RELATIVISTIC THEORY OF GRAVITY: GAUGE APPROACH

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Analysis of spontaneous symmetry breaking and energy-momentum conservation laws in gauge gravitation theory leads to the affine-metric modification of the Relativistic theory of gravity by A.A.Logunov in the presence of fermion fields.

1. Introduction

Identification of a gravitational field with a pseudo-Riemannian metric is the cornerstone of Einstein's General Relativity (GR) and many other gravitation models. This identification however is sufficient, but not necessary condition for that the dynamics of matter in the presence of a gravitational field looks like motion in some effective pseudo-Riemannian geometry. The Relativistic theory of gravity (RTG) by A.A.Logunov [5, 6] exemplifies a gravitation model where the effective metric is composition of a background pseudo-Riemannian metric and a non-metric gravitational field. To analize this problem, we appeal to gauge gravitation theory where a pseudo-Riemannian metric is not introduced by hand, but appears as a Higgs field corresponding to spontaneous breaking of world symmetries.

We follow the well-known mathematical formulation of gauge theory where classical fields are represented by sections of a bundle

$$\pi : Y \to X, \qquad s : X \ni x \mapsto s(x) \in \pi^{-1}(x)$$

over a world manifold X. The base X is a locally compact paracompact oriented connected 4-manifold. It is assumed to be a noncompact manifold with a trivial tangent bundle TX in order that a pseudo-Riemannian metric, a spinor structure and a space-time structure can exist on X. A bundle $Y \to X$ is provided with fibred coordinates

$$(x^\mu, y^i), \qquad x'^\lambda = \phi^\lambda(x^\mu), \qquad y'^i = \phi^i(x^\mu, y^j),$$

where (x^μ) are coordinates of X.

Gauge gravitation theory is called into play when gauge transformations act on the partial derivative operators $\{\partial_\lambda\}$ which form holonomic frames of the tangent bundle TX. One separates gauge transformations which send holonomic frames into holonomic frames. They are general covariant transformations. Invariance of Lagrangians under general covariant transformations lead to the energy-momentum conservation laws in gravitation theory [1, 3, 13].

To construct gauge gravitation theory, we are based on the fact that Dirac fermion fields possess only Lorentz symmetries and therefore imply breaking of world symmetries. This symmetry breaking can occur iff a pseudo-Riemannian metric exists on a world manifold X [9, 10]. This metric h defines the representation of the holonomic coframes $\{dx^\lambda\}$ in the cotangent bundle T^*X by the Dirac γ-matrices

$$\gamma_h(dx^\lambda) = h^\lambda_a\gamma^a, \quad h^{\mu\nu} = h^\mu_a h^\nu_b \eta^{ab}, \tag{1}$$

where h^λ_a are the tetrad functions of h and η is the Minkowski metric. One needs this representation in order to construct the Dirac operator

$$\mathcal{D}^h = \gamma_h(D_\lambda dx^\lambda) = h^\lambda_a\gamma^a(\partial_\lambda - A_\lambda), \tag{2}$$

where A is a spinor connection associated with h. The key point lies in the fact that different pseudo-Riemannian metrics define nonequivalent representations (1), for there is no spinor transformation ρ which obeys the condition

$$\rho\circ\gamma_h(dx^\lambda) = \gamma_{h'}(dx^\lambda)\circ\rho.$$

The problem is to determine general covariant transformations acting on the representation (1) and the Dirac operator (2). We observe the following two kinds of such transformations.

(i) General covariant transformations change a metric $h \to h'$ and bring the representation (1) into the nonequivalent representation

$$\gamma_{h'}(dx^\lambda) = h'^\lambda_a\gamma^a. \tag{3}$$

To describe such transformations, we consider the composite spinor bundle $S \to \Sigma \to X$, where Σ is a bundle whose sections are the pseudo-Riemannian h [11, 12]. In this case, a pseudo-Riemannian metric is a dynamic variable which can be identified with a gravitational field. As a result, gauge gravitation theory comes to the affine-metric gravitation theory where spinor fields are defined in the presence of a general linear connection on X [3].

