $A_n(x)$ can be calculated only by the Monte Carlo method.

It is natural to interpret the expression (3) for the measurement Y_n as the scalar product of the functions $S(x)$ and $A_n(x)$. Introducing this scalar product in the space of the functions $S(x)$, we transform it into a Hilbert space. Then any function $S(x)$ can be expressed as a sum of two components: $S_A(x)$, which belongs to the subspace spanned by the system of functions $A_n(x)$,

$$S_A(x) = \sum_{n=1}^{N} C_n A_n(x), \tag{4}$$

and $S_A^{\perp}(x)$, the component orthogonal to this subspace:

$$\int dx A_n(x) S_A^{\perp}(x) = 0. \tag{5}$$

It follows that the component $S_A^{\perp}(x)$ cannot be estimated on the basis of Y_n, alone, since it follows from (3) and (4) that Y_n does not depend on it. The function $S_A(x)$ is often called the "measured/visible" component of $S(x)$, and $S_A^{\perp}(x)$ is called the "unmeasured/invisible" component.

From these considerations there follows an important conclusion: **If in the framework of the model (3) $S_A^{\perp}(x) \not\equiv 0$, then the finiteness of the number of measurements leads to a bias in estimate of the function $S(x)$.**

But if we have a priori information, say function with unknown parameters: $S(x) = f(x, \vec{a})$ the last statement is not true. We can write $f(x, \vec{a})$ as the sum of orthogonal components

$$f(x, \vec{a}) = f_A(x, \vec{a}) + f_A^{\perp}(x, \vec{a}).$$

If measurements contain enough information to estimate parameters $\vec{a}$ in $f_A(x, \vec{a})$, we can estimate $S(x)$, including its "invisible" component.

The set of functions $A_n(x)$ after discretization forms smearing matrix A, rows are apparatus functions ($n = 1, \ldots, N$), columns ($i = 1, \ldots, I$) are resolution functions. Smearing matrix is calculated by Monte Carlo simulation:

– we simulate K events with given x_i or $x \in [x_i, x_{i+1}]$,

– by use of program that simulates the detector we calculate measurements for each event,

– reconstruction program gives us a set of $\{x'\}$,

– we calculate the number of events k in n-th histogram bin in x',

– and obtain element of smearing matrix:

$$A_{ni} = A_n(x_i) = \frac{\int_{x'_n}^{x'_{n+1}} dx' \int_{x_i}^{x_{i+1}} K(x'|x) S(x) dx}{\int_{x_i}^{x_{i+1}} S(x) dx} \approx \frac{\int_{x'_n}^{x'_{n+1}} dx' \int_{x_i}^{x_{i+1}} K(x'|x) S^{MC}(x) dx}{\int_{x_i}^{x_{i+1}} S^{MC}(x) dx} = \frac{k}{K}.$$

Matrix A is not with necessity square. Binning in x: number of bins $I \geq N$, equicontent high statistics bins. The bin width in x' must be close to the experimental resolution. For several reasons smoothing of apparatus functions used (p.e. by B-splines).

In presenting methods and numerical illustrations they usually use the example:

$$S(x) = C\Big[A_1 \frac{G_1^2}{(x - x_1)^2 + G_1^2} + A_2 \frac{G_2^2}{(x - x_2)^2 + G_2^2}\Big],$$

$$E(x) = 1 - \frac{x - x_1}{x_2 - x_1 + 2G_2}, \quad \tilde{E}(x') = 1,$$

$$K(x'|x) = N(x'|x, \sigma^2),$$

where $x \in [4, 16]$, $A_1 = 2$, $A_2 = 1$, $G_1 = G_2 = 1$, $x_1 = 10$, $x_2 = 14$, $\sigma = 1.5$ and C was chosen using the condition $\int dx S(x) = 10^4$. Instead of *camelus bactrianus* we observe *camelus dromedarius.*

2. Purpose of a measurement → numerical method

The purpose of a measurement:

to $\boxed{\text{verify a theory}}$ – Testing a hypothesis. χ^2 comparison of the measurement and the simulation. Empirical distribution function (EDF) tests like Kolmogorov-Smirnov test.

to $\boxed{\text{determine one or several unknown parameters}}$ – Method to fit Monte Carlo distributions to data using least square method.

to $\boxed{\text{measure a distribution of some kinematical variable}}$ – Unfolding, i.e. inverse (ill-posed) problem.