(ii) We can consider gauge transformations which are general covariant transformations of matter fields, but keep a given metric h. Then, we have

$$\gamma_h(dx^\lambda) \to \gamma_h(\Phi^\lambda{}_\mu dx^\mu) = \Phi^\lambda{}_\mu h^\mu_a\gamma^a. \tag{4}$$

It is readily observed that the Dirac operator (2) is not invariant under these transformations. To overcome this difficulty, one may introduce an additional geometric field q represented locally by tensor functions $q^\lambda{}_\mu$ and then may rewrite the Dirac operator into the form

$$\mathcal{D}^{qh} = q^\lambda{}_\mu h^\mu_a \gamma^a D^h{}_\lambda, \tag{5}$$

which is invariant under the transformations (4):

$$\mathcal{D}^{qh} \to \Phi^\lambda{}_\nu q^\lambda{}_\mu h^\mu_a \gamma^a D'^h{}_\lambda = q'^\lambda{}_\mu h^\mu_a \gamma^a D'^h{}_\lambda.$$

As a result, we get the affine-metric modification of RTG where independent dynamic variables are gravitational fields q and general linear connections Γ on X in the presence of a background metric, e. g. the Minkowski metric. Since the above-mentioned gauge transformations do not act on the indices of the background Minkowski metric $\eta^{\mu\nu}$ and the low index μ of a gravitational field $q^\lambda{}_\mu$, it is the tensor field

$$g^{\mu\nu} = q^\mu{}_\alpha q^\nu{}_\beta h^{\alpha\beta} = H^\mu_a H^\nu_b \eta^{ab}, \tag{6}$$

which perform contraction of indices in the Lagrangians of matter fields and connections. Thus, the tensor field (6) plays the role of an effective metric, although it is not a true metric, for the coframes $H^a = H^a_\mu dx^\mu$ are carried by the γ-matrices

$$\gamma_h(H^a) = h_{\mu\nu}\eta^{ab} q^\nu{}_\alpha h^\alpha_b h^\mu_c \gamma^c$$

with respect to the same representation as the coframes $h^a = h^a_\mu dx^\mu$ are done. At the same time, there always exists a pseudo-Riemannian metric h' such that $h'^{\mu\nu} = g^{\mu\nu}$.

As a consequence, a gauge invariant Lagrangian of affine-metric RTG is the sum

$$L_{\rm RG} = L_{\rm AM}(g,\Gamma) + L_\psi + L_q(q,\eta) \tag{7}$$

(i) of a Lagrangian $L_{\rm AM}$ of affine-metric gravitation theory and a Lagrangian L_ψ of fermion fields which utilize the effective metric g (6) and are invariant under standard general covariant transformations, and (ii) of a Lagrangian L_q of fields q where contraction is performed by means of the background Minkowski metric. Therefore, L_q is not invariant under standard general covariant transformations. In particular, put

$$L_{\rm AM} = (-\lambda_1 R + \lambda_2)\sqrt{\mid g \mid}, \quad L_\psi = 0, \quad L_q = \lambda_3 \eta_{\mu\nu} g^{\mu\nu}\sqrt{\mid \eta \mid}, \tag{8}$$

where $R = g^{\mu\nu} R^\alpha{}_{\mu\alpha\nu}$ is the effective scalar curvature of a connection Γ. Then, the conventional RTG is recovered. Moreover, it is the Higgs-type massive term L_q (8) which provides the nonzero local part of the total energy-momentum flow, besides the superpotential term.

Note that additional tensor fields q and generalized tetrad fields $H^\mu_a = q^\mu{}_\nu h^\nu_a$ have been considered in the framework of both the $GL(4,\mathbf{R})$-gauge models [7, 4] and the translation group gauge model where q describe deformations of a world manifold [9, 10]. Moreover, since different pseudo-Riemannian metrics define nonequivalent representations (1) and so, do not satisfy the superposition principle, we come to RTG again in order to describe deviations of a metric gravitational field.

2. Symmetry breaking

Let LX be a principal linear frame bundle associated with the tangent bundle TX. Its structure group is $GL_4 = GL^+(4,\mathbf{R})$.

Recall that, given a principal bundle $\pi_P : P \to X$, its structure group G acts on P on the right in a canonical way

$$r_g : p \mapsto pg, \quad \pi_P(p) = \pi_P(pg), \quad p \in P, \quad g \in G. \tag{9}$$

Accordingly, every P-associated bundle $Y \to X$ with a standard fibre V is isomorphic to the quotient $Y = (P \times V)/G$ by identification of elements (p, v) and $(pg, g^{-1}v)$, $g \in G$.

In gauge gravitation theory as like as in other gauge models, symmetry breaking is formalized by the mathematical condition that there exists a principal subbundle L^hX of the frame bundle LX whose structure group is the Lorentz group $L = SO(3,1)$). In virtue of the well-known theorem, there is the 1:1 correspondence between the Lorentz subbundles L^hX of LX and the sections h of the quotient bundle

$$\Sigma := LX/L \to X \tag{10}$$

with the standard fibre GL_4/L. This bundle is the 2-fold covering of the bundle $B \subset \overset{2}{\vee} TX$ of the pseudo-Riemannian bilinear forms in tangent spaces to X. Therefore, every its global section h define a pseudo-Riemannian metric on X. We denote $\mid h \mid = -\det(h_{\mu\nu})$.