We should always remember that any numerical method works under definite conditions (some values considered big, some small, some are neglected, etc).

3. Comparing a measured distribution to a simulated one

3.1. χ^2 test

Y_n, Y_n^{MC} measured and predicted N bin histograms. Assuming that the simulation describes the data, the numbers Y_n will follow the Poisson distribution with mean Y_n^{MC} and variance Y_n^{MC}. Monte Carlo fluctuations are neglected here.

$$\chi^2 = \sum_{n=1}^{N} \frac{(Y_n - Y_n^{MC})^2}{Y_n^{MC}}.$$

The statistical variance of the χ^2 distribution is, for not too small a number of bins N equal to $2N$. Thus we roughly measure the χ^2-value to a precision of $\sqrt{2N}$ and we may argue that we can detect a systematic error, contributing $\sqrt{2N}$ to χ^2. For instance for systematic deviations D_n we would expect

$$< \chi^2 >= \sum_n \frac{< (Y_n + D_n - Y_n^{MC})^2 >}{Y_n^{MC}} = N + \sum_n \frac{< D_n^2 >}{Y_n^{MC}} = N(1+ < \frac{< D_n^2 >}{Y_n^{MC}} >).$$

In assumption that the systematic deviation is proportional to the number of events in a bin $D_n \approx d_0 Y_n^{MC}$. We set $N < Y_n^{MC} = N_{tot}$ – total number of events and get

$$< \chi^2 >= N + d_0^2 N_{tot}.$$

Following our argument we can detect the additional contribution if it exceeds $\sqrt{2N}$ and find for the minimum detectable deviation

$$d_0 \approx \frac{(2N)^{1/4}}{N_{tot}^{1/2}}.$$

A necessary condition for the validity is that systematic deviation is not oscillating, but extends over many bins and that N is large enough to approximate the χ^2 distribution by a Gaussian.

χ^2 test with large number of bins have little significance. For visual inspection of localized systematic deviation one should use narrow bins.

Advises:

1. The number of Monte Carlo events per bin should be large enough $>\approx 20$ to approximate Poisson distribution by a Gaussian and to replace the expectation in a bin by the observed number when estimating the error.

2. Nonregular binning or even equicontent binning (equal relative error!).

3.2. Kolmogorov-Smirnov test

The problem of binning is avoided in a number of empirical distribution functions (EDF) tests.

The theoretical prediction is normalized to one and thus transformed into a probability density, the integral of which is the continuously rising distribution function $F(x)$.

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

The test statistics is the maximum difference between the experimental sum

$$S(x) = \frac{\text{number of events with} x_i < x}{\text{total number of events}}$$

and the prediction $F(x)$

$$D_{max} = \sup |S(x) - F(x)|.$$

The test quantity is the maximum deviation D_{max} of two curves. The probability to find $\sqrt{N_{tot}}D_{max} > \alpha$ is for large $N_{tot} > 100$ is

$$\lim_{N\to\infty} P(\sqrt{N_{tot}}D_{max} > \alpha) = 2\sum_{m=1}^{\infty}(-1)^{m-1}e^{-2m^2\alpha^2}.$$

Thus we can convert the observed D_{max} into probability, i.e. we can obtain the probability to observe in a similar experiment a larger deviation than we actually found. Application of Smirnov-Kolmogorov test is restricted to univariate distributions, for multivariate You should apply it to appropriate projections.

3.3. Tests based on quadratic statistics

The test quantity Q is

$$Q = N\int_{-\infty}^{\infty}(S(x) - F(x))^2\psi(x)dF(x).$$

$\psi(x) = 1$ – Cramer-von Mises staistics, usually called W^2 statistics.

$\psi(x) = F/(1-F)$ corresponds to the Anderson-Darling statistics A^2.

W^2 and A^2 statistics have similar power, but A^2 should be recommended because it is more sensitive to deviations in the tails of distributions.

4. Inference of parameters

It seems to us that the most consistent approach in parametric analysis is to use the direct measurements Y_n. This means that the estimate of the parameter vector

$\vec{a}$ must be sought by the method of least squares (χ^2-method) from the minimum of the functional

$$\Phi = \sum_{n,m} \Big(Y_n - < A_n|f(x|\vec{a}) > \Big) K^{-1}_{nm} \Big(Y_n - < A_m|f(x|\vec{a}) > \Big). \qquad (6)$$

If we have some information, say from PDG or from other publications we can include these data in the functional by adding appropriate terms. (Fig. 3) About the weight matrix K_{nm}. We can take into account the finiteness of statistics in Monte Carlo simulation.