In particular, every metric field h yields an associated Lorentz reference frame given by a family of local sections

$$\{h_a(x) = h^\mu_a(x)\partial_\mu\} \tag{11}$$

of LX which take their values into L^hX so that $\eta_{ab} = h^\mu_a h^\nu_b h_{\mu\nu}$. Lorentz reference frames are not holonomic in general. Therefore, given a Lorentz atlas and a holonomic atlas of the bundle LX, the metric field h can be represented by a family of the tetrad functions $h^\mu_a(x)$ (11).

Every metric field h separates also the subset of Lorentz connections associated with h. These are the connections on the principal bundle L^hX. Every such a connection can be extended to a connection Γ on the principal bundle LX so that $\Gamma^{ab}{}_\lambda = -\Gamma^{ba}{}_\lambda$ with respect to the Lorentz frames (11). At the same time, every connection Γ on LX projected onto the Lorentz Lie algebra defines the connection associated with h:

$$A^{ab}{}_\lambda = \frac{1}{2}(\Gamma^{ab}{}_\lambda - \Gamma^{ba}{}_\lambda). \qquad (12)$$

3. Spinor structure

We follow the standard machinery of the Clifford algebra $\mathbf{C}_{1,3}$ in order to describe Dirac's spinors [2, 8]. This algebra contains both the Minkowski space M of generating elements and the spinor space V of its minimal left ideal so that $M \subset \mathbf{C}_{1,3}$ acts on V on the left by the representation

$$\gamma : M \otimes V \to V, \qquad \gamma(e^a \otimes v^A) = \gamma^{aA}{}_B v^B, \qquad (13)$$

where $\{e^a\}$ is the basis of M, $\{v^A\}$ is the basis of V, and γ^a are the Dirac's matrices. The invertible elements of $\mathbf{C}_{1,3}$ which preserve the representation (13) form the Klifford group whose action on the Minkowski space M however fails to be effective. One considers usually its subgroup $SL(2, \mathbf{C})$ with the generators

$$I_{ab} = \frac{1}{4}[\gamma_a, \gamma_b].$$

For the sake of simplicity, let us identify $SL(2, \mathbf{C})$ with the Lorentz group L (we refer to [10] for the technical details).

To describe spinor fields, let us consider a bundle of the Clifford algebras $\mathbf{C}_{3,1}$ over X. It contains both a spinor bundle $S \to X$ and the bundle of Minkowski spaces $Y_M \to X$. There is the fibre-to-fibre morphism

$$\gamma_M : Y_M \otimes S \to S, \qquad (14)$$

which represent the elements of the bundle Y_M by the γ-matrices on elements of S.

To describe the Dirac fermion fields, we must require that the bundle Y_M is isomorphic to the cotangent bundle T^*X. It takes place if the spinor bundle S is associated with some Lorentz subbundle L^hX of the frame bundle LX. Then, we have

$$Y_M = (L^hX \times \mathbf{R}^4)/L = T^*X,$$
$$S = (L^hX \times V)/L = S_h, \qquad (15)$$

and the fibre-to-fibre morphism (1) can be defined as follows:

$$\gamma_h : T^*X \otimes S_h = (L^hX \times (\mathbf{R}^4 \otimes V))/L \to (L^hX \times \gamma(\mathbf{R}^4 \otimes V))/L = S_h. \qquad (16)$$

Given the representation (16), one says that sections ψ_h of the spinor bundle S_h (15) describe Dirac fermion fields in the presence of the metric field h.

Moreover, since the representations γ_h and $\gamma_{h'\neq h}$ are not equivalent, every Dirac fermion field can be considered only in a pair (ψ_h, h) with a certain metric field h.

4. Gauge transformations

Let us examine isomorphisms of the principal frame bundle LX.

Recall that an isomorphism Φ of a bundle $Y \to X$ over a diffeomorphism of X, by definition, is the fibre-to-fibre morphism which send the fibres $\pi^{-1}(x)$ into the fibres $\pi^{-1}(f(x))$. If $f = \mathrm{Id}\, X$, we call Φ the vertical isomorphism. In particular, every isomorphism Φ of a principal bundle P, by definition, is equivariant under the canonical action (9), that is, $r_g \circ \Phi = \Phi \circ r_g$. It generates the corresponding isomorphism

$$\Phi_Y : (P \times V)/G \to (\Phi(P) \times V)/G \tag{17}$$

of a bundle Y associated with P.

We can restrict our consideration to 1-parameter groups of bundle isomorphisms. Every 1-parameter group $G[\alpha]$ of isomorphisms of a bundle $Y \to X$ over diffeomorphisms $f[\alpha]$ of X yelds the corresponding fundamental vector field

$$u = u^\mu(x)\partial_\mu + u^i(y)\partial_i \tag{18}$$

on Y which is the generator of $G[\alpha]$. This vector field is projected onto the vector field $\tau_u = u^\mu\partial_\mu$ on X which is the generator of the group $f[\alpha]$. For instance, if $G[\alpha]$ are vertical isomorphisms, the vector field $u = u^i(y)\partial_i$ (18) is vertical. Given a vector field u (18), the Lie derivative $\mathbf{L}_u L$ of a Lagrangian L is equal to zero iff L is invariant under the coresponding 1-parameter group $G[\alpha]$. This condition is sufficient in order to get conservation laws in field theory.

Turn now to the frame bundle LX. With respect to a holonomic reference frame $\{\partial_\mu\}$, every element $\{H_a\}$ of LX is given by expression $H_a = S^\mu{}_a\partial_\mu$ where $S^\mu{}_a$ are matrices of the representation of the group GL_4 in $\mathbf{R}^4$. They form the fibred coordinates

$$(x^\lambda, S^\mu{}_a), \qquad S'^\mu{}_a = \frac{\partial x'^\mu}{\partial x^\lambda} S^\lambda{}_a, \tag{19}$$

of the bundle LX. Relative to these coordinates, the canonical action (9) of GL_4 on LX reads $r_g : S^\mu{}_a \mapsto S^\mu{}_b g^b{}_a$. In particular, the Lorentz subbundle $L^hX \subset LX$ consists of the elements with the coordinates $(x^\lambda, h^\lambda{}_b(x)g^b{}_a)$, $g \in L$.

Every diffeomorphism f of X gives rise to the general covariant transformation

$$\widetilde{f} : (x^\lambda, S^\lambda{}_a) \mapsto (f^\lambda(x), \partial_\mu f^\lambda S^\mu{}_a) \tag{20}$$

of the principal frame bundle LX and to the corresponding isomorphisms (17) of associated bundles.

The lift (20) implies the canonical lift $\widetilde{\tau}$ of every vector field τ on X onto the principal bundle LX and onto associated bundles, in particular, onto a tensor bundle $T = (\overset{m}{\otimes} TX) \otimes (\overset{k}{\otimes} T^*X)$:

$$\widetilde{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^{\alpha_1} \dot{x}^{\nu\alpha_2 \cdots \alpha_m}_{\beta_1 \cdots \beta_k} + \ldots - \partial_{\beta_1} \tau^\nu \dot{x}^{\alpha_1 \cdots \alpha_m}_{\nu \beta_2 \cdots \beta_k} - \ldots] \frac{\partial}{\partial \dot{x}^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_k}}.$$

In virtue of the equivariance property, every isomorphism $\widetilde{f}$ (20) of LX sends a Lorentz subbundle $L^h X$ into another Lorentz subbundle $L^{h'} X$ and, accordingly, send the associated spinor bundle S_h (15) into the spinor bundle $S_{h'}$:

$$\widetilde{f} : S_h \ni (p \times v)/L \mapsto (\widetilde{f}(p) \times v)/L \in S_{h'}.$$

It follows that we must consider all of the spinor bundles S_h and all of the pairs (ψ_h, h) in order to examine general covarian transformations in gauge gravitation theory where metric fields are identified with gravitational fields.

5. Gauge gravitation theory

Following the general procedure of analysis of gauge models with symmetry breaking [11], the fermion-gravitation pairs (ψ_h, h) can be described by sections of the composite spinor bundle

$$S \to \Sigma \to X. \tag{21}$$

where $\Sigma \to X$ is the quotient bundle (10) and $S \to \Sigma$ is the spinor bundle

$$S_\Sigma = (LX \times V)/L$$

associated with the Lorentz principal bundle $LX \to \Sigma$ [3, 11]. It is readily observed that, given a section h of Σ, the restriction of S_Σ to $h(X) \subset \Sigma$ is the subbundle S_h (15) of the bundle (21).