Consider Poisson distribution in bins of experimental histogram:

$$K_{nm} \approx \delta_{nm}(Y_n + \sum_{i=1}^{I} S_{ni} \Big[\frac{f_i(x|\vec{a})}{I_i}\Big]^2),$$

where

$$S_{ni} = L \int_{x'_n}^{x'_{n+1}} dx' \tilde{E}(x') \int_{x_i}^{x_{i+1}} dx K(x'|x) E(x) f(x|\vec{a}), \quad I_i = \sum_n S_{ni}.$$

The functional (6) follows χ^2_{N-r}-distribution, where r is the number of parameters being estimated.

For multinomial distribution (number of events in input Monte Carlo vector is fixed)

$$K_{nm} \approx Y_n(1 - p_n) + \sum_{i=1}^{I} S_{ni}(1 - P_{ni}) \Big[\frac{f(x_i|\vec{a})}{I_i}\Big]^2, \text{ for } n = m$$

and

$$K_{nm} \approx -Y_n p_n - \sum_{i=1}^{I} S_{ni} P_{ni} \Big[\frac{f(x_i|\vec{a})}{I_i}\Big]^2, \text{ for } n \neq m,$$

where

$$p_n = \frac{Y_n}{\sum_{l=1}^{N} Y_l}, \; P_{ni} = \frac{S_{ni}}{\sum_{i=1}^{N} S_{ni}}.$$

The functional (6) follows χ^2_{N-r-1}-distribution.

5. Unfolding

5.1. Matrix inversion

In vector notation the observed distribution $\vec{Y}$ is related to the true distribution $\vec{S}$ by smearing matrix A:

$$\begin{aligned} \vec{Y} &= A\vec{S}, \\ A^+ \cdot \vec{Y} &= A^+ A \cdot \vec{S}, \\ (A^+ A)^{-1} A^+ \cdot \vec{Y} &= \vec{S}. \end{aligned}$$

So we get an estimate $\widehat{\vec{S}}$ of $\vec{S}$ using the Monte Carlo estimate $\widehat{A}$ of smearing matrix A –

$$\widehat{\vec{S}} = (\widehat{A}^{+}\widehat{A})^{-1}\widehat{A}^{+} \cdot \vec{Y} = \widehat{T}\vec{Y}.$$

Covariance matrix of S is

$$\begin{aligned} C_{ii} &= \textstyle\sum_k \widehat{T}_{ik}^2 Y_k, \\ C_{ij} &= \textstyle\sum_k \widehat{T}_{ik}\widehat{T}_{jk} Y_k. \end{aligned}$$

But we should remember about possible oscilillations (i.e. statistical errors), bias and taking into account finiteness of Monte Carlo statistics. There were special attempts based on SVD-inversion, iterative inversion, etc.

5.2. Linear estimates, regularization

We consider linear estimates to be a reliable method because this approach provides properties of estimates obtained.

General form of linear estimate $\hat{S}(x)$ of $S(x)$ is

$$\hat{S}(x) = \sum_{n=1}^{N} a_n(x) Y_n. \tag{7}$$

It follows from (4) and (5) that for reasonable estimates that do not use *a priori* information, $a_n(x)$ must lie in the subspace spanned by $A_n(x)$. So

$$\hat{S}(x) = \sum_n a_n(x) \int dx' A_n(x') S(x') + \sum_n a_n(x) \varepsilon_n.$$

Averaging over the errors of measurements we obtain

$$\bar{\hat{S}}(x) = \sum_n a_n(x) \int dx' A_n(x') S(x') = \int dx' A_R(x, x') S(x'), \tag{8}$$

where the function

$$A_R(x, x') = \sum_n a_n(x) A_n(x')$$

is so called residual apparatus function.

It follows from (8), that if $A_R(x, x')$ does not possess the property

$$\int dx' A_R(x, x') S_A(x') = S_A(x), \tag{9}$$

then the estimate $\hat{S}(x)$ will possess an additional bias due to the particular method of estimation, i.e. due to the specific choice of the functions $a_n(x)$ in (7).