The composite spinor bundle (21) is coordinatized by $(x^\lambda, \sigma^\mu_a, \psi^A)$ where $(x^\lambda, \sigma^\mu_a)$ are the coordinates of Σ such that the frame fields $\{h_a = \sigma^\mu_a \partial_\mu\}$ constitute an atlas of the Lorentz bundle $LX \to \Sigma$. Given a section h of Σ, the coordinate functions $(\sigma^\lambda_a \circ h)(x) = h^\lambda_a(x)$ are exactly the tetrad functions (11) of h.

Since the bundle $LX \to \Sigma$ is trivial, the transition functions of the coordinates ψ_A are independent on the coordinates σ^a_μ, and one can define the representation

$$\gamma_\Sigma : T^*X \underset{\Sigma}{\otimes} S = (LX \times (M \otimes V))/L \to (LX \times \gamma(M \otimes V))/L = S, \tag{22}$$

$$\gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a.$$

It is readily observed that, given a metric field h, the restriction of the representation (22) onto $h(X) \subset \Sigma$ recovers the representation γ_h (16).

Let us construct the Dirac operator on the composite bundle (21) and the Lagrangian of fermion-gravitation pairs. We utilize the standard machinery of jet manifolds.

As a shorthand, one can say that the k-order jet manifold J^kY of a bundle $Y \to X$ comprises the equivalence classes $j^k_x s$, $x \in X$, of sections s of Y identified by the first $k+1$ terms of their Taylor series at a point x [14, 12]. Given fibered coordinates (x^λ, y^i) of $Y \to X$, the k-jet manifold is coordinatized by

$$(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda\mu}, \ldots, y^i_{\lambda_1\cdots\lambda_k}), \quad (y^i, y^i_\lambda, \ldots)(j^k_x s) = (s^i(x), \partial_\lambda s^i(x), \ldots).$$

In particular, the first order jet manifold J^1Y is coordinatized by $(x^\lambda, y^i, y^i_\lambda)$ where

$$y'^i_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda}(\partial_\mu + y^j_\mu \partial_j) y'^i.$$

Differential operators and the Lagrangian formalism are phrased conventionally in the jet terms. In particular, every connection $\Gamma = \Gamma^i_\lambda dx^\lambda \otimes \partial_i$ on a bundle $Y \to X$ yields the first order differential operator

$$D = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i \tag{23}$$

called the covariant differential. In the first order Lagrangian formalism, a Lagrangian is defined to be a horizontal density

$$L = \mathcal{L}(x^\mu, y^i, y^i_\mu)\omega, \quad \omega = dx^0 \wedge \cdots \wedge dx^3.$$

on the jet manifold J^1Y which plays the role of a finite-dimensional configuration space of fields.

The percularity of dynamics on the composite manifold (21) lies in the fact that the covariant differential (23) depends on components of the connection along the gravitational coordinates σ^λ_a and on those of a general connection on X [3].

Let the Lorentz principal bundle $LX \to \Sigma$ be provided with a connection

$$\widetilde{A} = (\widetilde{A}^{ab}{}_\mu dx^\mu + A^{abc}{}_\mu d\sigma^\mu_c) \otimes I_{ab}, \tag{24}$$

$$\widetilde{A}^{ab}{}_\mu = \frac{1}{2}\Gamma^\nu{}_{\lambda\mu}\sigma^\lambda_c(\eta^{ca}\sigma^b_\nu - \eta^{cb}\sigma^a_\nu), \tag{25}$$

$$A^{abc}{}_\mu = \frac{1}{2}(\eta^{ca}\sigma^b_\mu - \eta^{cb}\sigma^a_\mu), \tag{26}$$

where Γ is a connection on LX, (25) is the Lorentz connection (12) defined by Γ, and (26) is the canonical curvature-free connection on the bundle $GL_4 \to GL_4/L$. The corresponding covariant differential on the bundle S (21) reads

$$D = dx^\lambda \otimes [\partial_\lambda - \frac{1}{2}A^{abc}{}_\mu(\sigma^\mu_{c\lambda} + \Gamma^\mu{}_{\nu\lambda}\sigma^\nu_c) I_{ab}{}^A{}_B \psi^B \partial_A]. \tag{27}$$

Given a section $h(x)$ of Σ, the connection (24) comes to a Lorentz connection on the subbundle L^hX of LX, and the covariant differential (27) recovers the covariant differential (2).

Thus, we get the affine-metric gravitation theory in the presence of fermion fields where independent variables are metric gravitational fields h and principal connections Γ on the principal frame bundle LX. Such connections are represented by sections of the bundle

$$C = J^1LX/GL_4 \to X$$

coordinatized by $(x^\lambda, k^\mu{}_{\nu\lambda})$ so that, given a section Γ of C, its coordinates $k^\mu{}_{\nu\lambda}\circ\Gamma = \Gamma^\mu{}_{\nu\lambda}$ are exactly the components of the linear connection Γ on TX [3, 12, 13].

The total configuration space of fermion-gravitation pairs is the jet manifold J^1Y of the bundle

$$Y = C \underset{\Sigma}{\times} S \underset{\Sigma}{\oplus} S^+ \tag{28}$$

coordinatized by $(x^\mu, \sigma^\mu_a, k^\mu{}_{\nu\lambda}, \psi^A, \psi^+_A)$. A total Lagrangian on this configuration space is given by the sum

$$L = L_{\rm AM} + L_\psi \tag{29}$$

(i) of the affine-metric Lagrangian $L_{\rm AM}(R^\alpha{}_{\beta\nu\lambda}, \sigma^{\mu\nu})$ expressed into the curvature tensor

$$R^\alpha{}_{\beta\nu\lambda} = k^\alpha{}_{\beta\lambda\nu} - k^\alpha{}_{\beta\nu\lambda} + k^\alpha{}_{\varepsilon\nu}k^\varepsilon{}_{\beta\lambda} - k^\alpha{}_{\varepsilon\lambda}k^\varepsilon{}_{\beta\nu}.$$

and the metric tensor $\sigma^{\mu\nu} = \sigma^\mu_a\sigma^\nu_b\eta^{ab}$, and (ii) of the Lagrangian of fermion fields

$$\begin{aligned} L_\psi = \{&\frac{i}{2}[\psi^+_A(\gamma^0\gamma^\lambda)^A{}_B(\psi^B_\lambda - \frac{1}{2}A^{abc}{}_\mu(\sigma^\mu_{c\lambda} + k^\mu{}_{\nu\lambda}\sigma^\nu_c)I_{ab}{}^B{}_C\psi^C) - \\ &-(\psi^+_{A\lambda} - \frac{1}{2}A^{abc}{}_\mu(\sigma^\mu_{c\lambda} + k^\mu{}_{\nu\lambda}\sigma^\nu_c)\psi^+_C I^{+C}_{ab}{}_A)(\gamma^0\gamma^\lambda)^A{}_B\psi^B] \\ &-m\psi^+_A(\gamma^0)^A{}_B\psi^B\}\sigma^{-1}\omega, \end{aligned} \tag{30}$$

where $\gamma^\mu = \sigma^\mu_a\gamma^a$ and $\sigma = \det(\sigma^\mu_a)$.

The Lagrangian (29), by construction, is invariant under general covariant transformations whose generator is the canonical lift

$$\begin{aligned} \widetilde\tau = {}&\tau^\mu\partial_\mu + (\partial_\nu\tau^\alpha k^\nu{}_{\beta\mu} - \partial_\beta\tau^\nu k^\alpha{}_{\nu\mu} - \partial_\mu\tau^\nu k^\alpha{}_{\beta\nu} - \partial_{\beta\mu}\tau^\alpha)\frac{\partial}{\partial k^\alpha{}_{\beta\mu}} + \\ &+\partial_\nu\tau^\mu\sigma^\nu_c\frac{\partial}{\partial\sigma^\mu_c} + \frac{1}{2}\partial_\alpha\tau^\beta(\eta^{ap}\sigma^\alpha_p\sigma^b_\beta - \eta^{bp}\sigma^\alpha_p\sigma^a_\beta)[(\eta_{bc}\delta^d_a - \eta_{ac}\delta^d_b)\sigma^\mu_d\frac{\partial}{\partial\sigma^\mu_c} + \\ &+I_{ab}{}^A{}_B\psi^B\partial_A + I^{+A}_{ab}{}_B\psi^+_A\partial^B]. \end{aligned} \tag{31}$$

of a vector field τ on X onto the bundle (28).

6. Relativistic theory of gravity

There is another lift of diffeomorphisms of the world manifold X onto LX which preserve a given metric field h and differs from general covariant transformations (20) in vertical gauge transformations.