The noise component of $\hat{S}(x)$ is characterized, by an error operator

$$\Sigma(x, x') \equiv \overline{\left(\hat{S}(x) - \bar{\hat{S}}(x)\right)\left(\hat{S}(x') - \bar{\hat{S}}(x')\right)} = \sum_{n,m} a_n(x) K_{nm} a_m(x').$$

Thus the linear estimate of the general form (7) can be interpreted as the result of "new experiment" with new apparatus function $A_R(x,x')$ and noise component $\xi(x)=\sum_n a_n(x)\varepsilon_n$ with error operator $\Sigma(x,x')$.

We consider the mean square of the distance between $S_A(x)$ and $\hat{S}(x)$:

$$\overline{D^2} = \overline{\int dx\Big[S_A(x)-\hat{S}(x)\Big]^2} =$$

$$= \overline{\int dx\Big[\Big(S_A(x)-\bar{\hat{S}}(x)\Big)+\Big(\bar{\hat{S}}(x)-\hat{S}(x)\Big)\Big]^2} =$$

$$= \int dx[S_A(x)-A_R S]^2 + Sp\Sigma.$$

It can be seen from this that it is convenient to characterize the noise component of the estimate $\hat{S}(x)$ as a whole by the value $Sp\Sigma$. The resolution of the "new experiment" (the bias of the estimate $\hat{S}(x)$) can be conveniently characterized by the value SpA_R.

Least squares method. On the basis of the arguments about the "measured" component $S(x)$, we shall seek an estimate of $S(x)$ in the form

$$\hat{S}(x)=\sum_n C_n A_n(x)$$

from the condition of a minimum of the functional

$$\Phi=\sum_{n,m}\Big[Y_n-\int dx A_n(x)\hat{S}(x)\Big]W_{nm}\Big[Y_n-\int dx A_m(x)\hat{S}(x)\Big],$$

which in this case takes the form

$$\Phi=\sum_{n,m}\Big(Y_n-\sum_k G_{nk}C_k\Big)W_{nm}\Big(Y_m-\sum_l G_{ml}C_l\Big),$$

where $W=K^{-1}$, $G_{nk}=<A_n|A_k>=\int dx A_n(x)A_k(x)$ – Gram matrix of the functions $A_n(x)$. Differentiation of this functional with respect C_i leads to the system of equations for the vector C:

$$G^T W G\hat{C}=G^T W Y.$$

Since the matrices G and W are positive definite, the solution can be written in the form

$$\hat{C}=G^{-1}Y.$$

Therefore, the least squares estimate $S(x)$ is

$$\hat{S}^+(x)=\sum_{n,m}A_n(x)G^{-1}_{nm}Y_m.$$

It is readily seen that $\int dx A_n(x)\hat{S}^+(x)=Y_n$ and the functional vanishes.

The residual apparatus function $A_R(x,x')$ in the least squares method,

$$A^+_R(x,x') = \sum_{n,m} A_n(x) G^{-1}_{nm} A_m(x')$$

possesses the maximum resolution that can be achieved, since it does not distort the "measured" component of $S(x)$ (the relation (9) is satisfied):

$$\begin{array}{rcl} \int dx' A^+_R(x,x') S_A(x') & = & \int dx' \sum_{n,m} A_n(x) G^{-1}_{nm} A_m(x') \sum_l C_l A_l(x') \\ & = & \sum_{n,m,l} C_l A_n(x) G^{-1}_{nm} G_{ml} = \sum_l C_l A_l(x) = S_A(x). \end{array}$$

This means that the least squares estimate $\hat{S}^+(x)$ is an unbiased estimate of the projection of $S(x)$ onto the subspace spanned by the system of functions $A_n(x)$, i.e. the subspace that is accessible for "measurement" by experiment: $\overline{\hat{S}^+}(x) = S_A(x)$.

We calculate the characteristics of the resolution introduced earlier:

$$SpA^+_R = \int dx \sum_{n,m} A_n(x) G^{-1}_{nm} A_m(x) = N.$$

This agrees with the natural assertion that the greater the number of measurements, the greater the "dimension" of the measured space and the better, in principle, one can construct the estimate $\hat{S}(x)$ on the basis of measurements Y_n.