Let h be a pseudo-Riemannian metric and Γ_h some associated Lorentz connection (12) on LX. Any 1-parameter group $f[\alpha]$ of diffeomorphisms of X whose generator is a vector field τ on X gives rise to the group of nonholonomic isomorphisms $f_\Gamma[\alpha]$ of LX whose generator is the vector field

$$\tau_\Gamma = \tau^\lambda(\partial_\lambda - {\Gamma_h}^\mu{}_{\nu\lambda} S^\nu{}_a \frac{\partial}{\partial S^\mu{}_a})$$

on LX. Isomorphisms $f_\Gamma[\alpha]$ keep the Lorentz subbundle L^hX and the metric field h. Given a 1-parameter group $\widetilde{f}[\alpha]$ of general covariant transformations over diffeomorphisms $f[\alpha]$, its elements are represented by composition

$$\widetilde{f}[\alpha] = \Phi[\alpha] \circ f_\Gamma[\alpha] \tag{32}$$

of $f_\Gamma[\alpha]$ and the vertical isomorphisms $\Phi[\alpha]$ of LX with the generator

$$u = D_\nu \tau^\mu S^\nu{}_a \frac{\partial}{\partial S^\mu{}_a},$$

where by D_ν are meant the covariant derivatives with respect to the connection Γ_h.

Recall the following construction. Let $P \to X$ be a principal bundle with a structure group G and $\widetilde{P} \to X$ the associated group bundle with the standard fibre G on which the structure group acts by the adjoint representation. Given a bundle Y associated with P, there is the canonical fibre-to-fibre morphism $\widetilde{P} \times Y \to Y$. As a consequence, every vertical isomorphism Φ_Y (17) of an associated bundle Y is brought into the form

$$\Phi_Y(y) = q_\Phi(\pi(y)) \cdot y, \qquad y \in Y,$$

where $q_\Phi(x)$ is a global section of the group bundle $\widetilde{P}$. In particular, we can show that the matrix functions $\Phi^\lambda{}_\mu(x)$ in the expression (4) are local functions of a section $q_{\Phi[\alpha]}$ of the group bundle $\widetilde{LX} \subset TX \otimes T^*X$ which corresponds to the vertical isomorphism $\Phi[\alpha]$ in the expression (32).

Therefore, let us introduce additional dynamic fields q represented by sections of the group bundle $\widetilde{LX}$ and consider the compositions $q \cdot h$. Then, every vertical gauge transformation of $q \cdot h$ reads

$$\Phi(q \cdot h) = q_\Phi \cdot q \cdot h = q' \cdot h.$$

The field q in the expression (5) exemplifies such a section of $\widetilde{LX}$.

Thus, we come to RTG in case of a background metric field h, the dynamic gravitational fields q and the effective metric fields $q \cdot h$. The total configuration space of RTG in the presence of fermion fields is the jet manifold of the bundle

$$Y = \widetilde{LX} \underset{X}{\times} C \underset{X}{\times} S_h \underset{X}{\oplus} S_h^+ \tag{33}$$

coordinatized by $(x^\mu, q^\mu{}_\nu, k^\mu{}_{\nu\alpha}, \psi^A, \psi^+_A)$. Let us consider the gauge transformations of the bundle (33) whose generator is the vector field

$$\widetilde{\tau} = \tau^\mu \partial_\mu + (\partial_\nu \tau^\alpha k^\nu{}_{\beta\mu} - \partial_\beta \tau^\nu k^\alpha{}_{\nu\mu} - \partial_\mu \tau^\nu k^\alpha{}_{\beta\nu} - \partial_{\beta\mu}\tau^\alpha)\frac{\partial}{\partial k^\alpha{}_{\beta\mu}} + \tag{34}$$
$$(\partial_\nu \tau^\mu q^\nu{}_\alpha + \tau^\lambda \Gamma_h{}^\nu{}_{\alpha\lambda} q^\mu{}_\nu)\frac{\partial}{\partial q^\mu_\alpha} +$$
$$\frac{1}{2}\partial_\alpha \tau^\beta (\eta^{ap} h^\alpha_p h^b_\beta - \eta^{bp} h^\alpha_p h^a_\beta)[I_{ab}{}^A{}_B \psi^B \partial_A + I^{+A}_{ab}{}_B \psi^+_A \partial^B].$$

These transformations act on the metric functions $h^{\mu\nu}$ and the tetrad functions h^μ_a of a background metric h by the law

$$h^{\mu\nu} \to \tau^\lambda(\partial_\lambda h^{\mu\nu} - \Gamma_h{}^\mu{}_{\alpha\lambda} h^{\alpha\nu} - \Gamma_h{}^\nu{}_{\alpha\lambda} h^{\mu\alpha}),$$
$$h^\mu_a \to \tau^\lambda(\partial_\lambda h^\mu_a - \Gamma_h{}^\mu{}_{\alpha\lambda} h^\alpha_a) + \partial_\alpha \tau^\beta (\eta^{dp}\eta_{ba} h^\alpha_p h^b_\beta - h^\alpha_a h^d_\beta) h^\mu_d.$$

At the same time, they act on the effective metric field g in accordance with the standard law of general covariant transformations

$$g^{\mu\nu} \to \tau^\lambda \partial_\lambda g^{\mu\nu} + \partial_\alpha \tau^\mu g^{\alpha\nu} + \partial_\alpha \tau^\nu g^{\mu\alpha},$$
$$H^\mu_a \to \tau^\lambda \partial_\lambda H^\mu_a + \partial_\nu \tau^\mu H^\nu_a + \partial_\alpha \tau^\beta (\eta^{dp}\eta_{ba} h^\alpha_p h^b_\beta - h^\alpha_a h^d_\beta) H^\mu_d.$$

The total gauge-invariant Lagrangian of RTG can be given by the sum (7) of the Lagrangian (29) where the metric field h is replaced by the effective metric g (6) and the Lagrangian $L_q(q,h)$ of gravitational fields q where contraction of indices is performed by means of the background metric h. Note that also the effective tetrad functions H^μ_a and the tensor fields $\Omega^\alpha{}_{\beta\nu} = k^\alpha{}_{\beta\nu} - \Gamma_h{}^\alpha{}_{\beta\nu}$ can be chosen as dynamic variables of the affine-metric RTG. In absence of fermion fields, we have the solution $k^\alpha{}_{\beta\nu} = \{^\alpha{}_{\beta\nu}\}$ where $\{^\alpha{}_{\beta\nu}\}$ are the Cristoffel symbols of the effective metric g.

7. Energy-momentum conservation laws

As it has been showen [3, 13], if the Lagrangian (29) of the gauge gravitation theory is invariant under general covariant transformations, that is, its Lie derivatives along the vector fields (31) vanish, then there is the energy-momentum conservation law

$$d_\mu \mathcal{T}^\mu \approx 0, \qquad \mathcal{T}^\mu \approx d_\lambda U^{\mu\lambda}, \tag{35}$$

on shell where the energy-momentum flow is reduced to the generalized Komar superpotential

$$U^{\mu\lambda} = 2\frac{\partial\mathcal{L}}{\partial R^\alpha{}_{\nu\mu\lambda}}(\partial_\nu\tau^\alpha + \Gamma^\alpha{}_{\sigma\nu}\tau^\sigma). \tag{36}$$

Note that fermion fields do not make any contribution into the conservation law (35) and into the superpotential (36).

Similarly, let us examine the energy-conservation laws in RTG. We get the relation

$$d_\lambda(2\tau^\mu h^{\lambda\alpha}\frac{\partial\mathcal{L}_{\rm RG}}{\partial h^{\alpha\mu}} + d_\mu U^{\lambda\mu}) + (\tau^\lambda\partial_\lambda h^{\mu\nu} - \partial_\alpha\tau^\mu h^{\alpha\nu} + \partial_\alpha\tau^\nu h^{\mu\alpha})\frac{\partial\mathcal{L}_{\rm RG}}{\partial h^{\mu\nu}} \approx 0.$$

A glance at this relation shows that, if the Lagrangian $L_{\rm RG}$ contains the Higgs term L_q, the energy-momentum flow does not reduce to the superpotential, and we have the standard covariant conservation law

$$\nabla_\alpha T^\alpha_\lambda \approx 0, \qquad \sqrt{\mid h\mid}T^\alpha_\lambda = 2h^{\alpha\mu}\frac{\partial\mathcal{L}_q}{\partial h^{\mu\nu}}, \tag{37}$$

where ∇_α are the covariant derivatives with respect to the Christoffel symbols of the background metric h. In particular, (37) is brought into the conservation law if h is the Minkowski metric.

Note that one can express the energy-momentum tensor T (37) into the tensor

$$\sqrt{\mid h\mid}\widetilde{T}^\alpha_\lambda = 2g^{\alpha\mu}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}$$

owing to the field equations

$$\frac{\partial\mathcal{L}_{\rm RG}}{\partial g^{\alpha\mu}} = \frac{\partial\mathcal{L}}{\partial g^{\alpha\mu}} + \frac{\partial\mathcal{L}_q}{\partial h^{\alpha\mu}} = 0. \tag{38}$$

In particular, if L_q is equal to the Higgs massive term (8), we have the relation

$$\sqrt{\mid h\mid}T^\alpha_\lambda = \sqrt{\mid g\mid}(\widetilde{T}^\alpha_\lambda + \frac{1}{2}\delta^\alpha_\lambda\widetilde{T}^\mu_\mu).$$

Let us emphasize that this relation does not depend on the constant λ_3 which therefore can be as small as will. At the same time, different constants λ_3 lead to different solutions of the gravitation equations (38).

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