The error operator and its spur for the least squares estimate:

$$\Sigma^+(x,x') = \sum_{n,m,l,k} A_n(x) G^{-1}_{nm} K_{ml} G^{-1}_{lk} A_k(x'), Sp\Sigma^+ = SpKG^{-1}.$$

We introduce eigenvectors $|g^{(l)}>$ and eigenvalues λ_l of the Gram matrix G:

$$G|g^{(l)} = \lambda_l |g^{(l)}>, \quad <g^{(l)}|g^{(m)}> = \delta_{lm}.$$

Using the theorem on the spectral representation of G^{-1}, we can write

$$G^{-1} = \sum_{l=1}^{N} \frac{|g^{(l)}><g^{(l)}|}{\lambda_l}.$$

Substituting this into expression for $\hat{S}^+(x)$, we obtain

$$\hat{S}^+(X) = \sum_{n,m,l} A_n(x) g^{(l)}_n g^{(l)}_m \frac{(\bar{Y}_m + \varepsilon_m)}{\lambda_l}.$$

It can be seen from this that the noise component of $\hat{S}^+(x)$ is proportional to ε_m/λ_l, which for small λ_l may be very large. Poorly conditioned matrix has relatively small eigenvalues, usually the reason is in large value of nondiagonal elements – this corresponds to the case when functions $A_n(x), A_m(x), n \neq m$ overlap strongly.

Thus, although the estimate $\hat{S}^+(x)$ has a finite noise component ($\lambda_{min} \neq 0$), its large value makes it necessary to use regularization methods developed for the solution of ill-posed problems.

Spectral window method. As a regularization method, the spectral window method reduces to supression of the contribution of small λ_l by introduction of a factor $f(\lambda_l)$ under the summation sign:

$$\tilde{G}^{-1} = \sum_{l=1}^{N} f(\lambda_l) \frac{|g^{(l)} >< g^{(l)}|}{\lambda_l},$$

such that $f(\lambda) \leq 1$ and $f(\lambda)$ is small for small eigenvalues. For the further analysis of the resolution and noise of the estimate obtained from $\hat{S}^+(x)$ by replacing G^{-1} by $\tilde{G}^{-1}$, it is convenient to introduce the functions

$$\Psi_l(x) = \frac{1}{\sqrt{\lambda_l}} \sum_n g_n^{(l)} A_n(x).$$

One can show that $\Psi_l(x)$ form an orthonormal basis in the "measured" space:

$$< \Psi_l|\Psi_k >= \sum_{n,m} \int dx \frac{1}{\sqrt{\lambda_l}} g_n^{(l)} A_n(x) \frac{1}{\sqrt{\lambda_k}} g_m^{(k)} A_m(x) =$$

$$= \sum_{n,m} \frac{1}{\sqrt{\lambda_l \lambda_k}} g_n^{(l)} G_{nm} g_m^{(k)} = \sum_n \frac{1}{\sqrt{\lambda_l \lambda_k}} g_n^{(l)} \lambda_k g_n^{(k)} = \delta_{lk}.$$

Making the substitution $G^{-1} \rightarrow \tilde{G}^{-1}$ in the expression for $\hat{S}^+(x)$, we obtain

$$\hat{S}(x) = \sum_l \frac{1}{\sqrt{\lambda_l}} \Psi_l(x) f(\lambda_l) < g^{(l)}|Y > .$$

From this expression we can see that

$$A_R(x, x') = \sum_l \Psi_l(x) f(\lambda_l) \Psi_l(x'), \tag{10}$$

$$SpA_R = \sum_l f(\lambda_l) < SpA_R^+ = N,$$

$$\Sigma(x, x') = \sum_{l,n,m,k} \Psi_l(x) \frac{1}{\sqrt{\lambda_l}} f(\lambda_l) g_n^{(l)} K_{nm} g_m^{(k)} f(\lambda_k) \frac{1}{\sqrt{\lambda_k}} \Psi_k(x'),$$

$$Sp\Sigma = \sum_{l,m,n} \frac{1}{\lambda_l} f^2(\lambda_l) g_n^{(l)} K_{nm} g_m^{(l)}.$$

For $K_{nm} = \sigma^2 \delta_{nm}$ we have

$$Sp\Sigma = \sum_l^N f^2(\lambda_l) \frac{\sigma^2}{\lambda_l} < Sp\Sigma^+ = \sum_l \frac{\sigma^2}{\lambda_l}.$$

The last assertion ($Sp\Sigma < Sp\Sigma^+$) also holds for a general form of matrix K.

Thus, the spectral-window method leads, on the one hand, to a detorioration of the resolution A_R, i.e. the estimate $\hat{S}(x)$ will have an additional bias compared with $\hat{S}^+(x)$ as a result of regularization. On the other hand, the noise component of $\hat{S}(x)$ is smaller than the noise component of $\hat{S}^+(x)$.

It is obviuos that for $f(\lambda) = 1$ expressions for $\hat{S}(x)$ and $Sp\Sigma$ are those for least squares method.

We consider the simplest form for $f(\lambda) : f(\lambda) = 1$ for $\lambda \geq \lambda_{N_R}$, $N_R < N$ and $f(\lambda) = 0$ for $\lambda < \lambda_{N_R}$. Assuming that λ_l are arranged in descending order, we take only first N_R terms in sums. N_R is the regularization parameter (discrete and rather coarse in this case). **Choice of the regularization parameter is the balance between the bias and the noise**. Several other regularization methods are in use: iterations, Tikhonov's regularization, dual. Among them Tikhonov's considered to be optimal.

5.3. Correcting factors method

Nonlinear estimates are nonlinear functions of measurements, this leads to considerable difficulties in obtaining $\Sigma(x, x')$ and to absence $A_R(x, x')$. Therefore in our opinion their use is justified when one uses the *a priori* information $S(x) > 0$, and when without it the linear estimates are unacceptably bad. One, very popular method is the correcting factor method. There exist two forms. We begin with one most widely used – the bin to bin correction method, which proceeds as follows:

– the Monte Carlo method is used to generate events whose distribution with respect to variable x is described by function $S_0(x)$;

– from it the histogram S_n^0 in variable x at the entrance to the detector is constructed;

– then the histogram $Y_n^{(0)}$ filled after the analysis of response of the detector;

– as corrected data ones takes the histogram:

$$\hat{S}_n^{(1)} = \frac{S_n^{(0)}}{Y_n^{(0)}} Y_n \equiv \frac{S_n^{(0)}}{\int dx A_n(x) S_0(x)} Y_n.$$

Bad resolution and appreciable deviation of $S_0(x)$ from unknown true $S(x)$ have the consequence that $\hat{S}_n^{(1)}$ do not depart from the original measurements Y_n.

Another more complicated form of correction factors method considers previous expression as the first step of iteration process:

$$S_n^{(i)} = \frac{S_n^{(i-1)}}{\int dx A_n(x) S^{(i-1)}(x)} Y_n,$$

where $S^{(i-1)}(x) = \sum_n S_n^{(i-1)} \chi_n(x)$, $\chi_n(x) = 1/(x_{n+1} - x_n)$ if $x \in [x_n, x_{n+1}], = 0$ otherwise. Solution $\hat{S}_n^{(\infty)}$, obtained at the end of iteration process on the condition $max|(S_n^{(i)} - S_n^{(i-1)})/S_n^{(i)}| < 0.01$ – stabilization. But exists instability with respect to the fluctuations in Y_n (as in least squares method). If we rewrite the denominator of last relation in the form

$$\int dx A_n(x) \sum_m S_m^{(i-1)} \chi_m(x) = \sum_m A_{nm} S_m^{(i-1)},$$

where $A_{nm} =< A_n|\chi_m >$. The condition for stabilization of iteration process means that

$$\frac{Y_n}{\sum_m A_{nm} S_m^{(i-1)}} \approx 1$$

i.e., on stabilization we have approximate equation

$$\sum_m A_{nm} S_m^{(i-1)} \approx Y_n.$$

However, the matrix A_{nm}, like Gram matrix G_{nm} may have small eigenvalues that leads to an unacceptably large noise component of $\hat{S}_n^{(\infty)}$.

Finally, for the solution by correction factors method one can calculate the noise characteristics after one iteration, but the solution contains a systematic error, which depends on the function $S_0(x)$ (model dependence).

With regard to the solution of the correction factors method in accordance with the iteration scheme, we can draw the following conclusions:

– the solution converges to true one in the absence of noise in Y_n;

– the exit from the iteration process using the stabilization criterion can lead to an unacceptably large statistical error;

– by chosing a smaller then optimal number of iterations, one can make the error acceptable;

– it is difficult to interpret the estimate based on the correction factors method, since the procedure being nonlinear, it is impossible to calculate the characteristics of noise and the bias;

– the estimate by Tikhonov's method was found to be better on test and for it one can calculate the characteristics of the noise and the bias.

Conclusion

Finally, we recommend the reader to trust the data from the publication if:

– **the method used is proved and illustrated on realistic examples**, or

– **the method is cited from the well-known reliable source**.

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